Intermediate Algebra

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 102

1. Linear Equations and Inequalities in One Variable
   - Linear Equations in One Variable
   - Formulas and Functions
   - Applications
   - Inequalities
   - Compound Inequalities
   - Absolute Value Equations and Inequalities

Subsection 1

Linear Equations in One Variable

Equations

- An **equation** is a sentence expressing the equality of two algebraic expressions, e.g., $3x+1=16$; Since $3\cdot 5+1=16$, we say that 5 **satisfies** the equation; An equation may be satisfied by zero, one or multiple numbers; These are the **solutions** or **roots** of the equation;
- The set of all solutions of an equation is called the **solution set**;
- Example:
  - Is the equation $3x+7=-8$ satisfied by -5? We replace x by -5:

$$\begin{aligned} 3\cdot(-5)+7 &= -8 \\ -15+7 &= -8 \\ -8 &= -8 \end{aligned}$$

    Thus, -5 satisfies $3x+7=-8$.
  - Is the equation $2(x-1)=2x+3$ satisfied by 4? We replace x by 4:

$$\begin{aligned} 2(4-1) &= 2\cdot 4+3 \\ 2\cdot 3 &= 8+3 \\ 6 &= 11 \end{aligned}$$

    Thus, 4 does not satisfy $2(x-1)=2x+3$.

Solving Equations

- Key Properties we Apply:
  - **Addition Property**: If $a = b$, then $a + c = b + c$: i.e., adding the same number to both sides of an equation does not change the solution set of the equation;
  - **Multiplication Property**: If $a = b$ and $c \neq 0$, then $ca = cb$; i.e., multiplying both sides of an equation by the same non-zero number does not change the solution set of the equation;
- Example: Solve the equation $6 - 3x = 8 - 2x$;
  Our goal is to get x alone on one of the two sides; We do this by carefully applying one of the previous two operations at each step:

$$\begin{aligned} 6 - 3x &= 8 - 2x \\ 6 - 3x - 6 &= 8 - 2x - 6 \\ -3x &= 2 - 2x \\ -3x + 2x &= 2 - 2x + 2x \\ -x &= 2 \\ x &= -2. \end{aligned}$$

Another Example

- Solve the equation $2(x-4)+5x=34$;

$$\begin{array}{rcl}
2(x-4)+5x & = & 34 \\
2x-8+5x & = & 34 \\
7x-8 & = & 34 \\
7x-8+8 & = & 34+8 \\
7x & = & 42 \\
\frac{1}{7}\cdot 7x & = & \frac{1}{7}\cdot 42 \\
x & = & 6;
\end{array}$$

Another Example

- Solve the equation $\frac{x}{2} - \frac{1}{3} = \frac{x}{3} + \frac{5}{6}$;

$$\begin{array}{rcl}
\frac{x}{2} - \frac{1}{3} & = & \frac{x}{3} + \frac{5}{6} \\
6(\frac{x}{2} - \frac{1}{3}) & = & 6(\frac{x}{3} + \frac{5}{6}) \\
6 \cdot \frac{x}{2} - 6 \cdot \frac{1}{3} & = & 6 \cdot \frac{x}{3} + 6 \cdot \frac{5}{6} \\
3x - 2 & = & 2x + 5 \\
3x - 2 - 2x & = & 2x + 5 - 2x \\
x - 2 & = & 5 \\
x - 2 + 2 & = & 5 + 2 \\
x & = & 7;
\end{array}$$

One More Example

- Solve the equation $x - 0.1x = 0.75x + 4.5$;

$$\begin{aligned}
x - 0.1x &= 0.75x + 4.5 \\
0.9x &= 0.75x + 4.5 \\
0.9x - 0.75x &= 0.75x + 4.5 - 0.75x \\
0.15x &= 4.5 \\
\frac{100}{15} \cdot 0.15x &= \frac{100}{15} \cdot 4.5 \\
x &= 30;
\end{aligned}$$

Types of Equations

- Equations are classified depending on the number of their solutions:
  - **Identity**: is satisfied by every number for which both sides are defined;
  - **Conditional**: is satisfied by at least one number but is not an identity;
  - **Inconsistent**: has no solutions;
- Example: Classify $8-3(x-5)+7=3-(x-5)-2(x-11)$;

$$\begin{aligned} 8-3(x-5)+7 &= 3-(x-5)-2(x-11) \\ 8-3x+15+7 &= 3-x+5-2x+22 \\ -3x+30 &= -3x+30 \end{aligned}$$

  So this equation is an identity;
- Example: Classify $5-3(x-6)=4(x-9)-7x$;

$$\begin{aligned} 5-3(x-6) &= 4(x-9)-7x \\ 5-3x+18 &= 4x-36-7x \\ -3x+23 &= -3x-36 \\ -3x+23+3x &= -3x-36+3x \\ 23 &= -36 \end{aligned}$$

  So this equation is inconsistent;

Steps for Solving an Equation

- A **linear equation in one variable** is one of the form $ax = b$, where a, b are real numbers with $a \neq 0$;
- **Solution Strategy for Linear Equations**:
  1. To eliminate fractions multiply both sides by the least common denominator; to eliminate decimals, multiply both sides by an appropriate power of 10;
  2. Distribute to remove parentheses;
  3. Combine like terms;
  4. Take all variables to one side and all constants on the other;
  5. Multiply to solve for the variable;
- Example: Solve $\frac{x}{2} - \frac{x-4}{5} = \frac{23}{10}$;

$$\begin{aligned} \frac{x}{2} - \frac{x-4}{5} &= \frac{23}{10} \\ 10\left(\frac{x}{2} - \frac{x-4}{5}\right) &= 10 \cdot \frac{23}{10} \\ 5x - 2(x-4) &= 23 \end{aligned}$$

$$\begin{aligned} 5x - 2x + 8 &= 23 \\ 3x + 8 &= 23 \\ 3x &= 15 \\ x &= 5. \end{aligned}$$

An Application: Enrollment in Public Schools

- Suppose that the expression $0.45x + 39.05$ approximates in millions the total enrollment in public elementary and secondary schools in the year $1985 + x$.
  - What was the enrollment in 1992?
    Year 1992 corresponds to $x = 7$; Compute
    $$0.45 \cdot 7 + 39.05 = 42.2 \text{ million.}$$
  - In which year will enrollment reach 50 million?
    We need to solve $0.45x + 39.05 = 50$;
    Subtract 39.05: $0.45x = 10.95$;
    Divide by 0.45: $x = 24.333$;
    Therefore enrollment will reach 50 million in the year
    $1985 + 25 = 2010$;

Subsection 2

Formulas and Functions

Solving for a Variable

- Solve the Celsius - Fahrenheit formula $C = \frac{5}{9}(F-32)$ for F;

$$\begin{array}{rcl} C & = & \frac{5}{9}(F-32) \\ \frac{9}{5} \cdot C & = & \frac{9}{5} \cdot \frac{5}{9}(F-32) \\ \frac{9}{5}C & = & F-32 \\ \frac{9}{5}C+32 & = & F \end{array}$$

Therefore $F = \frac{9}{5}C + 32$;

Functions

- A **function** is a rule that determines uniquely the value of one variable a from the values of one or more other variables; In that case, it is said that a **is a function of** the other variable(s);
- Example: Suppose that $7a - 5b = 35$. Write a formula expressing a as a function of b and a formula expressing b a function of a;

$$\begin{aligned} 7a - 5b &= 35 \\ 7a &= 5b + 35 \\ a &= \tfrac{1}{7}(5b + 35) \\ a &= \tfrac{5}{7}b + 5; \end{aligned}$$

For b as a function of a:

$$\begin{aligned} 7a - 5b &= 35 \\ -5b &= -7a + 35 \\ b &= -\tfrac{1}{5}(-7a + 35) \\ b &= \tfrac{7}{5}a - 7; \end{aligned}$$

Two More Examples

- Suppose that $A = P + Prt$; Write a formula expressing P as a function of A, r and t;

$$\begin{aligned} A &= P + Prt \\ A &= P(1+rt) \\ \frac{A}{1+rt} &= P \end{aligned}$$

  Therefore $P(A, r, t) = \frac{A}{1+rt}$;

- Suppose that $3a + 7 = -5ab + b$; Solve for a;

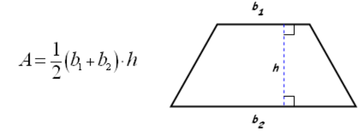
$$\begin{aligned} 3a + 7 &= -5ab + b \\ 3a + 5ab &= b - 7 \\ a(3 + 5b) &= b - 7 \\ a &= \frac{b-7}{3+5b}; \end{aligned}$$

Determining the Value of a Variable

- Use the formula $-2x+3y=9$ to determine the value of y, given $x=-3$;
  We first express y as a function of x:
  $$\begin{aligned} -2x+3y &= 9 \\ 3y &= 2x+9 \\ y &= \tfrac{2}{3}x+3 \end{aligned}$$
  Now compute $y=\frac{2}{3}\cdot(-3)+3=1$;
- Recall that the interest amount I as a function of the principal amount P, the interest rate r and the time t is given by $I=Prt$; If the interest is \$50, the principal is \$500 and the time is 2 years, what is the annual interest rate?
  $$I=Prt \quad \Rightarrow \quad r=\frac{I}{Pt} \quad \Rightarrow \quad r=\frac{50}{500\cdot 2} \quad \Rightarrow \quad r=0.05;$$
  Therefore the annual interest rate is 5%;

Area of Trapezoid

- Recall that the area of a trapezoid with height h, and lengths of bases b_1 and b_2 is given by $A = \frac{1}{2}h(b_1 + b_2)$;



If a trapezoid has area 30 square kilometers, the length of one of its bases is 10 kilometers and its height is 5 kilometers, what is the length b_2 of its other base?

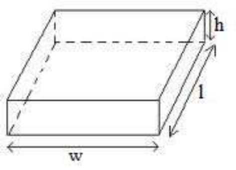
We first solve the area formula for b_2:

$$\begin{aligned} A &= \tfrac{1}{2}h(b_1 + b_2) \\ 2A &= h(b_1 + b_2) \\ \tfrac{2A}{h} &= b_1 + b_2 \\ b_2 &= \tfrac{2A}{h} - b_1; \end{aligned}$$

Therefore, $b_2 = \frac{2 \cdot 30}{5} - 10 = 12 - 10 = 2$ kilometers;

Volume of Rectangular Solid

- Katie has poured 14 cubic yards of concrete to construct a rectangular driveway:



If the concrete is 4 inches thick and the driveway is 18 feet wide, how long is Katie's driveway? (Recall: 1 yard = 3 feet = 36 inches)
We have $V = l \cdot w \cdot h$;
Therefore $l = \dfrac{V}{w \cdot h}$;
This gives $l = \dfrac{14}{6 \cdot \frac{1}{9}} = 21$ yards.

Subsection 3

Applications

Writing Algebraic Expressions

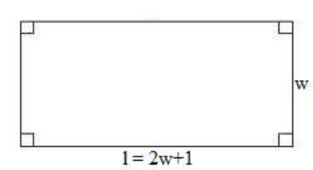
- Convert the following verbal statements into algebraic expressions:
  - Two numbers that differ by 12;
    Assume the numbers are x and y; Then their difference $x - y$ is 12; $x - y = 12$ gives $y = x - 12$; Therefore, the numbers are $x - 12$ and x;
  - Two consecutive even integers;
    If one even integer is x, the other is $x + 2$;
  - Two investments that total \$5,000
    Assume the investments are x and y; Then their sum $x + y$ is \$5,000; $x + y = 5000$ gives $y = 5000 - x$; Therefore, the investments are x and $5000 - x$;
  - The length of a rectangle if its width is x meters and its perimeter is 10 meters;
    Suppose the length is l meters; Then, the perimeter is $P = 2l + 2x$; Therefore, $l = \frac{1}{2}P - x$ or $l = 5 - x$ meters;

Number Problems

- If the sum of three consecutive integers is 228, what are the three integers?
  Suppose the integers are $x, x+1$ and $x+2$;
  Then $x+(x+1)+(x+2)=228$;
  This gives $3x+3=228$;
  Therefore, $3x=225$
  and, hence, $x=75$;
  The three integers are $75, 76$ and 77;
- If twice a number increased by 6 is 114, what is the number?
  If the number is x, we have
  $$2x+6=114;$$
  Therefore, $2x=108$, giving $x=54$;

Geometric Problem

- The length of a rectangular piece of property is 1 foot more than twice the width. If the perimeter is 302 feet, find the length and the width.



The perimeter of the rectangle is given by $P = 2l + 2w = 302$; Therefore, since $l = 2w + 1$, we get

$$\begin{aligned} 2(2w+1)+2w &= 302 \\ 4w+2+2w &= 302 \\ 6w &= 300 \\ w &= 50. \end{aligned}$$

Thus, the rectangle has length $l = 2w + 1 = 101$ feet and width $w = 50$ feet.

An Investment Problem (Problem 37, Page 97)

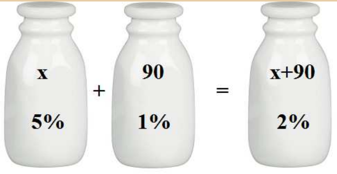
- Bob invested some money at 5% and some at 9% simple interest. If the amount he invested at the higher rate was twice what he invested at the lower rate and the total interest from the investments for 1 year was \$ 920, how much did he invest at each rate?
  If he invested \$$x$ at the lower rate, then he invested \$$2x$ at the higher rate.
  The return from the first investment in 1 year was $0.05x$ and the return from the second was $0.09(2x)$;
  Since the total return was \$ 920, we must have

$$\begin{aligned} 0.05x + 0.09(2x) &= 920 \\ 0.05x + 0.18x &= 920 \\ 0.23x &= 920 \\ x &= 4,000 \end{aligned}$$

Therefore, Bob invested \$ 4,000 at 5% and \$ 8,000 at 9%.

A Mixture Problem

- How many gallons of milk containing 5% butterfat must be mixed with 90 gallons containing 1% fat to obtain a 2% mixture?



The amount of fat in the first jar is $0.05x$ gallons; That in the second jar is $0.01 \cdot 90 = 0.9$ gallons; The amount of fat in the final mixture is $0.02(x+90)$ gallons;

Since no fat is created or lost in the mixing process, we must have

$$\begin{aligned} 0.05x + 0.9 &= 0.02(x+90) \\ 0.05x + 0.9 &= 0.02x + 1.8 \\ 0.03x &= 0.9 \\ x &= 30; \end{aligned}$$

Thus, one has to mix 30 gallons of the 5% milk.

A Uniform Motion Problem

- Ayrton drove for 3 hours and 30 minutes in a dust storm. When the skies cleared, he increased his speed by 35 miles per hour and drove for 4 more hours. If he drove a total of 365 miles, how fast did he drive during the dust storm?

Recall that multiplying the speed by the time yields the distance. Thus, if Ayrton drove with speed v miles per hour for 3.5 hours during the storm, covering x miles, we have $3.5v = x$; Moreover, he drove with speed $v + 35$ miles per hour after the storm for 4 hours, covering y miles, whence $4(v + 35) = y$. But $x + y = 365$, so

$$\begin{aligned} 3.5v + 4(v + 35) &= 365 \\ 3.5v + 4v + 140 &= 365 \\ 7.5v &= 225 \\ v &= 30; \end{aligned}$$

Thus, Ayrton was traveling at 30 miles per hour during the storm.

A Commission Problem

- Anna is selling her house through a real estate agent whose commission is 6% of the selling price. What should the selling price be so that Anna can get \$ 169,200?

Suppose \$ x is the price that should be asked; Then, the commission will be \$$0.06x$; Subtracting the commission from the price will yield Anna's share, which should be \$ 169,200:

$$\begin{aligned} x - 0.06x &= 169,200 \\ 0.94x &= 169,200 \\ x &= 180,000; \end{aligned}$$

Therefore, Anna should set the price at \$ 180,000.

Subsection 4

Inequalities

The Inequality Symbols and their Meaning

- Recall the inequality symbols and their meanings:

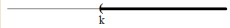
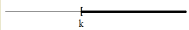
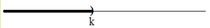
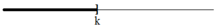
Symbol	Meaning
$<$	less than
$\leq$	less than or equal to
$>$	greater than
$\geq$	greater than or equal to

- True or False?

$-3 < -9$	✠
$-6 \geq -8$	✓
$(-1)(-3) < (-1)(5)$	✠
$(-4)(-5) + 2 \geq 21$	✓

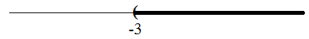
Interval Notation and Graphs

- The following table summarizes the forms of the various unbounded intervals (i.e., intervals that extend to $\pm\infty$):

Inequality	Interval Notation	Graph
$x > k$	(k, ∞)	
$x \geq k$	$[k, \infty)$	
$x < k$	$(-\infty, k)$	
$x \leq k$	$(-\infty, k]$	

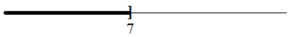
- Example: Write in interval notation and graph $x > -3$;

$(-3, \infty)$



- Example: Write in interval notation and graph $x \leq 7$;

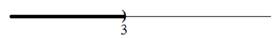
$(-\infty, 7]$



Solving Linear Inequalities

- A **linear inequality** in a single variable x is one of the form $ax < b$, with $a \neq 0$ and where $<$ may be replaced by $\leq, >$ or $\geq$;
- To solve such an inequality, we take advantage of
  - **Addition Property**: Adding the same number to both sides of an inequality preserves the inequality!
  - **Multiplication Property**: Multiplying by the same nonzero number both sides of an inequality either
    - preserves the inequality if the number is positive or
    - reverses the inequality if the number is negative.
- Example: Solve the inequality $2x - 7 < -1$, express the solution set in interval notation and graph it;

$$\begin{aligned} 2x - 7 &< -1 \\ 2x &< 6 \\ x &< 3 \end{aligned}$$

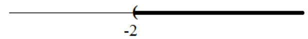
Thus, the solution set is $(-\infty, 3)$: 

More Examples

- Example: Solve the inequality $5 - 3x < 11$, express the solution set in interval notation and graph it;

$$\begin{aligned} 5 - 3x &< 11 \\ -3x &< 6 \\ x &> -2 \end{aligned}$$

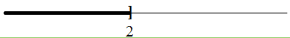
Thus, the solution set is $(-2, \infty)$:



- Example: Solve the inequality $6 - x \geq 4$, express the solution set in interval notation and graph it;

$$\begin{aligned} 6 - x &\geq 4 \\ -x &\geq -2 \\ x &\leq 2 \end{aligned}$$

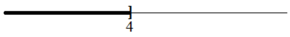
Thus, the solution set is $(-\infty, 2]$:



Examples Involving Fractions

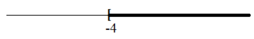
- Example: Solve the inequality $\frac{8+3x}{-5} \geq -4$, express the solution set in interval notation and graph it;

$$\begin{array}{rcl} \frac{8+3x}{-5} & \geq & -4 \\ 8+3x & \leq & 20 \\ 3x & \leq & 12 \\ x & \leq & 4 \end{array}$$

Thus, the solution set is $(-\infty, 4]$: 

- Example: Solve the inequality $\frac{1}{2}x - \frac{2}{3} \leq x + \frac{4}{3}$, express the solution set in interval notation and graph it;

$$\begin{array}{rcl} \frac{1}{2}x - \frac{2}{3} & \leq & x + \frac{4}{3} \\ -\frac{2}{3} & \leq & \frac{1}{2}x + \frac{4}{3} \\ -2 & \leq & \frac{1}{2}x \\ -4 & \leq & x \end{array}$$

Thus, the solution set is $[-4, \infty)$: 

Writing Inequalities

- Example: Introduce a variable and write an inequality that describes the following situation:
  - Chris paid more than \$200 for a suit;
    If p is the price of the suit, $p > 200$;
  - A candidate for U.S. President must be at least 35 years old;
    If a is the age of the candidate, $a \geq 35$;
  - The capacity of an elevator is at most 1500 pounds;
    If c is the capacity, $c \leq 1500$;
  - The company must hire no fewer than 10 programmers;
    If n is the number of programmers to be hired, $n \geq 10$;

Applications of Inequalities

- Example: Rose plans to spend less than \$ 500 on a dryer, including the 9% sales tax and a \$64 setup charge. In what range is the selling price of the dryer that she can afford?
  Suppose that \$$x$ is the selling price.
  Then, the sales tax is $0.09x$ and the total bill (including the setup charge) is $x + 0.09x + 64$;
  Since she intends to spend less than \$500, we must have

$$\begin{aligned} x + 0.09x + 64 &< 500 \\ 1.09x + 64 &< 500 \\ 1.09x &< 436 \\ x &< 400. \end{aligned}$$

  Thus, Rose can afford a dryer costing less that \$400.

Another Application

- Ingrid owns a piece of land on which she owes \$12,760 to a bank. She wants to sell the land for enough money to at least pay off the mortgage. The real estate agent gets 6% of the selling price and the city has a \$400 real estate transfer tax paid by the seller. What should be the range of the selling price for Ingrid to get at least enough money to pay off her mortgage?
  Suppose that $\$x$ is the selling price.
  Then, the real estate agent would get $0.06x$ and the city would get \$400; Thus her net income would be $x - 0.06x - 400$;
  Since she intends to get at least \$12,760, we must have

$$\begin{aligned} x - 0.06x - 400 &\geq 12,760 \\ 0.94x - 400 &\geq 12,760 \\ 0.94x &\geq 13,160 \\ x &\geq 14,000. \end{aligned}$$

  Thus, Ingrid needs to sell for at least \$14,000.

Subsection 5

Compound Inequalities

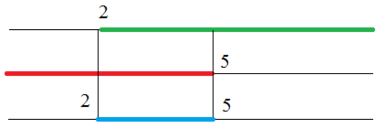
Compound Inequalities

- A **compound inequality** is obtained by joining two ordinary inequalities by the "and" or the "or" logical connective;
- If the "and" connective is used, the compound inequality is true if both inequalities joined are true;
  For instance, "$5 > 3$ and $-2 \leq 7$" is true because both "$5 > 3$" and "$-2 \leq 7$" are true;
- If the "or" connective is used, the compound inequality is true if at least one of the inequalities joined are true;
  For instance, "$5 > 3$ or $-2 \geq 7$" is true because "$5 > 3$" is true;
- Example: Determine the truth value:

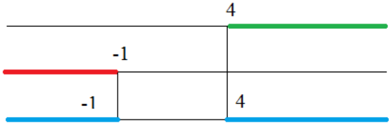
Inequality	Truth Value
$3 > 2$ and $3 < 5$	✓
$6 > 2$ and $6 < 5$	✠
$2 < 3$ or $2 > 7$	✓
$4 < 3$ or $4 \geq 7$	✠

Graphing the Solution Set

- Example: Graph the solution set of "$x > 2$ and $x < 5$";
  We first graph $x > 2$; Then we graph $x < 5$; Finally, because of the "and", we include in the graph of the compound inequality that part of the line included in both the first and the second solution sets:

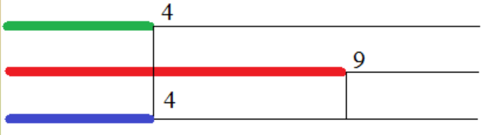


- Example: Graph the solution set of "$x > 4$ or $x < -1$";
  We first graph $x > 4$; Then we graph $x < -1$; Finally, because of the "or", we include in the graph of the compound inequality that part of the line included in at least one of the two solution sets:

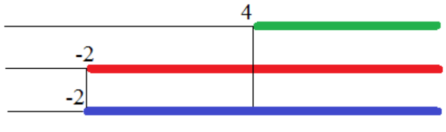


Two More Examples

- Example: Graph the solution set of "$x < 4$ and $x < 9$";
  We first graph $x < 4$; Then we graph $x < 9$; Finally, because of the "and", we include in the graph of the compound inequality that part of the line included in both the first and the second solution sets:



- Example: Graph the solution set of "$x > 4$ or $x > -2$";
  We first graph $x > 4$; Then we graph $x > -2$; Finally, because of the "or", we include in the graph of the compound inequality that part of the line included in at least one of the two solution sets:



More Complicated Examples I

- Example: Graph the solution set of "$x + 2 > 3$ and $x - 6 < 7$";
We first rewrite the compound inequality in a simpler form by solving each component for x:

$$x > 1 \quad \text{and} \quad x < 13.$$

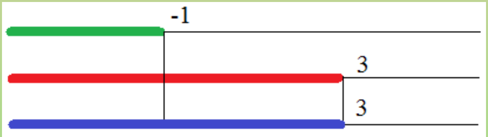
We now graph $x > 1$; Then we graph $x < 13$; Finally, because of the "and", we include in the graph of the compound inequality that part of the line included in both the first and the second solution sets:

More Complicated Examples II

- Example: Graph the solution set of "$5 - 7x \geq 12$ or $3x - 2 < 7$";
  We first rewrite the compound inequality in a simpler form by solving each component for x:

$$\begin{array}{rcl} -7x \geq 7 & \text{or} & 3x < 9 \\ x \leq -1 & \text{or} & x < 3 \end{array}$$

  We now graph $x \leq -1$; Then we graph $x < 3$; Finally, because of the "or", we include in the graph of the compound inequality that part of the line included in at least one of the two solution sets:



Alternative Notation

- Example: Graph the solution set of $-2 \leq 2x - 3 < 7$;
(**Important Remark**: The meaning of this expression is the same as

$$-2 \leq 2x - 3 \quad \text{and} \quad 2x - 3 < 7$$

Be careful with the and!!)
We first rewrite the compound inequality in a simpler form by solving each component for x:

$$\begin{aligned} -2 \leq 2x - 3 &< 7 \\ 1 \leq 2x &< 10 \\ \tfrac{1}{2} \leq x &< 5. \end{aligned}$$

We now graph $\frac{1}{2} \leq x$; Then we graph $x < 5$; Finally, because of the "and", we include in the graph of the compound inequality that part of the line included in both the first and the second solution sets:

An Application: Figuring Out a Final Exam Score

- Rachel got a 76/100 on her midterm. To get a final B grade in the course the average of her midterm and final exam scores must fall between 80 and 89 inclusive. What possible score over a 100 in the final exam would give Rachel an overall B in the course?

  Let f be her final exam score over 100. The requirement is that the average of her midterm and final exam scores must be between 80 and 89 inclusive:
  $$\begin{gathered} 80 \leq \frac{76+f}{2} \leq 89 \\ 160 \leq 76+f \leq 178 \\ 84 \leq f \leq 102. \end{gathered}$$

  Therefore, Rachel needs at least an 84/100 score in her final exam.

Subsection 6

Absolute Value Equations and Inequalities

Absolute Value Equations

- Recall that, if x is a real number, the **absolute value** $|x|$ of x is the distance of x from 0 on the real line;
- Examples: $|-7| = 7$; $|-3.9| = 3.9$; $|\frac{7}{3}| = \frac{7}{3}$; $|2.1| = 2.1$; $|0| = 0$;
- Example: Solve the following equations:
  - $|x| = 5$;
    Since the distance of x from 0 is 5, we get $x = -5$ or $x = 5$;
  - $|x| = -7$;
    The distance of a number from 0 cannot be negative; Therefore, this equation has no solutions;
  - $|x| = 0$;
    The only number x whose distance from 0 is 0 is $x = 0$;
  - $|x| = 27$;
    $x = -27$ or $x = 27$;

Solving Absolute Value Equations

- In general if $|a| = c > 0$, then we get $a = -c$ or $a = c$;
- Examples: Solve the equations:
  - $|x-7| = 2$;
    Since the distance of $x-7$ from 0 is 2, we have $x-7 = -2$ or $x-7 = 2$; Therefore, $x = 5$ or $x = 9$;
  - $|3x-5| = 7$;
    Since the distance of $3x-5$ from 0 is 7, we have $3x-5 = -7$ or $3x-5 = 7$; Therefore, $3x = -2$ or $3x = 12$ whence $x = -\frac{2}{3}$ or $x = 4$;
  - $|2(x-6)+7| = 0$;
    Since the distance of $2(x-6)+7$ from 0 is 0, we have $2(x-6)+7 = 0$; Therefore, $2x-12+7 = 0$ whence $2x = 5$, i.e., $x = \frac{5}{2}$;
  - $|x-9| = -13$;
    Since the distance cannot be negative, this equation has no solutions;
  - $-5|3x-7|+4 = 14$;
    First, isolate the absolute value: Subtract 4 to get $-5|3x-7| = 10$; Divide by -5 to get $|3x-7| = -2$; Absolute value cannot be negative, so we have no solutions!

Absolute Value on Both Sides

- Suppose that for two numbers a and b, $|a| = |b|$; This means that the distances of a and b from 0 are equal; What are then the possible positions of a and b on the real line? The answer is
$$a = b \quad \text{or} \quad a = -b.$$

- Example: Solve the equation $|2x - 1| = |x + 3|$;
Since $2x - 1$ and $x + 3$ are the same distance from 0, they are either equal or on opposite sides but at equal distance from 0!
$$\begin{gathered} 2x - 1 = x + 3 \quad \text{or} \quad 2x - 1 = -(x + 3); \\ x = 4 \quad \text{or} \quad 2x - 1 = -x - 3; \\ x = 4 \quad \text{or} \quad 3x = -2; \\ x = 4 \quad \text{or} \quad x = -\frac{2}{3}. \end{gathered}$$

Absolute Value Inequalities ($k > 0$)

- $|x| > k$ means that the distance of x from 0 is greater than k; Where could x be possibly located on the line?

$$x < -k \quad \text{or} \quad x > k;$$

- $|x| \geq k$ means the distance of x from 0 is greater than or equal to k; Where could x be possibly located on the line?

$$x \leq -k \quad \text{or} \quad x \geq k;$$

- $|x| < k$ means that the distance of x from 0 is less than k; Where could x be possibly located on the line?

$$-k < x < k;$$

- $|x| \leq k$ means that the distance of x from 0 is less than or equal to k; Where could x be possibly located on the line?

$$-k \leq x \leq k;$$

Solving Absolute Value Inequalities

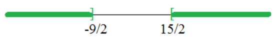
- Solve the following inequalities and graph the solution set:
  - $|x-7|<3$;
    The distance of $x-7$ from 0 is less than 3; Therefore $x-7$ has to be between -3 and 3;
    $$-3<x-7<3$$
    $$4<x<10;$$
    

  - $|3x+5|>2$;
    The distance of $3x+5$ from 0 is greater than 2; Therefore $3x+5$ has to be outside the interval $[-2,2]$;
    $$\begin{array}{ccc} 3x+5<-2 & \text{or} & 3x+5>2 \\ 3x<-7 & \text{or} & 3x>-3 \\ x<-\frac{7}{3} & \text{or} & x>-1 \end{array}$$
    

Two More Examples

- Solve the following inequalities and graph the solution set:
  - $|5-3x| \le 6$;
    The distance of $5-3x$ from 0 is less than or equal to 6; Therefore $5-3x$ has to be between -6 and 6;

$$\begin{aligned} -6 &\le 5-3x \le 6 \\ -11 &\le -3x \le 1 \\ -\tfrac{1}{3} &\le x \le \tfrac{11}{3}; \end{aligned}$$



  - $2|3-2x|-6 \ge 18$;
    First, isolate the absolute value:

$$2|3-2x|-6 \ge 18 \quad \Rightarrow \quad 2|3-2x| \ge 24 \quad \Rightarrow \quad |3-2x| \ge 12.$$

The distance of $3-2x$ from 0 is greater than or equal to 12; Therefore $3-2x$ has to be outside the interval $(-12, 12)$;

$$\begin{aligned} 3-2x \le -12 \quad &\text{or} \quad 3-2x \ge 12 \\ -2x \le -15 \quad &\text{or} \quad -2x \ge 9 \\ x \le -\tfrac{9}{2} \quad &\text{or} \quad x \ge \tfrac{15}{2} \end{aligned}$$