Intermediate Algebra

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

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Subsection 1

The Real Numbers

The Number Systems

- The set of **natural numbers** N is the set
$$N = \{1, 2, 3, \ldots\};$$
- The set of **whole numbers** W is the set consisting of the natural numbers with 0 appended:
$$W = \{0, 1, 2, 3, \ldots\};$$
- The set of **integers** Z is the set consisting of the whole numbers together with the negatives of the natural numbers:
$$Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\};$$
- The set of **rational numbers** Q consists of all numbers that can be written as quotients of integers:
$$Q = \{\frac{a}{b} : a, b \text{ integers, with } b \neq 0\};$$

Alternative Description of the Rationals

- Recall

$$Q = \{\frac{a}{b} : a, b \text{ integers, with } b \neq 0\};$$

- Examples: The following numbers are rational numbers:

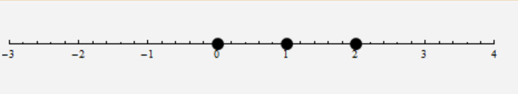
$$5, \quad \frac{11}{6}, \quad -\frac{12}{17}, \quad 0, \quad \frac{-6}{-12};$$

- Another way to describe this set is by way of their decimal form;
  - If we divide the numerator by the denominator, the division terminated for some numbers and continues indefinitely for others;
  - The **rational numbers** are those decimal numbers whose digits either repeat or terminate;
- Examples:

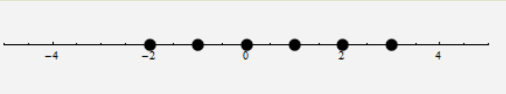
$$\frac{37}{100} = 0.37, \quad \frac{11}{1} = 11 \quad \frac{3}{5} = 0.6, \quad \frac{1}{3} = 0.333\ldots, \quad \frac{37}{99} = 0.3737\ldots$$

Examples

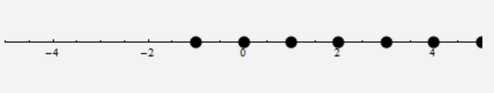
- Consider the set $\{x : x \text{ is a whole number less than } 3\}$;
  In list notation this set is $\{0, 1, 2\}$; In graphical notation, we have



- Consider the set $\{x : x \text{ is an integer number between -3 and } 4\}$;
  In list notation this set is $\{-2, -1, 0, 1, 2, 3\}$; In graphical notation, we have

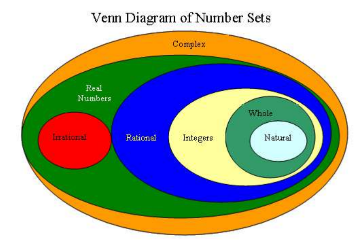


- Consider the set $\{x : x \text{ is an integer greater than -2}\}$;
  In list notation this set is $\{-1, 0, 1, 2, 3, \ldots\}$; In graphical notation, we have



The Irrationals and the Reals

- Recall that $Q = \{\frac{a}{b} : a, b \text{ integers, with } b \neq 0\}$;
- Recall also that the rational numbers are those that, when written in decimal, they either terminate or repeat;
- Those numbers that cannot be written as quotients of two integers or, equivalently, whose decimal representations are infinite and non-repeating, are called **irrational numbers**;
- Examples: The numbers $\sqrt{2}, \sqrt{5}, 0.15115111511115\ldots$ and $1.23456789101112\ldots$ are all irrational numbers;
- The sets of rational and irrational numbers taken together form the set of **real numbers** R;



Example

- Let us decide whether the following numbers belong to the sets listed in the columns of the table:

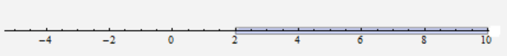
Number	Whole	Integer	Rational	Real
$-\sqrt{7}$	✠	✠	✠	✓
$-\frac{1}{4}$	✠	✠	✓	✓
0	✓	✓	✓	✓
$\sqrt{5}$	✠	✠	✠	✓
π	✠	✠	✠	✓
4.16	✠	✠	✓	✓
-12	✠	✓	✓	✓

Interval Notation for Intervals of Reals

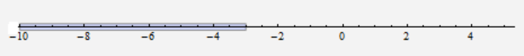
- The following notation is used for intervals of real numbers: Assume $a < b$ are two real numbers:
  - **Open Interval**: $(a, b) = \{x \text{ real} : a < x < b\}$;
  - **Semi-open** or **Semi-closed Intervals**: $(a, b] = \{x \text{ real} : a < x \leq b\}$; $[a, b) = \{x \text{ real} : a \leq x < b\}$;
  - **Closed Interval**: $[a, b] = \{x \text{ real} : a \leq x \leq b\}$;
- Moreover, we have the following types of **unbounded intervals**:
  - $(-\infty, b) = \{x \text{ real} : x < b\}$;
  - $(-\infty, b] = \{x \text{ real} : x \leq b\}$;
  - $(a, +\infty) = \{x \text{ real} : x > a\}$;
  - $[a, +\infty) = \{x \text{ real} : x \geq a\}$;
  - $(-\infty, +\infty) = R$;

Examples I

- Write in set notation, interval notation and graph the following sets:
  - The set of real numbers greater than or equal to 2;
    In set notation $\{x \text{ real} : x \geq 2\}$;
    In interval notation $[2, +\infty)$;
    In graphical form:

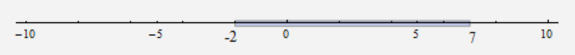
    

  - The set of real numbers less than -3;
    In set notation $\{x \text{ real} : x < -3\}$;
    In interval notation $(-\infty, -3)$;
    In graphical form:

Examples II

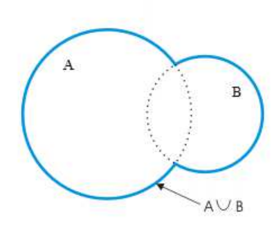
- Write in set notation, interval notation and graph the following sets:
  - The set of real numbers between -2 and 7 inclusive;
    In set notation $\{x \text{ real} : -2 \leq x \leq 7\}$;
    In interval notation $[-2, 7]$;
    In graphical form:

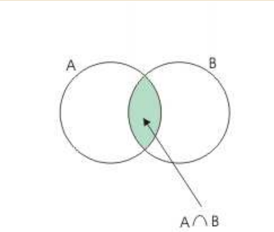


  - The set of all real numbers greater than 1 and less than or equal to 5;
    In set notation $\{x \text{ real} : 1 < x \leq 5\}$;
    In interval notation $(1, 5]$;
    In graphical form:

Union and Intersections of Intervals

- The **union** $A \cup B$ of two sets A and B consists of those elements that are in at least one of A or B:





- The **intersection** $A \cap B$ of two sets A and B consists of those elements that are in both A and B:
- These operations apply to intervals:
  - $(2,4) \cup (3,6) = (2,6)$;
  - $(2,4) \cap (3,6) = (3,4)$;
  - $(-1,2) \cup [0,+\infty) = (-1,+\infty)$;
  - $(-1,2) \cap [0,+\infty) = [0,2)$;

Subsection 2

Operations on the Real Numbers

Absolute Value

- Geometrically, the **absolute value** $|x|$ of a real number x is its distance from the origin;
- E.g., we have

$$|-1.2| = 1.2, \quad |3.7| = 3.7, \quad \left|-\frac{7}{4}\right| = \frac{7}{4}, \quad |-128| = 128;$$

- Algebraically, this relationship may be expressed by the formula:

$$|x| = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

- E.g., since $-13 < 0$, we have $|-13| = -(-13) = 13$;

Addition

- The following rules for addition apply:
  - To find the sum of two numbers with the same sign, we add the absolute values and assign to the sum the same sign as the original numbers;
  - To find the sum of two numbers having opposite signs, we subtract their absolute values and we assign to the sum the sign of the number with the largest absolute value;
- Examples:

$$(-9)+(-7)=-16, \quad -7+10=3, \quad -6+13=7,$$
$$-9+(-7)=-16, \quad -35.4+2.5=-32.9,$$
$$-7+0.05=-6.95, \quad \frac{1}{5}+(-\frac{3}{4})=-\frac{11}{20};$$

Subtraction

- For any real numbers a and b

$$a - b = a + (-b);$$

- Examples:
  - $-7 - 3 = -7 + (-3) = -10;$
  - $7 - (-3) = 7 + 3 = 10;$
  - $48 - 99 = 48 + (-99) = -51;$
  - $-3.6 - (-7) = -3.6 + 7 = 3.4;$
  - $0.02 - 7 = 0.02 + (-7) = -6.98;$
  - $\frac{1}{5} - (-\frac{1}{7}) = \frac{1}{5} + \frac{1}{7} = \frac{12}{35}.$

Multiplication and Division

- To find the product of two real numbers we multiply the two absolute values and we assign to the product a positive sign if the two numbers have the same sign and a negative sign if the two numbers have different signs;
- Examples:

$$(-2)(-7) = 14, \quad -4 \cdot 12 = -48,$$
$$(-0.01)(0.7) = -0.007, \quad \tfrac{2}{7} \cdot (-\tfrac{1}{3}) = -\tfrac{2}{21};$$

- To divide $a \div b$, we multiply a by the **reciprocal** $\frac{1}{b}$ of the number b, i.e.,

$$a \div b = a \cdot \frac{1}{b};$$

- Examples:

$$-60 \div (-2) = -60 \cdot (-\tfrac{1}{2}) = 30,$$
$$-24 \div \tfrac{3}{8} = -24 \cdot \tfrac{8}{3} = -64,$$
$$(-6) \div (-0.2) = (-6) \cdot (-5) = 30;$$

Subsection 3

Evaluating Expressions and Order of Operations

Arithmetic Expressions

- An **arithmetic expression** is a meaningful combination of numbers using the ordinary operations of arithmetic; E.g., $5 + (7 \cdot 3)$ is an arithmetic expression;
- The expression is called a **sum, difference, product** or **quotient** if the last operation to be performed in the expression is addition, subtraction, multiplication or division, respectively;
  E.g. the expression $5 + (7 \cdot 3)$ is a sum, because the $+$ is the last operation to be performed;
- As in the expression above parentheses, brackets, angle brackets and other **grouping symbols** are used to indicate which operations are supposed to be performed first;
- Examples:
  - $5 + (7 \cdot 3) = 5 + 21 = 26;$
  - $(5 + 7) \cdot 3 = 12 \cdot 3 = 36;$

Additional Examples

- Let us evaluate the following expressions:
  - $8[(5 \cdot 2) - 3] = 8[10 - 3] = 8 \cdot 7 = 56$;
  - $2\{[4 \cdot (-5)] - |5 - 19|\} = 2\{-20 - |-14|\} = 2\{-20 - 14\} = 2(-34) = -68$;

Exponential Expressions and Roots

- If a is any real number and n is a natural number,

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}};$$

- The number a is the **base** and n the **exponent**;
- Examples: Evaluate the expressions:
  - $2^3 = 2 \cdot 2 \cdot 2 = 8$;
  - $(-3)^4 = (-3) \cdot (-3) \cdot (-3) \cdot (-3) = 81$;
  - $(-\frac{1}{2})^5 = (-\frac{1}{2})(-\frac{1}{2})(-\frac{1}{2})(-\frac{1}{2})(-\frac{1}{2}) = -\frac{1}{32}$;
- If $a^2 = b$, then a is called the **square root** of b; If $a > 0$, we write $\sqrt{b} = a$;
- Examples: Evaluate the expressions:
  - $\sqrt{64} = 8$;
  - $\sqrt{9+16} = \sqrt{25} = 5$;
  - $\sqrt{3(17-5)} = \sqrt{3 \cdot 12} = \sqrt{36} = 6$;

Order of Operations

- To simplify expressions, some grouping symbols are omitted;
- To avoid ambiguities, we follow the **order of operations**:
    1. Evaluate expressions inside grouping symbols first;
    2. Evaluate exponential expressions from left to right;
    3. Perform multiplications and divisions from left to right;
    4. Perform additions and subtractions from left to right.
- Examples: Evaluate the expressions:
    - $5 + 2 \cdot 3 = 5 + 6 = 11$;
    - $7 \cdot 3^2 = 7 \cdot 9 = 63$;
    - $(5 - 3^2)^2 = (5 - 9)^2 = (-4)^2 = 16$;
    - $40 \div 8 \cdot 2 \div 5 \cdot 3 = 5 \cdot 2 \div 5 \cdot 3 = 10 \div 5 \cdot 3 = 2 \cdot 3 = 6$;

Order of Negative Signs and Fractions

- Examples: Evaluate the expressions:
  - $-2^4 = -16$;
  - $-5^2 = -25$;
  - $(3-5)^2 = (-2)^2 = 4$;
  - $-(5^2 - 4 \cdot 7)^2 = -(25 - 4 \cdot 7)^2 = -(25-28)^2 = -(-3)^2 = -9$;
- Examples: Evaluate the quotients:
  - $\frac{10-8}{6-8} = \frac{2}{-2} = -1$;
  - $\frac{-6^2 + 2 \cdot 7}{4 - 3 \cdot 2} = \frac{-36 + 2 \cdot 7}{4-6} = \frac{-36+14}{-2} = \frac{-22}{-2} = 11$;

Algebraic Expressions (Expressions with Variables)

- Examples: Evaluate the following expressions assuming that $a = 2, b = -3$ and $c = 4$:
  - $a - c^2 = 2 - 4^2 = 2 - 16 = -14$;
  - $a - b^2 = 2 - (-3)^2 = 2 - 9 = -7$;
  - $b^2 - 4ac = (-3)^2 - 4 \cdot 2 \cdot 4 = 9 - 4 \cdot 2 \cdot 4 = 9 - 32 = -23$;
  - $\frac{a - b}{c - b} = \frac{2 - (-3)}{4 - (-3)} = \frac{5}{7}$;
- Examples: Evaluate the following expressions assuming that $a = 5, b = -2$ and $c = 7$:
  - $a - c^2 = 5 - 7^2 = 5 - 49 = -44$;
  - $a - b^2 = 5 - (-2)^2 = 5 - 4 = 1$;
  - $b^2 - 4ac = (-2)^2 - 4 \cdot 5 \cdot 7 = 4 - 4 \cdot 5 \cdot 7 = 4 - 140 = -136$;
  - $\frac{a - b}{c - b} = \frac{5 - (-2)}{7 - (-2)} = \frac{7}{9}$;