Intermediate Algebra

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LSSU Math 102

1. Exponential and Logarithmic Functions
   - Exponential Functions and Applications
   - Logarithmic Functions and Applications
   - Properties of Logarithms
   - Solving Equations and Applications

Subsection 1

Exponential Functions and Applications

Exponential Functions

- An **exponential function** is one of the form

$$f(x) = a^x,$$

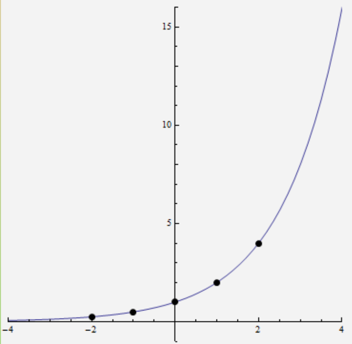
where $0 < a \neq 1$ and x is a real number;

- Example: Consider $f(x) = 2^x$, $g(x) = (\frac{1}{4})^{1-x}$ and $h(x) = -3^x$; Compute the following values:
  - $f(\frac{3}{2}) = 2^{3/2} = \sqrt{2^3} = \sqrt{2^2}\sqrt{2} = 2\sqrt{2}$;
  - $f(-3) = 2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$;
  - $g(3) = (\frac{1}{4})^{1-3} = (\frac{1}{4})^{-2} = 4^2 = 16$;
  - $h(2) = -3^2 = -9$;
- Two important exponentials for applications are the base 10 exponential $f(x) = 10^x$ (called **common base**), and the base e exponential $f(x) = e^x$ (called **natural base**);

Graphs of Exponentials (Exponential Growth)

- When the base a is such that $a > 1$, then $f(x) = a^x$ has an increasing graph (going up as we move from left to right);
- As an example, we'll use a few points to sketch the graph of $f(x) = 2^x$;

x	$y = 2^x$
-2	$1/4$
-1	$1/2$
0	1
1	2
2	4

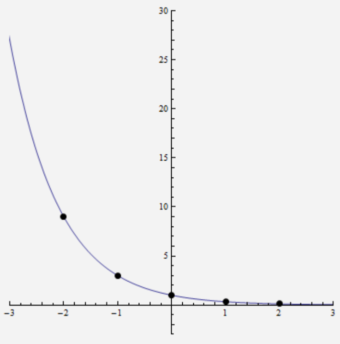


- Note that the x-axis is a horizontal asymptote as $x \to -\infty$;

Graphs of Exponentials (Exponential Decay)

- When the base a is such that $0 < a < 1$, then $f(x) = a^x$ has a decreasing graph (going down as we move from left to right);
- As an example, we'll use a few points to sketch the graph of $f(x) = \left(\frac{1}{3}\right)^x$;

x	$y = \left(\frac{1}{3}\right)^x$
-2	9
-1	3
0	1
1	1/3
2	1/9

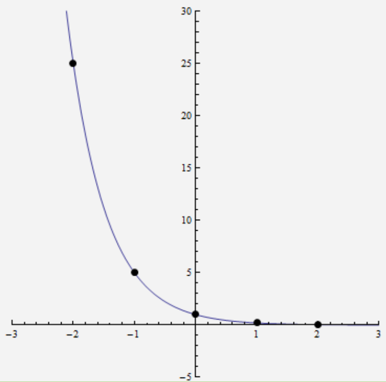


- Note that the x-axis is a horizontal asymptote as $x \to +\infty$;

Another Graph of an Exponential

- Let us use a few points to sketch the graph of $f(x) = 5^{-x}$;

x	$y = 5^{-x}$
-2	25
-1	5
0	1
1	$1/5$
2	$1/25$

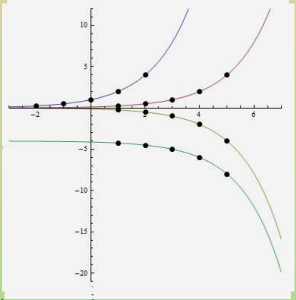


- The x-axis is a horizontal asymptote as $x \to +\infty$;

Using Transformations to Graph

- This method of graphing starts with a known graph and through a series of reflections, shifts and, possibly, expansions/contractions leads to the graph of a function not known in advance;
- We illustrate with an example;

  The graph of $y = 2^x$ is known to us; Suppose that, based on it, we would like to obtain the graph of $f(x) = -2^{x-3} - 4$, which is not known to us; From $y = 2^x$, by shifting 3 to the right, we get $y = 2^{x-3}$;



  Then, by reflecting with respect to the x-axis, we get $y = -2^{x-3}$; Finally, by shifting down 4, we get $y = -2^{x-3} - 4$;

Exponential Equations

One-to-One Property of Exponentials

$$a^m = a^n \quad \text{implies} \quad m = n.$$

- Example: Solve each equation:
  - $3^{2x-1} = 243$
    $3^{2x-1} = 243 \Rightarrow 3^{2x-1} = 3^5 \Rightarrow 2x - 1 = 5 \Rightarrow 2x = 6 \Rightarrow x = 3$;
  - $8^{|x|} = 2$
    $8^{|x|} = 2 \Rightarrow (2^3)^{|x|} = 2^1 \Rightarrow 2^{3|x|} = 2^1 \Rightarrow 3|x| = 1 \Rightarrow |x| = \frac{1}{3} \Rightarrow x = -\frac{1}{3}$ or $x = \frac{1}{3}$;
  - $\frac{1}{125} = 25^{x+7}$
    $\frac{1}{125} = 25^{x+7} \Rightarrow \frac{1}{5^3} = (5^2)^{x+7} \Rightarrow 5^{-3} = 5^{2(x+7)} \Rightarrow -3 = 2x + 14 \Rightarrow 2x = -17 \Rightarrow x = -\frac{17}{2}$;
- Example: Suppose $f(x) = 2^x$ and $g(x) = (\frac{1}{3})^{1-x}$; Find x if
  - $f(x) = 128$
    $f(x) = 128 \Rightarrow 2^x = 2^7 \Rightarrow x = 7$;
  - $g(x) = 81$
    $g(x) = 81 \Rightarrow (\frac{1}{3})^{1-x} = 3^4 \Rightarrow 3^{-(1-x)} = 3^4 \Rightarrow -1 + x = 4 \Rightarrow x = 5$;

Application: Compounding of Interest

- If P is the principal amount invested, i the interest rate per period and n the number of periods, then the amount A accrued at the end of n periods is
$$A = P(1+i)^n;$$
- Example: If \$1,200 is deposited in an account paying 2% compounded quarterly, how much will be in the account at the end of 5 years?

  We have $P = 1200$, $i = \frac{0.02}{4} = 0.005$ and $n = 5 \cdot 4 = 20$;
  Therefore
$$A = P(1+i)^n = 1200(1+0.005)^{20} = 1200 \cdot 1.005^{20} \approx 1,325.87;$$

Application: Continuous Compounding of Interest

- If P is the principal amount invested, r the annual interest rate compounded continuously and t the number of years, then the amount A accrued at the end of t years is
$$A = Pe^{rt};$$
- Example: If \$1,200 is deposited in an account paying annually 2% compounded continuously, how much will be in the account at the end of 5 years?

  We have $P = 1200$, $r = 0.02$ and $t = 5$;
  Therefore
$$A = Pe^{rt} = 1200 \cdot e^{0.02 \cdot 5} = 1200 \cdot e^{0.1} \approx 1,326.21;$$

Subsection 2

Logarithmic Functions and Applications

Logarithmic Functions

- The logarithm $y = \log_a x$ (read logarithm to base a of x) is the exponent to which one must raise the base a to get x;
- More formally,

$$y = \log_a x \quad \text{if and only if} \quad a^y = x.$$

- Example: Convert each logarithmic equation to an exponential one and vice-versa:
  - $5^3 = 125 \quad \Longleftrightarrow \quad \log_5 125 = 3$;
  - $(\frac{1}{4})^6 = x \quad \Longleftrightarrow \quad 6 = \log_{1/4} x$;
  - $(\frac{1}{2})^m = 8 \quad \Longleftrightarrow \quad m = \log_{1/2} 8$;
  - $7 = 3^z \quad \Longleftrightarrow \quad z = \log_3 7$;

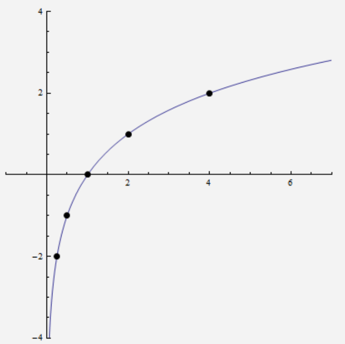
Evaluating Logarithms

- Evaluate each logarithm:
  - $\log_5 25 = 2$; (since $5^2 = 25$)
  - $\log_2 \frac{1}{8} = -3$; (since $2^{-3} = \frac{1}{8}$)
  - $\log_{1/2} 4 = -2$; (since $(\frac{1}{2})^{-2} = 4$)
  - $\log_{10} 0.001 = -3$; (since $10^{-3} = 0.001$)
  - $\log_9 3 = \frac{1}{2}$; (since $9^{1/2} = 3$)
- The logarithm to base 10 is called the **common logarithm**, denoted $\log x$ (i.e., this means $\log_{10} x$); The logarithm to base e is called the **natural logarithm**, denoted $\ln x$ (i.e., this means $\log_e x$);
- Evaluate each logarithm:
  - $\log 1000 = 3$; (since $10^3 = 1000$)
  - $\ln e = 1$; (since $e^1 = e$)
  - $\log \frac{1}{10} = -1$; (since $10^{-1} = \frac{1}{10}$)

Graphs of Logarithmic Functions to Base $a > 1$

- When the base a is such that $a > 1$, then $f(x) = \log_a x$ has an increasing graph (going up as we move from left to right);
- As an example, we'll use a few points to sketch the graph of $f(x) = \log_2 x$;

x	$y = \log_2 x$
$1/4$	-2
$1/2$	-1
1	0
2	1
4	2

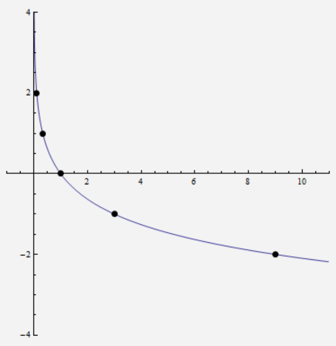


- Note that the y-axis is a vertical asymptote;

Graphs of Logarithmic Functions to Base $0 < a < 1$

- When the base a is such that $0 < a < 1$, then $f(x) = \log_a x$ has a decreasing graph (going down as we move from left to right);
- As an example, we'll use a few points to sketch the graph of $f(x) = \log_{1/3} x$;

x	$y = \log_{1/3} x$
$1/9$	2
$1/3$	1
1	0
3	-1
9	-2



- Note that the y-axis is a vertical asymptote;

Logarithmic Equations

One-to-One Property of Logarithms

$$\log_a m = \log_a n \quad \text{implies} \quad m = n.$$

- Example: Solve each equation:
  - $\log_3 x = -2$
    $\log_3 x = -2 \Rightarrow x = 3^{-2} \Rightarrow x = \frac{1}{9}$;
  - $\log_x 8 = -3$
    $\log_x 8 = -3 \Rightarrow x^{-3} = 8 \Rightarrow (x^{-3})^{-1/3} = 8^{-1/3} \Rightarrow x = \frac{1}{8^{1/3}} \Rightarrow x = \frac{1}{\sqrt[3]{8}} \Rightarrow x = \frac{1}{2}$;
  - $\log(x^2) = \log 4$
    $\log(x^2) = \log 4 \Rightarrow x^2 = 4 \Rightarrow x = \pm\sqrt{4} \Rightarrow x = -2$ or $x = 2$;

Application: Find Time in Continuous Compounding

- Recall: If P is the principal amount invested, r the annual interest rate compounded continuously and t the number of years, then the amount A accrued at the end of t years is
$$A = Pe^{rt};$$

- Example: How long does it take for \$200 to grow to \$600 at 3% annually compounded continuously?

  We have $P = 200$, $r = 0.03$ and $A = 600$;
  We would like to compute t;
$$\begin{aligned} A = Pe^{rt} \quad &\Rightarrow \quad 600 = 200e^{0.03t} \\ &\Rightarrow \quad e^{0.03t} = 3 \\ &\Rightarrow \quad 0.03t = \ln 3 \\ &\Rightarrow \quad t = \frac{100}{3}\ln 3 \approx 36.62 \text{ years;} \end{aligned}$$

Subsection 3

Properties of Logarithms

The Inverse Properties

- Recall the definition of the logarithm to base a of x:

$$y = \log_a x \quad \text{if and only if} \quad a^y = x;$$

- This property gives

$$\boxed{a^{\log_a x} = x \quad \text{and} \quad \log_a (a^y) = y;}$$

- Example: Simplify:
  - $\ln(e^7) = 7$; (recall this means $\log_e(e^7)$)
  - $2^{\log_2 13} = 13$;

The Product Rule for Logarithms

- For $M > 0$ and $N > 0$, we have

$$\boxed{\log_a (M \cdot N) = \log_a M + \log_a N.}$$

- Example: Write as a single logarithm:
  - $\log_2 7 + \log_2 5 = \log_2 (7 \cdot 5) = \log_2 35;$
  - $\ln(\sqrt{2}) + \ln(\sqrt{3}) = \ln(\sqrt{2} \cdot \sqrt{3}) = \ln(\sqrt{2 \cdot 3}) = \ln(\sqrt{6});$

The Quotient Rule for Logarithms

- For $M > 0$ and $N > 0$, we have

$$\boxed{\log_a\left(\frac{M}{N}\right) = \log_a M - \log_a N.}$$

- Example: Write as a single logarithm:
  - $\log_2 3 - \log_2 7 = \log_2\left(\frac{3}{7}\right)$;
  - $\ln(w^8) + \ln(w^2) = \ln\left(\frac{w^8}{w^2}\right) = \ln(w^{8-2}) = \ln(w^6)$;

The Power Rule for Logarithms

- For $M > 0$, we have

$$\boxed{\log_a\left(M^N\right) = N \cdot \log_a M.}$$

- Example: Write in terms of $\log 2$:
  - $\log(2^{10}) = 10\log 2$;
  - $\log(\sqrt{2}) = \log(2^{1/2}) = \frac{1}{2}\log 2$;
  - $\log(\frac{1}{2}) = \log(2^{-1}) = -\log 2$;

Summary of the Properties of Logarithms

Properties of Logarithms

Assuming M, N and a are positive, with $a \neq 1$, we have

1. $\log_a a = 1$;
2. $\log_a 1 = 0$;
3. $\log_a (a^y) = y$; $\quad$ (y any real)
4. $a^{\log_a x} = x$; $\quad$ (x any positive real)
5. $\log_a (M \cdot N) = \log_a M + \log_a N$;
6. $\log_a \left(\frac{M}{N}\right) = \log_a M - \log_a N$;
7. $\log_a \left(\frac{1}{N}\right) = -\log_a N$;
8. $\log_a (M^N) = N \log_a M$;

Using the Properties

- Rewrite each expression in terms of $\log 2$ and/or $\log 3$:
  - $\log 6 = \log(2 \cdot 3) = \log 2 + \log 3$;
  - $\log 16 = \log(2^4) = 4\log 2$;
  - $\log(\frac{9}{2}) = \log 9 - \log 2 = \log(3^2) - \log 2 = 2\log 3 - \log 2$;
  - $\log(\frac{1}{3}) = -\log 3$;
- Rewrite each expression as a sum or difference of multiples of logarithms:
  - $\log\left(\dfrac{xz}{y}\right) = \log(xz) - \log y = \log x + \log z - \log y$;
  - $\log_3\left(\dfrac{(x-3)^{2/3}}{\sqrt{x}}\right) = \log_3((x-3)^{2/3}) - \log_3(x^{1/2}) = \frac{2}{3}\log_3(x-3) - \frac{1}{2}\log_3 x$;
- Rewrite each expression as a single logarithm:
  - $\frac{1}{2}\log x - 2\log(x+1) = \log(x^{1/2}) - \log((x+1)^2) = \log\left(\dfrac{\sqrt{x}}{(x+1)^2}\right)$;
  - $3\log y + \frac{1}{2}\log z - \log x = \log(y^3) + \log(z^{1/2}) - \log x = \log(y^3\sqrt{z}) - \log x = \log(\frac{y^3\sqrt{z}}{x})$;

Subsection 4

Solving Equations and Applications

Logarithmic Equations with Only One Logarithm

- Solve $\log(x+3)=2$;

  The technique involves rewriting as an exponential equation using the property

  $$y=\log_a x \quad \text{if and only if} \quad a^y=x;$$

  $$\begin{aligned} &\log(x+3)=2 \\ \Rightarrow &\quad 10^2=x+3 \\ \Rightarrow &\quad x=100-3=97; \end{aligned}$$

  Now check whether the solution is admissible; i.e., is $\log(97+3)=2$? Yes! since $10^2=100$; So, the solution $x=97$ is admissible.

Using the Product Rule to Solve an Equation

- Solve $\log_2(x+3) + \log_2(x-3) = 4$;

  The technique involves using the Product Rule $\log_a(M \cdot N) = \log_a M + \log_a N$ to combine the sum into a single logarithm; Then, we apply the technique for solving equations involving a single logarithm;

$$
\begin{aligned}
&\log_2(x+3) + \log_2(x-3) = 4 \\
\Rightarrow \quad & \log_2((x+3)(x-3)) = 4 \\
\Rightarrow \quad & 2^4 = (x+3)(x-3) \\
\Rightarrow \quad & 16 = x^2 - 9 \\
\Rightarrow \quad & x^2 = 25 \\
\Rightarrow \quad & x = \pm\sqrt{25} = \pm 5;
\end{aligned}
$$

  Now check whether the solutions are admissible; $x = -5$ is not admissible; Therefore, $x = 5$ is the only admissible solution!

Using the One-to-One Property to Solve an Equation

- Solve $\log x + \log(x-1) = \log(8x-12) - \log 2$;

  The technique involves using the Product/Quotient Rule for Logarithms to combine the sum/difference into a single logarithm and, then, the One-to-One Property

  $$\log_a m = \log_a n \quad \text{implies} \quad m = n;$$

  $$\begin{aligned} & \log x + \log(x-1) = \log(8x-12) - \log 2 \\ \Rightarrow \quad & \log(x(x-1)) = \log\left(\tfrac{8x-12}{2}\right) \\ \Rightarrow \quad & x(x-1) = 4x-6 \\ \Rightarrow \quad & x^2 - x = 4x - 6 \\ \Rightarrow \quad & x^2 - 5x + 6 = 0 \\ \Rightarrow \quad & (x-2)(x-3) = 0 \\ \Rightarrow \quad & x = 2 \text{ or } x = 3; \end{aligned}$$

  Now check whether the solutions are admissible; Both $x = 2$ and $x = 3$ are admissible solutions!

A Single Exponential Equation

- Solve $2^x = 10$;

  The technique involves rewriting as a logarithmic equation using the property

  $$y = \log_a x \quad \text{if and only if} \quad a^y = x;$$

  $$\begin{aligned} 2^x &= 10 \\ \Rightarrow \quad x &= \log_2 10; \end{aligned}$$

  Now check whether the solution is admissible; The solution $x = \log_2 10$ is admissible.

Powers of the Same Base

- Solve $2^{(x^2)} = 4^{3x-4}$;

  The technique involves rewriting the two sides as powers over the same base and, then, using the One-to-One Property

  $$a^m = a^n \quad \text{implies} \quad m = n;$$

  $$\begin{aligned} & 2^{(x^2)} = 4^{3x-4} \\ \Rightarrow \quad & 2^{(x^2)} = (2^2)^{3x-4} \\ \Rightarrow \quad & 2^{(x^2)} = 2^{2(3x-4)} \\ \Rightarrow \quad & x^2 = 6x - 8 \\ \Rightarrow \quad & x^2 - 6x + 8 = 0 \\ \Rightarrow \quad & (x-2)(x-4) = 0 \\ \Rightarrow \quad & x = 2 \text{ or } x = 4; \end{aligned}$$

  Now check whether the solutions are admissible; Both $x = 2$ and $x = 4$ are admissible solutions;

Exponential Equations Involving Different Bases

- Solve $2^{x-1} = 3^x$;

  The technique involves taking logarithms of both sides and, then, using the Power Property

$$\log_a(M^N) = N\log_a M;$$

$$\begin{aligned} & 2^{x-1} = 3^x \\ \Rightarrow \quad & \log(2^{x-1}) = \log(3^x) \\ \Rightarrow \quad & (x-1)\log 2 = x\log 3 \\ \Rightarrow \quad & x\log 2 - \log 2 = x\log 3 \\ \Rightarrow \quad & x\log 2 - x\log 3 = \log 2 \\ \Rightarrow \quad & x(\log 2 - \log 3) = \log 2 \\ \Rightarrow \quad & x = \frac{\log 2}{\log 2 - \log 3}; \end{aligned}$$

It is more difficult now to check whether the solution(s) is admissible.

Changing the Base

Change-Of-Base Formula

If a, b are positive numbers not equal to 1 and $M > 0$,

$$\boxed{\log_a M = \frac{\log_b M}{\log_b a};}$$

- Example: Use your calculator to compute $\log_7 99$ to four decimal places;

  Calculators usually allow us to compute common logarithms (base 10) and natural logarithms (base e);
  Thus, we need to use Change-Of-Base to convert $\log_7 99$ to an expression involving either common or natural logarithms:

  $$\log_7 99 = \frac{\ln 99}{\ln 7} \approx 2.3614;$$

Strategy for Solving Equations

1. If the equation has a single logarithm or a single exponential, rewrite using
$$y = \log_a x \quad \text{if and only if} \quad a^y = x;$$
2. Use the rules to combine logarithms as much as possible;
3. Use the One-to-One Properties
$$\log_a m = \log_a n \quad \text{implies} \quad m = n;$$
$$a^m = a^n \quad \text{implies} \quad m = n;$$
4. For exponential equations with different bases, take the common or natural logarithms of both sides of the equation;

Application:Finding Time in Finance

- If \$1,000 is deposited into an account paying 4% compounded quarterly, in how many quarters will the account have \$2,000 in it?

  Recall $A = P(1+i)^n$;
  We have $P = 1000$, $i = \frac{0.04}{4} = 0.01$ and $A = 2000$;
  We would like to determine the number of periods (quarters) n;

$$\begin{aligned} 2000 &= 1000(1+0.01)^n \\ \Rightarrow \quad 2 &= 1.01^n \\ \Rightarrow \quad n &= \log_{1.01} 2 \\ \overset{\text{Change Base}}{\Rightarrow} \quad n &= \frac{\ln 2}{\ln 1.01} \approx 69.66; \end{aligned}$$

  Thus, the account will have \$2,000 in approximately 70 quarters, i.e., in 17-and-a-half years;

Application: Finding the Rate of Radioactive Decay

- The number of grams of a radioactive substance that is present in an old bone after t years is given by

$$A = 8e^{rt},$$

where r is the decay rate.

(a) How many grams were present when the bone was in a living organism at $t = 0$?

(b) If it took 6300 years for the substance to decay from 8 grams to 4 grams, what is its decay rate?

(a) Plugging-in $t = 0$, we get $A = 8e^{r \cdot 0} = 8 \cdot e^0 = 8 \cdot 1 = 8$; Thus, when the organism was living the bone had 8 grams of the substance;

(b)

$$\begin{aligned} & 4 = 8e^{6300r} \\ \Rightarrow \quad & \tfrac{1}{2} = e^{6300r} \\ \Rightarrow \quad & 6300r = \ln\left(\tfrac{1}{2}\right) \\ \Rightarrow \quad & r = \tfrac{1}{6300} \ln\left(\tfrac{1}{2}\right) \approx -0.00011; \end{aligned}$$