Topics in Discrete Mathematics

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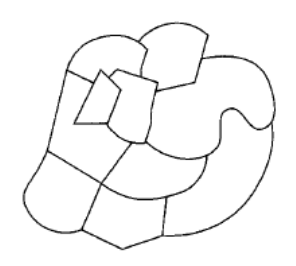
LSSU Math 216

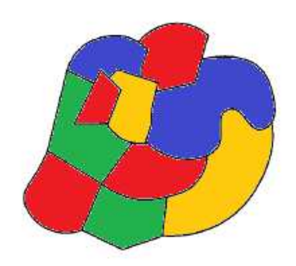
Subsection 1

Fundamentals of Graph Theory

Map Coloring I

- Imagine a map with several countries.





  To show the different countries clearly, we fill their regions with various colors.
- For simplicity, we use as few colors as possible.
- However, neighboring countries should not receive the same color so that they be distinguishable.
- What is the smallest number of colors we need to color he map?

Map Coloring II

- In the general version, we do require that some restrictions hold:
  - We do not allow countries that are disconnected.
  - Regions that touch at just one point need not receive different colors.
- The map in the figure is colored with just four colors.
- Several related questions can be asked:
  - Can this map be colored with fewer than four colors?
  - Is there another map that can be colored with fewer than four colors?
  - Is there a map that requires more than four colors?
- The third question is known as the **four color map problem**.
  - It was first posed in 1852 by Francis Guthrie.
  - Only in the mid 1970s, did Appel and Haken prove that every map can be colored using at most four colors.

Examination Scheduling

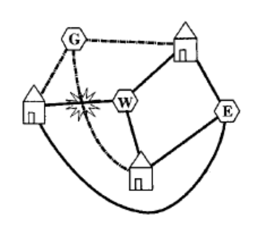
- At a university, there is an examination period at the end of a term in which each course has a 3-hour final exam. On any given day, the university can schedule two final exams. To avoid conflicts, the university wishes to devise a final examination schedule with the condition that if a student is enrolled in two courses, these courses must get different examination periods.
- A simple solution to this problem is to hold only one examination during any time slot. This would needlessly prolong the examination period! The solution the university prefers is to have the smallest possible number of examination slots.
- Even though this scheduling problem seems unrelated to map coloring, these problems are essentially the same!
  - In map coloring, we seek the least number of colors, subject to a special condition.
  - In exam scheduling, we seek the least number of time slots subject to a special condition.

Utilities and Printed Circuits

- We want to run connections from three utility (gas, water, electricity) plants to three homes, i.e., three electric wires, three water pipes, and three gas lines.

  We may place the houses and the utility plants anywhere, but no two wires/pipes/lines are allowed to cross!

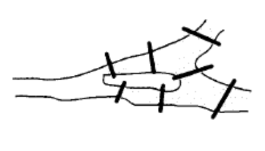
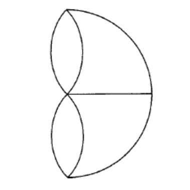
- A failed attempt to construct a suitable layout: No solution is possible.



- A printed circuit board is a flat piece of plastic on which various electronic devices (resistors, capacitors, integrated circuits, etc.) are mounted. Connections between devices are made by printing bare metal wires onto the surface of the board. No two of these wires may cross to avoid short circuits. Can we print the various connecting wires onto the board in such a way that there are no crossings?

Seven Bridges I

- In the 1700 city of Königsberg (now Kaliningrad), there are seven bridges:

  A count, who enjoyed a walk through the city wondered whether there was a tour he could take so that he could cross every bridge exactly once.
- No such tour is possible.
- Euler created a diagram in which each line represents a bridge.
- The problem of walking the seven bridges is now replaced by the problem of drawing the abstract figure without lifting the pencil from the paper and without redrawing a line. Can this figure be so drawn?

Seven Bridges II

- There are four places where lines come together; at each of these places, the number of lines is odd.
- Claim: A point where an odd number of lines meet must be either the starting point or the finishing point of a "proper" drawing.
- Since there are four points with the property, not every point in the diagram can be either the first or the last point in the drawing. It is impossible to tour the city of Königsberg and cross each of the seven bridges exactly once.
- Suppose a small city can afford only one garbage truck for garbage collection. Can we set the route the garbage truck is to follow? It needs to collect along every street in the city, while it would be wasteful to traverse the same street more than once. Is there a route for the garbage truck so that it travels only once down every street?
- If the city has more than two intersections where an odd number of roads meet, then such a tour is not possible.

Graphs

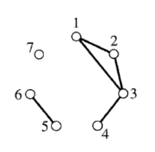
Definition (Graph)

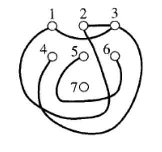
A **graph** is a pair $G = (V, E)$, where V is a finite set and E is a set of two-element subsets of V.

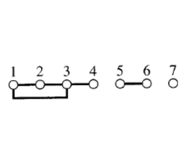
- Example: Let $G = (\{1,2,3,4,5,6,7\}, \{\{1,2\},\{1,3\},\{2,3\},\{3,4\}, \{5,6\}\})$. Here V is the finite set $\{1,2,3,4,5,6,7\}$ and E is a set containing 5 two-element subsets of V: $\{1,2\},\{1,3\},\{2,3\},\{3,4\}$ and $\{5,6\}$. Therefore $G = (V, E)$ is a graph.
- The elements of V are called the **vertices** (singular: **vertex**) of the graph; the elements of E are called the **edges** of the graph.
- The graph of the example has seven vertices and five edges.

Drawings of Graphs

- There is a nice way to draw pictures of graphs, which make graphs easier to understand.
- Keep in mind, however, that a picture of a graph is not the same thing as the graph itself!
- We draw a dot for each vertex in V and each edge $e = \{u, v\} \in E$ is drawn as a curve joining the dot for u to the dot for v.
- Example: The following three pictures all depict the graph from the previous example:







- Conversely, we can "read" the pictures and determine the vertices and edges of the graph.

Adjacency and Incidence

Definition (Adjacent)

Let $G = (V, E)$ be a graph and let $u, v \in V$. We say that u is **adjacent** to v provided $\{u, v\} \in E$. The notation $u \sim v$ means that u is adjacent to v.

- If $\{u, v\}$ is an edge of G, we call u and v the **endpoints** of the edge.
- For convenience, we avoid writing the curly braces for an edge $\{u, v\}$ and write uv in place of $\{u, v\}$.
- If v is a vertex and an endpoint of the edge e, we may write $v \in e$, since e is a two-element set, one of whose elements is v. We also say that v is **incident on** (or **incident with**) e.

Properties of Adjacency

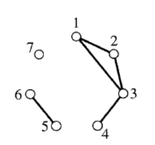
- Which of the various properties of relations does "is-adjacent-to" exhibit?
  - Is $\sim$ reflexive? No.
  - Is $\sim$ irreflexive? Yes!
  - Is $\sim$ symmetric? Yes!
    If $u \sim v$ in G, then $\{u, v\}$ is an edge of G. Of course, $\{u, v\}$ is the exact same thing as $\{v, u\}$, so $v \sim u$. Therefore, $\sim$ is symmetric.
  - Is $\sim$ antisymmetric? No (in general).
    But there are graphs with antisymmetric adjacency relations.
  - Is $\sim$ transitive? No (in general).
    But there are graphs with transitive adjacency relations.

(Simple) Graphs versus Multigraphs

- Note that it is never the case that $\{u, u\}$ is an edge of a graph. Thus a vertex is never adjacent to itself and therefore $\sim$ is irreflexive.
- According to our definition, an edge of a graph is a two-element subset of V.
- Some mathematicians use the word *graph* in a different way and allow the possibility that a vertex could be adjacent to itself; an edge joining a vertex to itself is called a **loop**.
  For us, graphs are not allowed to have loops.
- Some authors also allow more than one edge with the same endpoints; such edges are called **parallel edges**.
  For us, graphs may not have parallel edges.
- When we want to be perfectly clear, we use the term **simple graph**.
- If we wish to discuss a "graph" that may have loops and multiple edges, we use the word **multigraph**.

Degrees

- Let $G = (V, E)$ be a graph and suppose u and v are vertices of G. If u and v are adjacent, we also say that u and v are **neighbors**. The set of all neighbors of a vertex v is called the **neighborhood** of v and is denoted $N(v)$, i.e., $N(v) = \{u \in V : u \sim v\}$.
- Example: For the graph



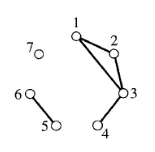
$N(1) = \{2, 3\}$, $N(5) = \{6\}$, $N(2) = \{1, 3\}$,
$N(6) = \{5\}$, $N(3) = \{1, 2, 4\}$, $N(7) = \emptyset$.

Definition (Degree)

Let $G = (V, E)$ be a graph and let $v \in V$. The **degree** of v is the number of edges with which v is incident. The degree of v is denoted $d_G(v)$ or, if there is no risk of confusion, simply $d(v)$.

- Example: In the previous graph, we have $d(1) = 2$, $d(2) = 2$, $d(3) = 3$, $d(4) = 1$, $d(5) = 1$, $d(6) = 1$, $d(7) = 0$.

Some Remarks on the Example



Note that
$\sum_{v \in V} d(v) = d(1) + d(2) + d(3) + d(4) + d(5) + d(6) + d(7) = 2 + 2 + 3 + 1 + 1 + 1 + 0 = 10$
and this is twice the number of edges in G.

- The following matrix is the adjacency matrix of G, i.e. has entry 1 in row i and column j if $v_i \sim v_j$ and 0, otherwise.

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Notice that for every edge of G there are exactly two 1's in the chart. So, the number of 1's in the matrix is exactly $2|E|$.
- Notice also that the number of 1's in each row is exactly the degree of that vertex. Therefore, the total number of 1's is the sum of all degrees.

A Theorem on the Sum of Degrees

Theorem

Let $G = (V, E)$ be a graph. The sum of the degrees of the vertices in G is twice the number of edges, i.e., $\sum_{v \in V} d(v) = 2|E|$.

- Suppose the vertex set is $V = \{v_1, v_2, \ldots, v_n\}$. Consider the adjacency matrix of the graph. How many 1's are in this matrix?
  - For every edge of G there are exactly two 1's in the matrix. Thus, the total number of 1's in this matrix is exactly $2|E|$.
  - Consider a given row corresponding to some vertex v_i. There is a 1 in this row exactly for those vertices adjacent to v_i. Thus, the number of 1's in this row is exactly the degree $d(v_i)$ of the vertex v_i. The total number of 1's in the matrix is the sum of the row subtotals, i.e., equals the sum of the degrees of the vertices of the graph.

These two answers are both correct counts of the number of 1's in the matrix. Thus, the sum of the degrees of the vertices of G equals twice the number of edges of G.

Additional Notation and Terminology

- Some additional graph theoretic terminology:
  - The **maximum degree** of a vertex in G is denoted $\Delta(G)$. The **minimum degree** of a vertex in G is denoted $\delta(G)$.
  - If all vertices in G have the same degree, we call G **regular**. If a graph is regular and all vertices have degree r, G is called r-**regular**.
  - Let $G = (V, E)$ be a graph. To show explicitly that V and E are the vertex and edge sets of G, respectively, we sometimes write $V = V(G)$ and $E = E(G)$.
  - Let $G = (V, E)$ be a graph. The **order** of G is the number of vertices in G, i.e., $|V(G)|$. The **size** of G is the number of edges, i.e., $|E(G)|$. Sometimes we use $\nu(G) = |V(G)|$ and $\epsilon(G) = |E(G)|$.
  - Let G be a graph. If all pairs of distinct vertices are adjacent in G, we call G **complete**. A complete graph on n vertices is denoted K_n. The opposite extreme is a graph with no edges. We call such graphs **edgeless**. A graph with no vertices (and hence no edges) is called an **empty graph**.

Subsection 2

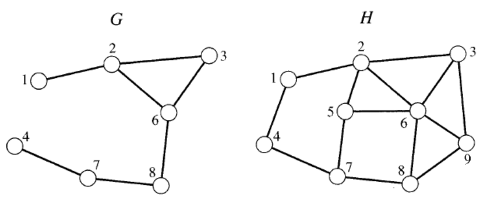
Subgraphs

Subgraphs

Definition (Subgraph)

Let G and H be graphs. We call G a **subgraph of** H provided $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$.

- Example: Let G and H be the following graphs: $V(G) = \{1,2,3,4,6,7,8\}$, $E(G) = \{\{1,2\},\{2,3\},\{2,6\},\{3,6\},\{4,7\},\{6,8\},\{7,8\}\}$, and $V(H) = \{1,2,3,4,5,6,7,8,9\}$, $E(H) = \{\{1,2\},\{1,4\},\{2,3\},\{2,5\},\{2,6\},\{3,6\},\{3,9\},\{4,7\},\{5,6\},\{5,7\},\{6,8\},\{6,9\},\{7,8\},\{8,9\}\}$. Notice that $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$, and so G is a subgraph of H.



Naturally, if G is a subgraph of H, we call H a **supergraph of** G.

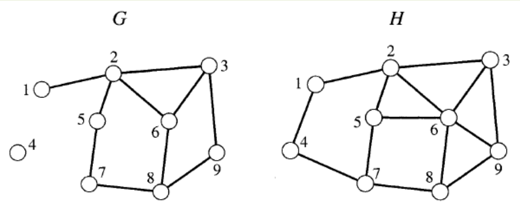
Spanning Subgraphs

- We form a subgraph G from a graph H by deleting some parts of H:
  - If e is an edge of H, then removing e from H results in a new graph that we denote $H - e$.

Definition (Spanning Subgraph)

Let G and H be graphs. We call G a **spanning subgraph of** H provided G is a subgraph of H, and $V(G) = V(H)$.

- Example: Let H be the graph of the previous slide and let G be the graph with $V(G) = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and $E(G) = \{\{1, 2\}, \{2, 3\}, \{2, 5\}, \{2, 6\}, \{3, 6\}, \{3, 9\}, \{5, 7\}, \{6, 8\}, \{7, 8\}, \{8, 9\}\}$.



G and H have the same vertex set, whence G is a spanning subgraph of H.

Induced Subgraphs

- Deleting vertices from a graph is more subtle: When we delete a vertex v from H, we must delete all edges that are incident with v. So the graph $H - v$ is defined by $V(H - v) = V(H) - \{v\}$, $E(H - v) = \{e \in E(H) : v \notin e\}$.

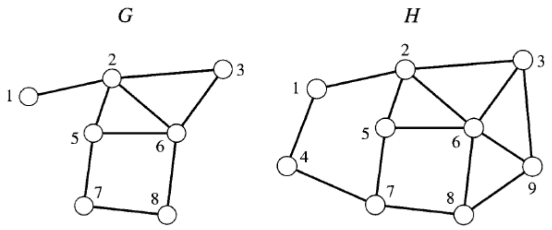
Definition (Induced Subgraph)

Let H be a graph and let A be a subset of the vertices of H. The **subgraph of** H **induced on** A is the graph $H[A]$ defined by $V(H[A]) = A$, and $E(H[A]) = \{xy \in E(H) : x \in A \text{ and } y \in A\}$.

- G is an **induced subgraph** of H, if $G = H[A]$ for some $A \subseteq V(H)$.
- The graph $H - v$ is an induced subgraph of H. If $A = V(H) - \{v\}$, then $H - v = H[A]$.

Induced Subgraphs: An Example

- Again, consider the graph H of the previous slide. Let G be the graph with $V(G) = \{1, 2, 3, 5, 6, 7, 8\}$, and $E(G) = \{\{1, 2\}, \{2, 3\}, \{2, 5\}, \{2, 6\}, \{3, 6\}, \{5, 6\}, \{5, 7\}, \{6, 8\}, \{7, 8\}\}$.



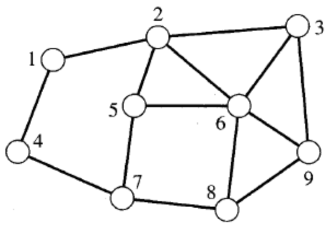
Note that G is a subgraph of H. From H we deleted vertices 4 and 9. We have included in G every edge of H except those edges incident with vertices 4 or 9. Thus G is an induced subgraph of H and $G = H[A]$ where $A = \{1, 2, 3, 5, 6, 7, 8\}$. We can also write $G = (H - 4) - 9 = (H - 9) - 4$.

Clique and Clique Number

Definition (Clique and Clique Number)

Let G be a graph. A subset of vertices $S \subseteq V(G)$ is called a **clique** provided any two distinct vertices in S are adjacent. The **clique number** of G is the size of a largest clique and is denoted $\omega(G)$.

- Note that $S \subseteq V(G)$ is a clique provided $G[S]$ is a complete graph.
- Example: Let H be the graph of the previous slide.



Some of his cliques are $\{1,4\}$, $\{2,5,6\}$, $\{9\}$, $\{2,3,6\}$, $\{6,8,9\}$, $\{4\}$, $\emptyset$. The largest size of a clique in H is 3, so $\omega(H) = 3$. The clique $\{1,4\}$ contains two vertices, so it does not have the largest possible size for a clique in H.

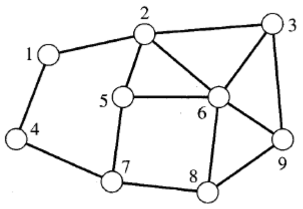
However, it cannot be extended. It is a maximal clique that does not have maximum size. Maximal means "cannot be extended". Maximum means "largest".

Independent Set and Independence Number

Definition (Independent Set, Independence Number)

Let G be a graph. A subset $S \subseteq V(G)$ is called an **independent set** provided no two vertices in S are adjacent. The **independence number** of G is the size of a largest independent set and is denoted $\alpha(G)$.

- A set $S \subseteq V(G)$ is independent provided $G[S]$ is an edgeless graph.
- Example: Let H be the same graph as before.



Some of his independent sets are $\{1,3,5\}$, $\{1,7,9\}$, $\{4\}$, $\{1,3,5,8\}$, $\{4,6\}$, $\{1,3,7\}$, $\emptyset$. The largest size of an independent set in H is 4, so $\alpha(H) = 4$.

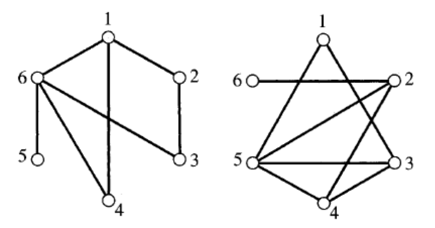
The independent set $\{4,6\}$ is not a largest independent set, but it is a maximal independent set.

Complements, Cliques and Independent Sets

Definition (Complement)

Let G be a graph. The **complement of** G is the graph, denoted $\overline{G}$, defined by $V(\overline{G}) = V(G)$, and
$E(\overline{G}) = \{xy : x, y \in V(G), x \neq y, xy \notin E(G)\}$.

- Example: The two graphs in the figure are complements of one another.



Proposition (Cliques and Independent Sets)

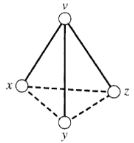
Let G be a graph. A subset of $V(G)$ is a clique of G if and only if it is an independent set of $\overline{G}$. Furthermore, $\omega(G) = \alpha(\overline{G})$ and $\alpha(G) = \omega(\overline{G})$.

A Taste of Ramsey Theory

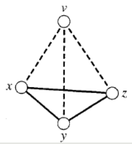
Proposition (Taste of Ramsey Theory)

Let G be a graph with at least six vertices. Then $\omega(G) \geq 3$ or $\omega(\overline{G}) \geq 3$.

- Equivalently, $\omega(G) \geq 3$ or $\alpha(G) \geq 3$.
- Let v be any vertex of G. We consider two possibilities: either $d(v) \geq 3$ or $d(v) < 3$.
  - If $d(v) \geq 3$, v has at least three neighbors x, y, z.



    - If one (or more) of xy, yz, or xz is an edge of G, then G contains a clique of size 3. So $\omega(G) \geq 3$.
    - If none of xy, yz, or xz is in G, then all three are edges of $\overline{G}$. So $\omega(\overline{G}) \geq 3$.
  - If $d(v) \leq 2$, there are three vertices x, y, z not adjacent to v.



    - If all of xy, yz, xz are edges of G, then clearly G has a clique of size 3. So $\omega(G) \geq 3$.
    - If one (or more) of xy, yz, or xz is not in G, then we have a clique of size 3 in $\overline{G}$. So $\omega(\overline{G}) \geq 3$.

Subsection 3

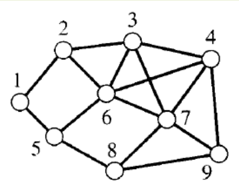
Connectedness

Walks

Definition (Walk)

Let $G = (V, E)$ be a graph. A **walk** in G is a sequence (or list) of vertices, with each vertex adjacent to the next, i.e., $W = (v_0, v_1, \ldots, v_\ell)$ with $v_0 \sim v_1 \sim v_2 \sim \cdots \sim v_\ell$. The **length** of this walk is ℓ. There are $\ell + 1$ vertices and ℓ edges on the walk.

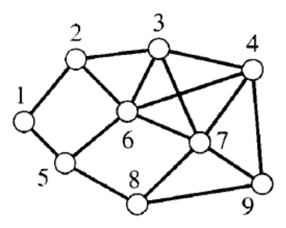
- Example: Consider the graph



The sequence $1 \sim 2 \sim 3 \sim 4$ is a walk of length three. It starts at vertex 1 and ends at vertex 4, and so we call it a $(1, 4)$-walk.

- In general, a (u, v)-**walk** is a walk in a graph whose first vertex is u and whose last vertex is v.

Reversals and Closed Walks



- $1 \sim 2 \sim 3 \sim 6 \sim 2 \sim 1 \sim 5$ is a walk of length six. There are seven vertices on this walk. We are permitted to visit a vertex more than once on a walk.
- $5 \sim 1 \sim 2 \sim 6 \sim 3 \sim 2 \sim 1$ is also a walk of length six. Notice that this sequence is exactly the reverse of that of the previous example.
- If $W = v_0 \sim v_1 \sim \cdots \sim v_{\ell-1} \sim v_\ell$, then its **reversal** is also a walk (because $\sim$ is symmetric). The **reversal of** W is $w^{-1} = v_\ell \sim v_{\ell-1} \sim \cdots \sim v_1 \sim v_0$.
- 9 is a walk of length zero.
- $1 \sim 5 \sim 1 \sim 5 \sim 1$ is a walk of length four. This walk is called **closed** because it begins and ends at the same vertex.
- Neither $(1, 1, 2, 3, 4)$ nor $(1, 6, 7, 9)$ is a walk ($1 \nsim 1$ and $1 \nsim 6$).

Concatenation

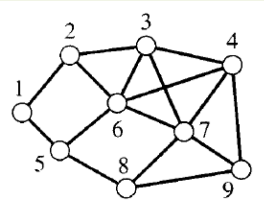
Definition (Concatenation)

Let G be a graph. Suppose W_1 and W_2 are the following walks:

$$\begin{aligned} W_1 &= v_0 \sim v_1 \sim \cdots \sim v_\ell \\ W_2 &= w_0 \sim w_1 \sim \cdots \sim w_k \end{aligned}$$

and suppose $v_\ell = w_0$. Their **concatenation**, denoted $W_1 + W_2$, is the walk

$$v_0 \sim v_1 \sim \cdots \sim (v_\ell = w_0) \sim w_1 \sim \cdots \sim w_k.$$

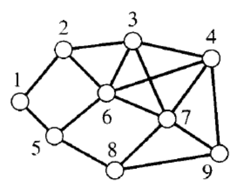


- Example: The concatenation of the walks $1 \sim 2 \sim 3 \sim 4$ and $4 \sim 7 \sim 3 \sim 2$ is the walk $1 \sim 2 \sim 3 \sim 4 \sim 7 \sim 3 \sim 2$.

Paths

Definition (Path)

A **path** in a graph is a walk in which no vertex is repeated.



- Example: The walk $1 \sim 2 \sim 6 \sim 7 \sim 3 \sim 4$ is a path. It is also called a $(1,4)$-path because it begins at vertex 1 and ends at vertex 4.
- In general, a (u,v)-**path** is a path whose first vertex is u and whose last vertex is v.
- If a walk (or path) is of the form $\cdots \sim u \sim v \sim \cdots$, then we say that the walk (or path) **used** or **traversed** the edge uv.

A Proposition

Proposition

Let P be a path in a graph G. Then P does not traverse any edge of G more than once.

- Suppose, for the sake of contradiction, that some path P in a graph G traverses the edge $e = uv$ more than once. Without loss of generality, we have $P = \cdots \sim u \sim v \cdots \sim u \sim v \sim \ldots$ or $P = \cdots \sim u \sim v \sim \cdots \sim v \sim u \sim \cdots$. In the first case, we have repeated both u and v, and in the second, we have repeated u, contradicting the fact that P is a path.

Definition (Path Graph)

A **path** is a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{v_i v_{i+1} : 1 \leq i < n\}$. A path with n vertices is denoted P_n.

- Given a sequence of vertices in G constituting a path, we can also view that sequence as a subgraph of G.

From Walks to Paths

Lemma

Let G be a graph and let $x, y \in V(G)$. If there is an (x, y)-walk in G, then there is an (x, y)-path in G.

- Suppose there is an (x, y)-walk in a graph G. The length of an (x, y)-walk is a natural number. Thus, by the Well-Ordering Principle, there is a shortest (x, y)-walk P.
- Claim: P is an (x, y)-path.
  Suppose, for the sake of contradiction, that P is not an (x, y)-path. Then there must be some vertex u that is repeated on the path. In other words, $P = x \sim \cdots \sim ? \sim u \sim \cdots \sim u \sim ?? \sim \cdots y$. We do not rule out the possibility that $u = x$ and/or $u = y$. Form a new walk P' by deleting $u \sim \cdots \sim u$. The result in a new walk in which ? and ?? are both adjacent to u. So the shortened sequence P' is still an (x, y)-walk. This contradicts the fact that P is a shortest (x, y)-walk.

Connection

Definition (Connected to)

Let G be a graph and let $u, v \in V(G)$. We say that u is **connected to** v provided there is a (u, v)-path in G, i.e., a path whose first vertex is u and whose last vertex is v.

- The relation "is-connected-to" is reflexive, but not (in general) irreflexive or antisymmetric.
- The "is-connected-to" is a symmetric relation. Suppose, in a graph G, vertex u is connected to vertex v. Thus, there is a (u, v)-path P in G. Its reversal P^{-1} is a (v, u)-path, and so v is connected to u.
- The "is-connected-to" relation is transitive. Suppose, in a graph G, x is connected to y and y is connected to z. Thus, there exist an (x, y)-path P and a (y, z)-path Q. The concatenation $P + Q$ is an (x, z)-walk, but nor necessarily an (x, z)-path. However, the existence of an (x, y)-walk implies the existence of an (x, y)-path! Therefore x is connected to z.

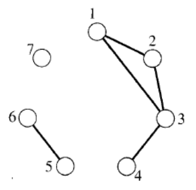
The Relation "is-connected-to" is an Equivalence

Theorem

Let G be a graph. The "is-connected-to" relation is an equivalence relation on $V(G)$.

- "is-connected-to" is reflexive, symmetric, and transitive.
- Since "is-connected to" is an equivalence relation, its equivalence classes form a partition of the vertex set. Let u and v be vertices of a graph G.
  - If $u, v \in V(G)$ are in the same equivalence class, then there is a path joining them.
  - On the other hand, if u and v are in different equivalence classes, there is no path joining u to v, or vice versa.
- Example:

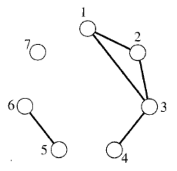
  The equivalence classes of the "is-connected-to" relation on this graph are $\{1,2,3,4\}$, $\{5,6\}$, and $\{7\}$.



Components and Connectedness

Definition (Components)

A **component of** G is a subgraph of G induced on an equivalence class of the "is-connected-to" relation on $V(G)$.



- The graph on the left has three components: $G[\{1,2,3,4\}]$, $G[\{5,6\}]$, and $G[\{7\}]$. The first has four vertices and four edges, the second has two vertices and one edge and the third component has just one vertex and no edges.
- If a graph is edgeless, then each vertex forms a component.
- If there is only one component, we call the graph connected.

Definition (Connected)

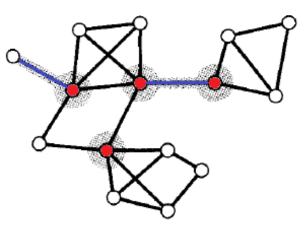
A graph is called **connected** provided each pair of vertices in the graph are connected by a path, i.e., for all $x, y \in V(G)$, there is an (x, y)-path.

Cut Vertex and Cut Edge

Definition (Cut Vertex, Cut Edge)

Let G be a graph. A vertex $v \in V(G)$ is called a **cut vertex** of G provided $G - v$ has more components than G.
Similarly, an edge $e \in E(G)$ is called a **cut edge** of G provided $G - e$ has more components than G.

- If G is a connected graph, a cut vertex v is a vertex such that $G - v$ is disconnected. Likewise e is a cut edge if $G - e$ is disconnected.
- Example:



The graph in the figure has two cut edges (blue) and four cut vertices (red).

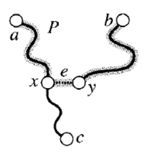
A Theorem

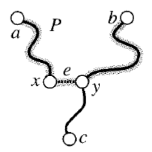
Theorem

Let G be a connected graph and suppose $e \in E(G)$ is a cut edge of G. Then $G - e$ has exactly two components.

- Let G be a connected graph and let $e \in E(G)$ be a cut edge. Because G is connected, it has exactly one component. Because e is a cut edge, $G - e$ has more components than G. Suppose $G - e$ has three or more components. Let a, b and c be three vertices of $G - e$, each in a separate component in $G - e$, i.e., there is no path in $G - e$ joining any pair of them. Let P be an (a, b)-path in G. Because there is no (a, b)-path in $G - e$, we know P must traverse the edge e. Suppose x and y are the endpoints of the edge e, and the path P traverses e in the order x, then y, i.e., $P = a \sim \cdots \sim x \sim y \sim \cdots \sim b$.

Proof of the Theorem (Cont'd)

- Similarly, since G is connected, there is a path Q from c to a that must use the edge $e = xy$. Which vertex, x or y, appears first on Q as we travel from c to a?
  - 

    If x appears before y on the (c, a)-path Q, then we have, in $G - e$, a walk from c to a: Use the (c, x)-portion of Q, concatenated with the (x, a)-portion of P^{-1}. This yields a (c, a)-walk in $G - e$ and, hence, a (c, a)-path in $G - e$, which is a contradiction.
  - 

    If y appears before x on the (c, a)-path Q, then we have, in $G - e$, a walk from c to b: Concatenate the (c, y)-section of Q with the (y, b)-section of P. This walk does not use the edge e. Therefore there is a (c, b)-walk in $G - e$ and hence a (c, b)-path in $G - e$, which is a contradiction.

Therefore, $G - e$ has at most two components.

Subsection 4

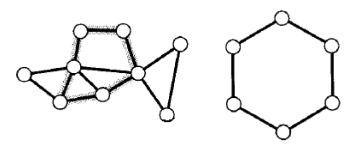
Trees

Cycles

Definition (Cycle)

A **cycle** is a walk of length at least three in which the first and last vertex are the same, but no other vertices are repeated. The term cycle also refers to a (sub)graph consisting of the vertices and edges of such a walk. In other words, a **cycle** is a graph of the form $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$, and $E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. A cycle (graph) on n vertices is denoted C_n.

- Example:



The left figure depicts a cycle of length six as a walk in a graph. The right figure shows the graph C_6.

Forests and Trees

Definition (Forest)

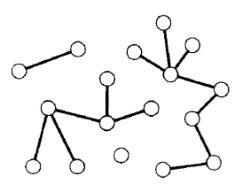
Let G be a graph. If G contains no cycles, then we call G **acyclic**. Alternatively, we call G a **forest**.

- A forest is (as in real life) a collection of trees.

Definition (Tree)

A **tree** is a connected, acyclic graph.

- Example: The forest in the figure contains four connected components. Each component of a forest is a tree.



K_1, K_2, P_3, P_4 and Star

- Note that a single isolated vertex, e.g., the graph K_1, is a tree.
- There is only one possible structure for a tree on two vertices: there must be an edge joining the two vertices. Any tree on two vertices must be a K_2.
- There is also only one possible structure for a tree on three vertices. Since the graph is connected, there certainly must be at least one edge, say joining vertices a and b. However, if there were only one edge, then the third vertex, c, would not be connected to either a or b, and so the graph would not be connected. Thus, there must be at least one more edge, say that it is the edge from b to c. So far we have $a \sim b \sim c$, but $ac \notin E$. Might we also add the edge ac? If we do, the graph is connected, but it is no longer acyclic, as we would have the cycle $a \sim b \sim c \sim a$. Any tree on three vertices is a P_3.
- However, on four vertices, we can have two different sorts of trees. We can have the path P_4 and we can have a **star**: a graph of the form $G = (V, E)$, where $V = \{a, x, y, z\}$ and $E = \{ax, ay, az\}$.

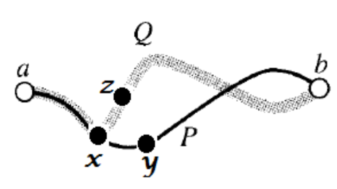
Uniqueness of (a, b)-Paths in Trees

Theorem

Let T be a tree. For any two vertices a and b in $V(T)$, there is a unique (a, b)-path. Conversely, if G is a graph with the property that, for any two vertices u, v, there is exactly one (u, v)-path, then G must be a tree.

- $(\Rightarrow)$ Suppose T is a tree and let $a, b \in V(T)$.
  - Existence: There exists an (a, b)-path because trees are connected.
  - Uniqueness: Suppose, for the sake of contradiction, there were two different (a, b)-paths P and Q in T. Since P and Q are different paths, at some point one of them traverses a different edge than the other.

  Let us say that from a to x the paths are the same, but then they traverse different edges, i.e., $P : a \sim \cdots \sim x \sim y \sim \cdots \sim b$ and $Q : a \sim \cdots \sim x \sim z \sim \cdots \sim b$.



  Now consider the graph $T - xy$.

Uniqueness of Paths in Trees (Cont'd)

- We are proving ($\Rightarrow$):
  - We finish uniqueness:
    - Claim: There is an (x, y)-path R in $T - xy$.
      Notice that there is an (x, y)-walk in $T - xy$: Start at x, follow P^{-1} from x to a, follow Q from a to b, and then follow P^{-1} from b to y. Thus, there is an (x, y)-path in $T - xy$.

    The path R must contain at least one vertex in addition to x and y because R does not use the edge xy. If we add the edge xy to the path R, we have a cycle, which is a contradiction.
- ($\Leftarrow$) Let G be a graph with the property that between any two vertices there is exactly one path. We must prove that G is a tree, i.e., connected and acyclic. Both are relatively easy to see.

An Alternative Characterization

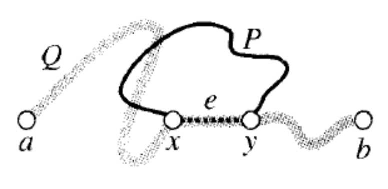
Theorem

Let G be a connected graph. Then G is a tree if and only if every edge of G is a cut edge.

- Let G be a connected graph.
  - ($\Rightarrow$) Suppose G is a tree. Let $e = xy$ be any edge of G. We must prove that $G - e$ is disconnected. In G there is an (x, y)-path $x \overset{e}{\sim} y$. Since there is no other (x, y)-path, if we delete e from G, there can be no (x, y)-paths. Therefore e is a cut edge.
  - ($\Leftarrow$) Suppose every edge of G is a cut edge. Since, by assumption, G is connected, we must show that G is acyclic. Suppose that G contains a cycle C. Let $e = xy$ be an edge of this cycle. The vertices and other edges of C form an (x, y)-path P. Since e is a cut edge of G, $G - e$ is disconnected. Thus, there exist vertices a, b for which there is no (a, b)-path in $G - e$. However, in G, there is an (a, b)-path Q. Hence Q must traverse the edge e.

Proof of the Alternative Characterization (Cont'd)

- We were showing that, if every edge of G is a cut edge, then G is acyclic.
  - We had created the following setting:



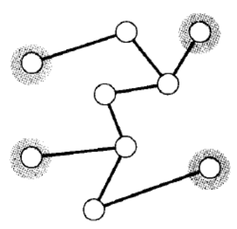
We argued that in G, there is an (a,b)-path Q that must traverse the edge e.
Without loss of generality, we traverse e from x to y as we step along Q: $Q = a \sim \cdots \sim x \sim y \sim \cdots \sim b$. In $G - e$ there is an (a,b)-walk. We traverse Q from a to x, then P from x to y, and then Q from y to b. This implies that in $G - e$ there is an (a,b)-path, contradicting the fact that there is no such path.

Leaves

Definition (Leaf)

A **leaf** of a graph is a vertex of degree 1.

- Leaves are also called **end vertices** or **pendant vertices**.



- Example: The tree in the figure has four leaves.

- The empty graph and the graph K_1 are trees, and they have no vertices of degree 1.
- Other than those two, every tree has a leaf.

Existence of Leaves

Theorem (Existence of Leaves)

Every tree with at least two vertices has a leaf.

- Let T be a tree with at least two vertices. Let P be a longest path in T, i.e., P is a path in T and there are no paths in T that are longer. Since T is connected and contains at least two vertices, P has two or more vertices. Suppose $P = v_0 \sim v_1 \sim \cdots \sim v_\ell$, where $\ell \geq 1$.

  Claim: The first and last vertices of P (v_0 and v_ℓ) are leaves of T. Suppose, for the sake of contradiction, that v_0 is not a leaf. Thus $d(v_0) \geq 2$ and v_0 has at least one neighbor other than v_1. Let x be another neighbor of v_0, i.e., $x \neq v_1$. Note that x is not a vertex on P, for otherwise we would have a cycle $v_0 \sim v_1 \sim \cdots \sim x \sim v_0$. Now $Q = x \sim v_0 \sim v_1 \sim \cdots \sim v_\ell$ is a path in T that is longer than P, which is a contradiction. Therefore v_0 is a leaf. Likewise v_ℓ is a leaf. Therefore T has at least two leaves.

Deleting a Leaf does not Spoil a Tree

Proposition

Let T be a tree and let v be a leaf of T. Then $T - v$ is a tree.

- We need to prove that $T - v$ is a tree.
  - $T - v$ is acyclic, since, if $T - v$ contained a cycle, that cycle would also exist in T.
  - We must show that $T - v$ is connected.
    Let $a, b \in V(T - v)$. We must show there is an (a, b)-path in $T - v$.
    Since T is connected, there is an (a, b)-path P in T.
    Claim: P does not include the vertex v.
    Otherwise, we would have $P = a \sim \cdots \sim v \sim \cdots \sim b$. Since v is neither the first nor the last vertex on this path, it has two distinct neighbors on the path. This contradicts the fact that $d(v) = 1$.
    Therefore P is an (a, b)-path in $T - v$, and so $T - v$ is connected and a tree.

Number of Edges in a Tree with n Vertices

Theorem (Number of Edges in a Tree)

Let T be a tree with $n \geq 1$ vertices. Then T has $n-1$ edges.

- We use induction on the number of vertices in T.
  - Basis Case: The theorem is true for all trees on $n = 1$ vertices. If T has only $n = 1$ vertex, then clearly it has $0 = n - 1$ edges.
  - Induction Hypothesis: Suppose the theorem is true for all trees on $n = k$ vertices.
  - Induction Step: Let T be a tree on $n = k + 1$ vertices. We need to prove that T has $n - 1 = k$ edges. Let v be a leaf of T and let $T' = T - v$. Note that T' is a tree with k vertices. By induction, T' satisfies the theorem, i.e., T' has $k - 1$ edges. Since v is a leaf of T, we have $d(v) = 1$. Hence, when we deleted v from T, we deleted exactly one edge. Therefore T has one more edge than T', i.e., T has $(k-1)+1 = k$ edges.

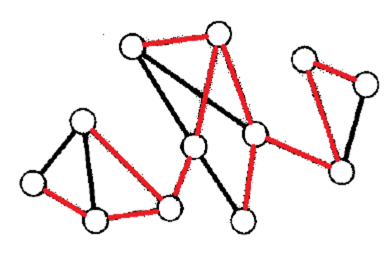
This concludes the induction.

Spanning Trees

Definition (Spanning Tree)

Let G be a graph. A **spanning tree of** G is a spanning subgraph of G that is a tree. (A spanning subgraph of G is a subgraph that has the same vertices as G.)

- Example: For the graph in the figure, we have colored in red one of its many spanning trees.



Existence of Spanning Trees

Theorem (Existence of Spanning Trees)

A graph has a spanning tree if and only if it is connected.

- ($\Rightarrow$) Suppose G is connected. Let T be a spanning connected subgraph of G with the least number of edges.
  Claim: T is a tree.
  By construction, T is connected. We claim that every edge of T is a cut edge. Otherwise, if $e \in E(T)$ were not a cut edge of T, then $T - e$ would be a smaller spanning connected subgraph of G, which is a contradiction. Therefore, every edge of T is a cut edge.
  By a previous theorem, T is a tree. So G has a spanning tree.
- ($\Leftarrow$) Suppose G has a spanning tree T. We must show that G is connected. Let $u, v \in V(G)$. Since T is spanning, we have $V(T) = V(G)$, whence $u, v \in V(T)$. Since T is connected, there is a (u, v)-path P in T. Since T is a subgraph of G, P is a (u, v)-path in G. Therefore G is connected.

Characterizing Trees Among Connected Graphs

Theorem

Let G be a connected graph on $n \geq 1$ vertices. Then G is a tree if and only if G has exactly $n-1$ edges.

- ($\Rightarrow$) This implication has already been proved.
- ($\Leftarrow$) Suppose G is a connected graph with n vertices and $n-1$ edges. We have shown that G has a spanning tree T. Thus, T is a tree, $V(T) = V(G)$, and $E(T) \subseteq E(G)$. Note, however, that

$$|E(T)| = |V(T)| - 1 = |V(G)| - 1 = |E(G)|.$$

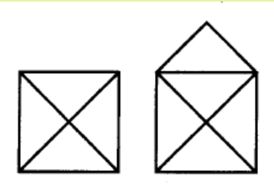
So we actually have $E(T) = E(G)$. Therefore, $G = T$, i.e., G is a tree.

Subsection 5

Eulerian Graphs

Eulerian Trails and Tours

- Consider the two figures shown below.



The figure on the left has four corners where an odd number of lines meet. Therefore, it is impossible to draw this figure without lifting the pencil or re-drawing a line.

- The figure on the right has only two corners with an odd number of lines. These points must be the first/last points in a drawing. The figure can be drawn without lifting the pencil or retracing a line.

Definition (Eulerian Trail, Tour)

Let G be a graph. A walk in G that traverses every edge exactly once is called an **Eulerian trail**. If, in addition, the trail begins and ends at the same vertex, we call the walk an **Eulerian tour**. Finally, if G has an Eulerian tour, we call G **Eulerian**.

Necessary Conditions

- A component of a graph is **trivial** if it contains only one vertex. Otherwise we call the component **nontrivial**.

Lemma

If G is Eulerian, then G has at most one nontrivial component.

- If the graph has two (or more) nontrivial components, it is impossible for the trail to visit more than one component, so there is no way we can traverse all the edges of the graph.

Lemma

If G has an Eulerian trail, then it has at most two vertices of odd degree.

- Suppose v is a vertex of a graph G in which there is an Eulerian trail W. If v is neither the first nor the last vertex on this trail, then v must have even degree: Every edge of the graph is traversed exactly once, and for every edge entering v there is another edge exiting v.

Additional Remarks

Lemma

If G has an Eulerian trail that begins at a vertex a and ends at a vertex b (with $a \neq b$), then vertices a and b have odd degree.

- There is one edge traversed from a when the trail begins. Then, every other time we visit a, an entering edge is paired with an exiting edge. Therefore, $d(a)$ is odd. The same is true for $d(b)$.

Lemma

If G has an Eulerian tour, i.e., if G is Eulerian, then all vertices in G have even degree.

- If the trail begins and ends at a, then $d(a)$ must be even.

A Final Remark

Lemma

If G is a connected Eulerian graph, then G has an Euler tour that begins/ends at any vertex.

- Suppose we have an Eulerian tour in a connected graph that begins and ends at a vertex a. Suppose b is the second vertex on this tour

$$W = a \sim b \sim \cdots \sim a.$$

We can, instead, begin the tour at b, follow the original tour until we get to the last visit to a, and finish at b:

$$W' = b \sim \cdots \sim a \sim b.$$

If we shift the tour repeatedly, we can begin an Eulerian tour at any vertex we choose.

Lemma I Supporting the Twin Theorems

Lemma

Let G be a graph all of whose vertices have even degree. Then no edge of G is a cut edge.

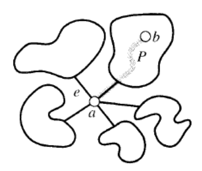
- Suppose $e = xy$ is a cut edge. Notice that $G - e$ has exactly two components and each of these components contains exactly one vertex of odd degree. This is a contradiction to having an even number of odd degree vertices.

Lemma II Supporting the Twin Theorems

Lemma

Let G be a connected graph with exactly two vertices of odd degree. Let a be a vertex of odd degree and suppose $d(a) \neq 1$. Then at least one of the edges incident with a is not a cut edge.

- Suppose that all edges incident at a are cut edges.



Let b be the other vertex of odd degree in G. Since G is connected, there is an (a, b)-path P in G. Exactly one edge incident at a is traversed by P. Let e be any other edge incident at a. Consider the graph $G' = G - e$. It has exactly two components. Since the path P does not use the edge e, vertices a and b are in the same component. In G', vertex a has even degree, and all other vertices in its component have not changed degree. Thus, in G', the component containing vertex a has exactly one vertex of odd degree, a contradiction.

Twin Theorems

Theorem

Let G be a connected graph all of whose vertices have even degree. For every vertex $v \in V(G)$, there is an Eulerian tour that begins and ends at v.

Theorem

Let G be a connected graph with exactly two vertices a and b of odd degree. Then G has an Eulerian trail that begins at a and ends at b.

- We give a single proof by induction on the number of edges in the graph.
  - Basis Case: Suppose G has 0 edges. Then G consists of just 1 isolated vertex v. The walk (v) is an Eulerian trail of G.
  - Another Basis Case: Suppose G has one edge. Since G is connected, the graph must consist of just two vertices, a and b, and a single edge joining them. Now G has exactly two vertices of odd degree, and $a \sim b$ is an Eulerian trail starting at one and ending at the other.

Proof of Twin Theorems I

- We continue the proof:
  - Induction Hypothesis: Suppose a connected graph has m edges.
    - If all of its vertices have even degree, then there is an Eulerian tour beginning/ending at any vertex.
    - If exactly two of its vertices have odd degree, then there is an Eulerian trail that begins at one of these vertices and ends at the other.
  - Induction Step: Let G be a connected graph with $m+1$ edges.
    - Case 1: All of G's vertices have even degree. Let v be an arbitrary vertex of G. Let w be any neighbor of v. Consider the graph $G' = G - vw$. Now G' has exactly two vertices of odd degree and is still connected. Thus, by induction, G' has an Eulerian trail W that begins at w and ends at v. If we add the edge vw to the beginning of W, the result is an Eulerian tour of G that begins/ends at v!

Proof of Twin Theorems II

- We continue the proof:
  - We are working on the Induction Step:
    - Case 2: Exactly two of G's vertices, a and b, have odd degree.
      Subcase 2a: Suppose $d(a) = 1$. In this case, a has exactly one neighbor x. It is possible that $x = b$ or $x \neq b$. Let $G' = G - a$. Then $d(x)$ drops by 1, while all other vertices have the same degree as before, G' has m edges and is connected.
      If $x = b$, then all vertices in G' have even degree. Therefore, by induction, G' has an Eulerian tour W that begins and ends at vertex b. If we insert the edge ab at the beginning of W, we have constructed an Eulerian trail that begins at a and ends at b.
      If $x \neq b$, then G' has exactly two vertices x and b of odd degree. Therefore, by induction, there is an Eulerian trail W that begins at x and ends at b. If we prepend the edge ax to W, we have an Eulerian trail in G that begins at a and ends at b.

Proof of Twin Theorems III

- We continue the proof:
  - We are working on the Induction Step:
    - Case 2: Exactly two of G's vertices, a and b, have odd degree.
      Subcase 2b: Suppose $d(a) > 1$. Since $d(a)$ is odd, we have $d(a) \geq 3$. In this case, at least one of the edges ax incident with a is not a cut edge. Let $G' = G - ax$. We might have $x = b$ or $x \neq b$.
      If $x = b$, then all vertices of G' have even degree. We can form, by induction, an Eulerian tour in G' that begins/ends at b. We, then, prepend the edge ab to form an Eulerian trail in G that begins at a and ends at b.
      If $x \neq b$, then we have exactly two vertices of odd degree in G', namely, x and b. By induction, we form, in G', an Eulerian trail that starts at x and ends at b. We prepend the edge ax to yield the requisite Euler trail in G.

  In all cases in the Induction Step, we constructed the required Eulerian trail/tour in G.

Fleury's Algorithm

Fleury's Algorithm

Let G be a graph all of whose vertices have even degrees or exactly two of whose vertices have odd degrees.

STEP 1: Choose any vertex v of G in the all-even case, or one of the two vertices with an odd degree in the other case.
Set $\text{CurrentVertex} = v$ and $\text{CurrentTrail} = \emptyset$.

Repeat

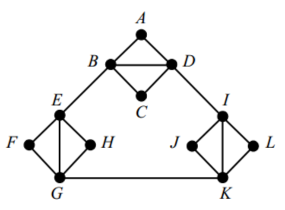
STEP 2: Select any edge $e = xy$ incident with the current vertex x but choosing a cut edge only if there is no alternative.
STEP 3: Add e to CurrentTrail and set $\text{CurrentVertex} = y$.
STEP 4: Delete e from the graph. together with any isolated vertices.

Until all edges have been deleted from G

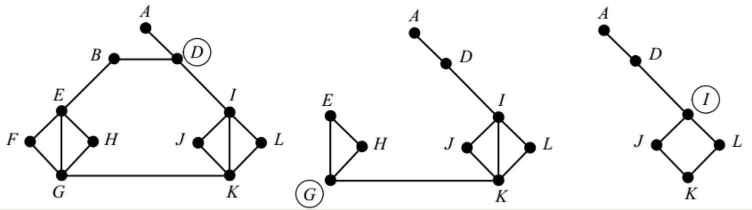
Return CurrentTrail.

Running Fleury's Algorithm

- Consider the graph



- In obtaining the trail *ABCDBEFGEHGKIJKLIDA* some of the intermediate steps are shown below:

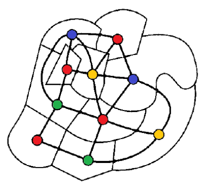


Subsection 6

Coloring

Graph Coloring

- Let G be a graph. To each vertex of G, we wish to assign a color.
  - The restriction is that adjacent vertices must receive different colors.
  - The objective is to use as few colors as possible.
- Example: Coloring a map can be converted to a graph-coloring problem by representing each country as a vertex of a graph.



Two vertices in this graph are adjacent exactly when the countries they represent share a common border. Thus coloring the countries on the map corresponds exactly to coloring the vertices of the graph.

Definition (Graph Coloring)

Let G be a graph and k a positive integer. A k-**coloring of** G is a function $f : V(G) \to \{1, 2, \ldots, k\}$. We call this coloring **proper** provided $\forall xy \in E(G), f(x) \neq f(y)$. If a graph has a proper k-coloring, we call it k-**colorable**.

Chromatic Number

- The central idea in the definition is the function f.
  - To each vertex $v \in V(G)$, the function f associates a value $f(v)$. The value $f(v)$ is the color of v. The palette of colors we use is the set $\{1, 2, \ldots, k\}$.
  - The condition $\forall xy \in E(G), f(x) \neq f(y)$ means that, if vertices x and y are adjacent in G, then $f(x) \neq f(y)$, i.e., they receive different colors.
- The definition does not require that all the colors be used, i.e., it does not require f to be onto.
- The number k refers to the size of the palette of colors available - it is not a demand that all k colors be used.
  - If, say, a graph is five-colorable, then it is also six-colorable.
- The goal in graph coloring is to use as few colors as possible.

Definition (Chromatic Number)

Let G be a graph. The smallest positive integer k for which G is k-colorable is called the **chromatic number of** G and is denoted $\chi(G)$.

Coloring Complete Graphs and Subgraphs

- Example: Consider the complete graph K_n. We can properly color K_n with n colors by giving every vertex a different color. Can we do better? No! Since every vertex is adjacent to every other vertex in K_n, no two vertices may receive the same color, and so n colors are required. Therefore, $\chi(K_n) = n$.
- For any graph G with n vertices, we have $\chi(G) \leq n$ because we can always give each vertex a separate color.

Proposition

Let G be a subgraph of H. Then $\chi(G) \leq \chi(H)$.

- Given a proper coloring of H, we can simply copy those colors to the vertices of G to achieve a proper coloring of G. So, if we used only $\chi(H)$ colors to color the vertices of H, we have used at most $\chi(H)$ colors in a proper coloring of G.

Coloring and Maximum Degree

Proposition

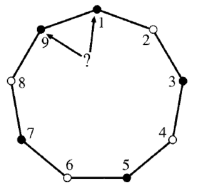
Let G be a graph with maximum degree Δ. Then $\chi(G) \leq \Delta + 1$.

- Suppose the vertices of G are $\{v_1, v_2, \ldots, v_n\}$ and we have a palette of $\Delta + 1$ colors. We color the vertices of G as follows:
  - Assign any color from the palette to vertex v_1.
  - Next, to color vertex v_2, take any color from the palette, as long as the coloring is proper. In other words, if $v_1 \sim v_2$ is an edge, we may not assign the same color to v_2 that we gave to v_1.
  - We continue in exactly this fashion through all the vertices. When we come to vertex v_j, we assign to vertex v_j any color from the palette, just making certain that the color on vertex v_j is not the same as any of its already colored neighbors.

  Are there sufficiently many colors so as not to get stuck? Every vertex has at most Δ neighbors and there are $\Delta + 1$ colors in the palette, so we can never get stuck. Since this procedure produces a proper $(\Delta + 1)$-coloring of the graph, $\chi(G) \leq \Delta + 1$.

Coloring Cycles on n Vertices

- What is the chromatic number of the cycle C_n?
  - If n is even, then we can alternate colors (black, white, black, white, etc.) around the cycle. When n is even, this yields a valid coloring.
  - However, if n is odd, then vertex 1 and vertex n would both be black if we alternated colors around the cycle.



Thus, for n odd, C_n is not 2-colorable.
It is, however, 3-colorable. We can alternately color vertices 1 through $n-1$ with black and white and then color vertex n with, say, blue.
This gives a proper three-coloring of C_n.

Thus
$$\chi(C_n) = \begin{cases} 2, & \text{if } n \text{ is even} \\ 3, & \text{if } n \text{ is odd} \end{cases}$$

- Note that the chromatic number of C_9 is 3, but C_9 does not contain K_3 as a subgraph.

Bipartite Graphs

Proposition

A graph G is one-colorable if and only if it is edgeless.

Definition (Bipartite Graphs)

A graph G is called **bipartite** provided it is 2-colorable.

- There is another useful way to describe bipartite graphs.

  Let $G = (V, E)$ be a bipartite graph and select a proper two-coloring. Let X be the set of all vertices that receive one of the two colors, and Y the set of all vertices that receive the other. Notice that $\{X, Y\}$ forms a partition of the vertex set V. Furthermore, if e is any edge of G, then e has one of its endpoints in X and its other endpoint in Y.
- The partition of V into the sets X and Y such that every edge of G has one end in X and one end in Y is called a **bipartition** of the bipartite graph.
- We know that even cycles are bipartite, but odd cycles are not.

Coloring Trees

Proposition

Trees are bipartite.

- The proof is by induction on the number of vertices in the tree.
  - Basis Case: Clearly a tree with only one vertex is bipartite. Indeed, $\chi(K_1) = 1 \leq 2$.
  - Induction Hypothesis: Every tree with n vertices is bipartite.
  - Induction Step: Let T be a tree with $n+1$ vertices. Let v be a leaf of T and let $T' = T - v$. Since T is a tree with n vertices, by induction T' is bipartite. Properly color T' using the two colors black and white. Now consider v's neighbor w. Whatever color w has, we can give v the other color. Since v has only one neighbor, this gives a proper two-coloring of T.

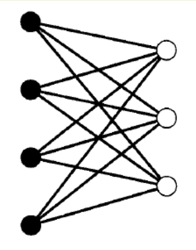
Complete Bipartite Graphs

Definition (Complete Bipartite Graphs)

Let n, m be positive integers. The **complete bipartite graph** $K_{n,m}$ is a graph whose vertices can be partitioned $V = X \cup Y$ such that:

- $|X| = n$, $|Y| = m$,
- for all $x \in X$ and for all $y \in Y$, xy is an edge, and
- no edge has both its endpoints in X or both its endpoints in Y.

- The graph in the figure is $K_{4,3}$:

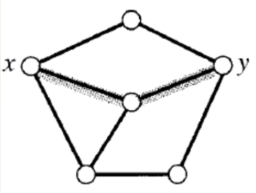


Distance

Definition (Distance)

Let G be a graph and let x, y be vertices of G. The **distance from** x **to** y **in** G is the length of a shortest (x, y)-path. In cases where there is no such path, we say that the distance is **undefined** or ∞. The distance from x to y is denoted $d(x, y)$.

- Example: Consider the following graph:



There are several (x, y)-paths. The shortest among them have length 2. Thus, $d(x, y) = 2$.

Characterization of Bipartite Graphs

Theorem (Characterization of Bipartite Graphs)

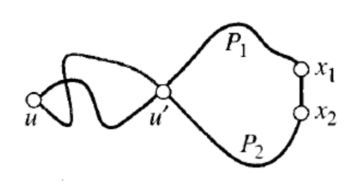
A graph is bipartite if and only if it does not contain an odd cycle.

- ($\Rightarrow$) Let G be a bipartite graph. Suppose that G contains an odd cycle C as a subgraph. Then $3 = \chi(C) \leq \chi(G) \leq 2$, a contradiction. Therefore G does not contain an odd cycle.
- ($\Leftarrow$) We show that if G does not contain an odd cycle, then G is bipartite. We begin by proving a special case: if G is connected and does not contain an odd cycle, then G is bipartite.
  Suppose G is connected and does not contain an odd cycle. Let u be any vertex in $V(G)$. Define two subsets of $V(G)$ as follows:
  - $X = \{x \in V(G) : d(u,x) \text{ is odd}\}$;
  - $Y = \{y \in V(G) : d(u,y) \text{ is even}\}$.

  Clearly, $u \in Y$ because $d(u,u) = 0$. Also, $V(G) = X \cup Y$ and $X \cap Y = \emptyset$. We color the vertices in X black and the vertices in Y white.

An Auxiliary Claim

- Claim: This gives a proper two-coloring of G.
  We must show that there are no two vertices in X that are adjacent and no two vertices in Y that are adjacent. Suppose $x_1, x_2 \in X$ with $x_1 \sim x_2$, and let P_1 be a shortest path from u to x_1. Because $x_1 \in X$, we know that $d(u, x_1)$ is odd, so the length of P_1 is odd. Likewise, if P_2 is a shortest (u, x_2)-path its length is also odd. Let u' denote the last vertex that P_1 and P_2 have in common.

  If we traverse P_1 from u' to x_1, then traverse the edge x_1x_2, and finally return to u' along P_2^{-1}, we have a cycle.

An Auxiliary Sub-Claim

- Sub-Claim: The cycle $u' \sim \overset{P_1}{\cdots} \sim x_1 \sim x_2 \sim \overset{P_2^{-1}}{\cdots} \sim u'$ is an odd cycle.
  The section of P_1 from u to u' is as short as possible. Otherwise, if there were a shorter path Q from u to u', we could concatenate Q with the (u', x_1)-section of P_1 and achieve a (u, x_1)-walk that is shorter than P_1, from which we could construct a (u, x_1)-path that is shorter than P_1, a contradiction. Likewise the (u, u')-section of P_2 is as short as possible. Hence the (u, u')-sections of P_1 and P_2 must have the same length.
  Now consider the (u, x_1)- and (u, x_2)-sections of P_1 and P_2. We know that P_1 and P_2 both have odd length. From them we delete the same length: their (u, u')-sections. Thus the two sections that remain have the same parity.
  We now conclude that the cycle C is an odd cycle, since it consists of the edge x_1x_2 and the two sections from u' of P_1 and P_2 having the same parity.

Gradual Ascend Back to the Theorem

- We showed that C is an odd cycle.
- But by hypothesis, G has no odd cycles. Therefore, there is no edge in G both of whose endpoints are in X. Similarly, there is no edge with both ends in Y. Therefore, the claim that we do have a proper two-coloring of G holds.
- Thus, G is bipartite. This establishes the $(\Leftarrow)$ part of the theorem, under the extra hypothesis that G is connected.
- We finally need to consider the case when G is disconnected. Suppose G is a disconnected graph that contains no odd cycles. Let $H_1, H_2, \ldots, H_c$ be its connected components. Note that since G does not contain an odd cycle, neither do any of its components. Hence, by the argument above, they are bipartite. Let $X_i \cup Y_i$ be a bipartition of $V(H_i)$ (with $1 \leq i \leq c$). Finally, let $X = X_1 \cup X_2 \cup \cdots \cup X_c$ and $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_c$.

Final Coup de Grâce

- Claim: $X \cup Y$ is a bipartition of $V(G)$.
  X and Y are pairwise disjoint and their union is $V(G)$.
  - There can be no edge between two vertices in X_i because $X_i \cup Y_i$ is a bipartition.
  - There can be no edge between vertices of X_i and X_j (with $i \neq j$) because these vertices are in separate components of G.

  Therefore no edge has both ends in X.
  Similarly, no edge has both ends in Y.
  Therefore $X \cup Y$ is a bipartition of $V(G)$, and so G is bipartite.

Deciding 2- and 3-Colorability

- The preceding theorem gives us a method for determining whether or not a graph is bipartite:

Backtraching Algorithm

1. Assign White to an arbitrarily chosen initial vertex (putting into set X).
2. Color all its neighbors with Black (putting them into set Y).
3. Color all neighbor's neighbors with White (putting into set X).
4. In this way, assign color to all vertices until either $X \cup Y = V(G)$ or a neighbor is found colored with same color as current vertex.
    - In the first case G is bipartite with bipartition $X \cup Y$.
    - In the latter, the graph is not bipartite.

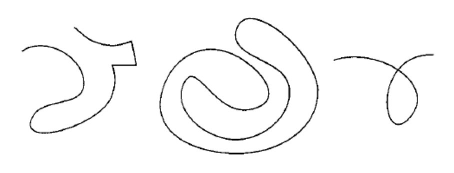
- Determining whether or not a graph is three-colorable is NP-complete: Intuitively, this means that it is difficult to color a graph properly with three colors or to show that no such coloring exists.

Subsection 7

Planar Graphs

Curves

- In this section, we study not only graphs, but their drawings as well.
- Intuitively, a curve is a "line" that may have corners and straight sections. We stipulate that a curve must be all in one piece.
- A simple curve is a curve that joins two distinct points in the plane and does not cross itself.
- Example: The left curve in the figure is simple; the other two are not.



- If a curve returns to its starting point, we call the curve closed.
- If the first/last point of the curve is the only point on the curve that is repeated, then we call the curve a simple closed curve.
- Example: The middle curve in the diagram is a simple closed curve. The third curve is neither simple nor closed.

Jordan Curve Theorem and Embeddings

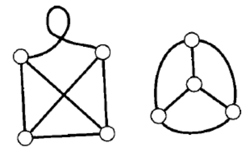
Theorem (Jordan Curve)

A simple closed curve in the plane divides the plane into two regions: the inside of the curve and the outside of the curve.

- An embedding of a graph is a collection of points and curves in a plane that satisfies the following conditions:
  - Each vertex is assigned a point in the plane; distinct vertices receive distinct points.
  - Each edge is assigned a curve in the plane. If the edge is $e = xy$, then the endpoints of the curve for e are exactly the points assigned to x and y. Furthermore, no other vertex point is on this curve.
- If all the curves are simple (do not cross themselves) and if the curves corresponding to two edges do not intersect (except at an endpoint, if they both are incident with the same vertex), then we call the embedding crossing-free.

Planar Graphs

- Example: The figure shows two embeddings of the graph K_4.



  The drawing on the right represents a crossing-free embedding on K_4.
- Not all graphs have crossing-free embeddings in the plane.

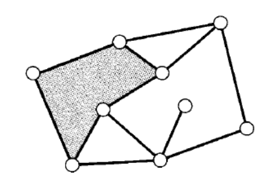
Definition (Planar Graph)

A **planar graph** is a graph that has a crossing-free embedding in the plane.

- Example: The graph K_4 is planar. The graph K_5 is not planar. The way to show this is to study properties of planar graphs and use that knowledge to prove that K_5 is not planar.
- We start this study with a classic result of Euler.

Euler's Formula

- Example: Let G be a planar graph and consider a crossing-free embedding of G:



The drawing has five faces, four bounded and one unbounded. The graph has $n = 9$ vertices, $m = 12$ edges, and $f = 5$ faces.

Theorem (Euler's Formula)

Let G be a connected planar graph with n vertices and m edges. Choose a crossing-free embedding for G, and let f be the number of faces in the embedding. Then $n - m + f = 2$.

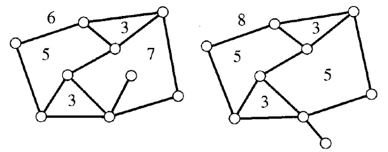
- The proof is by induction on the number of edges in the connected planar graph G.

Euler's Formula: The Proof

- Suppose G has n vertices. The basis case is when the number of edges is $n-1$, since the graph is connected.
  - Basis Case: Since G is connected and has $m = n-1$ edges, we know that G is a tree. In a drawing of a tree, there is only one face because there are no cycles to enclose additional faces. Thus $f = 1$. We therefore have $n - m + f = n - (n-1) + 1 = 2,$ as required.
  - Induction Hypothesis: Suppose all connected planar graphs with n vertices and m edges satisfy Euler's formula.
  - Induction Step: Let G be a planar graph with n vertices and $m+1$ edges. Choose a crossing-free embedding of G and let f be the number of faces in this embedding. Let e be an edge of G that is not a cut edge (exists since G is not a tree). Therefore $G-e$ is connected. If we erase e from the drawing of G, we have a crossing-free embedding of $G-e$, and so $G-e$ is planar. Notice that $G-e$ has n vertices and $(m+1)-1 = m$ edges. The drawing, we claim, has $f-1$ faces. The edge we deleted causes the two faces on either side of it to merge into a single face, so $G-e$'s drawing has one less face than G's. Now, by induction, we have $n - m + (f-1) = 2$, so $n - (m+1) + f = 2$.

Consequences of Euler's Formula

- Let G be a connected planar graph with n vertices and m edges. We can solve the equation $n - m + f = 2$ for f and we get $f = 2 - n + m$.
  - The number of vertices and edges are quantities that depend only on the graph G, not on how it is drawn in the plane.
  - On the other hand, the quantity f is the number of faces in a particular crossing-free drawing of G.
- The implication of Euler's formula is that regardless of how we draw the graph, the number of faces is always the same.
- Example: For example, consider the two drawings:



In both cases, the graph has $f = 2 - n + m = 2 - 9 + 12 = 5$ faces. The number inside each face indicates the number of edges that are on the boundary of that face.

It is called the **degree** of the face. Note the face with degree equal to 7; Why does it have degree equal to 7?

Sum of the Degrees of the Faces

Proposition

Let G be a planar graph. The sum of the degrees of the faces in a crossing-free embedding of G in the plane equals $2|E(G)|$.

- Since every edge has two sides, it contributes a total value of 2 to the degrees of the faces it touches.
  - If an edge only touches one face, then it counts twice toward that face's degree.
  - If it touches two faces, it counts once toward each of the two faces' degrees.

  Therefore, if we add the degrees of all the faces in the embedding, we get twice the number of edges in the graph.
- If the graph is K_1, then the only face has degree equal to 0.
- If the graph has just one edge, then the only face has degree 2.
- As soon as a planar graph has two (or more) edges, then all faces have degree 3 or greater.

A Corollary of Euler's Formula

Corollary

Let G be a planar graph with at least two edges. Then $|E(G)| \leq 3|V(G)| - 6$. Furthermore, if G does not contain K_3 as a subgraph, then $|E(G)| \leq 2|V(G)| - 4$.

- Without loss of generality, G is connected. So, let G be a connected planar graph with at least two edges. Pick a crossing-free embedding of G, with $f = 2 - |V(G)| + |E(G)|$ faces. We calculate the sum of the degrees of the faces in this embedding.
  - On the one hand the sum of the face degrees is $2|E(G)|$.
  - On the other hand, every face has degree at least 3, so the sum of the face degrees is at least $3f$.

  Therefore we have $2|E(G)| \geq 3f$, which yields $f \leq \frac{2}{3}|E(G)|$. Substituting into Euler's formula, we get $2 - |V(G)| + |E(G)| = f \leq \frac{2}{3}|E(G)|$, which rearranges to $2 - |V(G)| + \frac{1}{3}|E(G)| \leq 0$, which yields $|E(G)| \leq 3|V(G)| - 6$. The proof of the second inequality is similar.

Another Consequence of Euler's Formula

Corollary

Let G be a planar graph with minimum degree δ. Then $\delta \leq 5$.

- Let G be a planar graph. If G has fewer than two edges, clearly $\delta \leq 5$. So we may assume that G has at least two edges. Then, by the previous corollary, we have $|E(G)| \leq 3|V(G)| - 6$. The minimum degree δ cannot be greater than the average degree $\bar{d}$, i.e., $\delta \leq \bar{d}$. We now calculate

$$\begin{aligned} \delta &\leq \bar{d} = \frac{\sum_{v \in V} d(v)}{|V(G)|} = \frac{2|E(G)|}{|V(G)|} \\ &\leq \frac{2(3|V(G)| - 6)}{|V(G)|} = 6 - \frac{12}{|V(G)|} < 6. \end{aligned}$$

But since δ is an integer, we have $\delta \leq 5$.

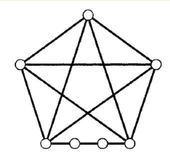
Non-Planarity of K_5

- A graph that is not planar is called **nonplanar**.

Proposition (Non-Planarity of K_5)

The graph K_5 is nonplanar.

- Suppose that K_5 were planar. By a previous corollary, we would have $10 = |E(K_5)| \leq 3|V(K_5)| - 6 = 3 \cdot 5 - 6 = 9,$ a contradiction.
- Example: Consider the graph:



It has 7 vertices and 12 edges. It satisfies $|E(G)| \leq 3|V(G)| - 6$, since $12 \leq 15 = 3 \cdot 7 - 6$. But it is non-planar: If it were planar, it would have a crossing-free embedding.

Given such an embedding, by ignoring the two vertices of degree 2, we would obtain a crossing-free planar embedding of K_5. However, since K_5 has no such embedding, neither does this graph.
- The graph in the figure is an example of a subdivision of K_5.

Subdivisions and Non-Planarity of $K_{3,3}$

- A **subdivision** of G is formed from G by replacing edges with paths.
- Clearly, if a graph is planar, so are its subdivisions.
- The converse of this statement is also true: If a graph is nonplanar, then all of its subdivisions are also nonplanar.
- Therefore, any subdivision of K_5 is nonplanar. Moreover, any graph that contains a subdivision of K_5 as a subgraph is also nonplanar.

Proposition (Non-Planarity of $K_{3,3}$)

The graph $K_{3,3}$ is nonplanar.

- Suppose that $K_{3,3}$ were planar. Since it does not contain K_3 as a subgraph, we would have $9 = |E(K_{3,3})| \leq 2|V(K_{3,3})| - 4 = 2 \cdot 6 - 4 = 8$, which is a contradiction.
- Not only is $K_{3,3}$ nonplanar, but so is any subdivision graph we can form from $K_{3,3}$. Furthermore, any graph that contains a subdivision of $K_{3,3}$ as a subgraph must be nonplanar as well.

Kuratowski's Theorem

Theorem (Kuratowski)

A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph.

- We have shown that, if G contains a subdivision of K_5 or $K_{3,3}$ as a subgraph, then G cannot be planar.
- The more difficult part is to prove that, if a graph does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph, then the graph is planar. This proof is a relatively advanced one in graph theory and is not presented here.
- Kuratowski 's Theorem is a marvelous characterization of planarity
  - To see that a graph is planar, we can present a crossing-free drawing.
  - To see that a graph is nonplanar, we find a subgraph which is a subdivision of K_5 or $K_{3,3}$.

The Four-Color Theorem

- Graphs that arise from maps must be planar.
  - We begin with a map.
  - We locate one vertex for each country at the capital city.
  - From that capital city, we draw curves out to its various borders. These curves fan out in a starlike pattern and do not cross each other. We connect capitals of neighbors through midpoints of borders. Thus, we construct a planar embedding of the graph.
- Thus, the map-coloring problem is equivalent to determining whether every planar graph is four-colorable.

Four-Color Theorem (Appel and Haken)

If G is a planar graph, then $\chi(G) \leq 4$.

- This theorem is best possible, since K_4 is planar, and $\chi(K_4) = 4$.
- The proof of the Four Color Theorem is long and complicated.
- We show, next, that every planar graph is six-colorable.
- Afterwards, we show that every planar graph is five-colorable.

Six-Color Theorem

Theorem (Six-Color Theorem)

If G is a planar graph, then $\chi(G) \leq 6$.

- The proof is by induction on the number of vertices in the graph.
  - Basis Case: The theorem is obviously true for all graphs on six or fewer vertices, because we can give each vertex a separate color.
  - Induction Hypothesis: Suppose the theorem is true for all graphs on n vertices (all planar graphs with n vertices are six-colorable).
  - Induction Step: Let G be a planar graph with $n+1$ vertices. By a preceding corollary, G contains a vertex v, with $d(v) \leq 5$. Let $G' = G - v$. Notice that G' is planar and has n vertices. By induction, G' is six-colorable. Properly color the vertices of G' using just six colors. We can extend this coloring to G by giving v a color. Notice that v has at most five neighbors, and so there is some other color that we can assign to v that is different from the colors of its neighbors. This yields a proper six-coloring of G.

Five-Color Theorem

Theorem (Five-Color Theorem)

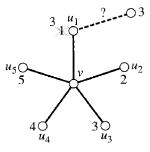
If G is a planar graph, then $\chi(G) \leq 5$.

- The proof is by induction on the number of vertices in the graph.
  - Basis Case: The theorem is obviously true for all graphs on five or fewer vertices, because we can give each vertex a separate color.
  - Induction Hypothesis: Suppose the theorem is true for all graphs on n vertices (all planar graphs with n vertices are five-colorable).
  - Induction Step: Let G be a planar graph with $n+1$ vertices. By a previous corollary, G contains a vertex v, with $d(v) \leq 5$. Let $G' = G - v$. G' is planar and has n vertices. By induction, G' is five-colorable. Properly color the vertices of G' using just five colors. To extend this coloring to G, consider the neighbors of v. If among the neighbors of v there are only four different colors, then we can assign to v the left-over color. This yields a proper five-coloring of G. The problem has been reduced to the case where $d(v) = 5$ and all five of its neighbors are different colors. To extend the coloring to vertex v, we need to recolor some vertices.

Recoloring in the Induction Step I

- Since G is planar, choose a crossing-free embedding of G. Every vertex of G, except v, has been colored using one of $\{1, 2, 3, 4, 5\}$. Let $u_1, u_2, \ldots, u_5$ be the five neighbors of v in clockwise order, and, without loss of generality, let us assume that u_i has color i (for $i = 1, 2, \ldots, 5$).

  Basic Idea: Change the color of one of v's neighbors.

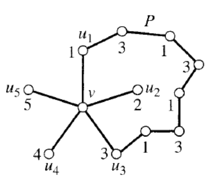
  

  Let $H_{1,3}$ be the subgraph of G induced by all vertices with color 1 or 3. If in one component of $H_{1,3}$, we exchange colors 1 and 3, then we still have a proper coloring of G'.

  We therefore exchange colors 1 and 3 in the component of $H_{1,3}$ that contains vertex u_1. This color exchange results in a proper coloring of G' in which vertex u_1 has color 3. We are all set to color vertex v with color 1. What if u_3 is in the same component of $H_{1,3}$ as vertex u_1? Then, v still has all five colors present on its neighbors.

Recoloring in the Induction Step II

- It remains to consider the case where u_1 and u_3 are in the same component of $H_{1,3}$ (i.e., there is a path P in $H_{1,3}$ from u_1 to u_3).



We argue as before, but now we attempt to recolor vertex u_2 with color 4. Let $H_{2,4}$ denote the subgraph of G induced on the vertices of color 2 or color 4.

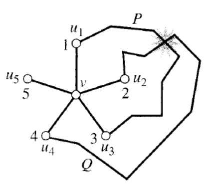
If u_2 and u_4 are in separate components of $H_{2,4}$, then we can recolor u_2's component, exchanging colors 2 and 4. The resulting modified coloring is a proper five-coloring of G' in which no neighbor of v has color 2. In this case, we can simply give vertex v color 2 and have a proper five-coloring of G.

The problem, as before, is that perhaps u_2 and u_4 are in the same component of $H_{2,4}$.

Claim: This cannot happen.

Proving the Final Claim

Claim: Vertices u_2 and u_4 cannot be in the same component of $H_{2,4}$.



Suppose there is a path Q, from u_2 to u_4. The vertices along Q are colored with colors 2 and 4, and the vertices on P are colored with colors 1 and 3. Thus P and Q have no vertices in common.

Furthermore, path P, together with vertex v, forms a cycle. This cycle becomes a simple closed curve in the plane. Notice that vertices u_2 and u_4 are on different sides of this curve! Therefore the path Q from u_2 to u_4 must pass from the inside of this simple closed curve to the outside, and where it does, there is an edge crossing. However, by construction, this embedding has no edge crossings! Therefore vertices u_2 and u_4 must be in separate components of $H_{2,4}$.
Thus, this recoloring technique of u_2 with color 4 may be used. So, we color vertex v with color 2, giving a proper five-coloring of G.