# Intermediate Algebra

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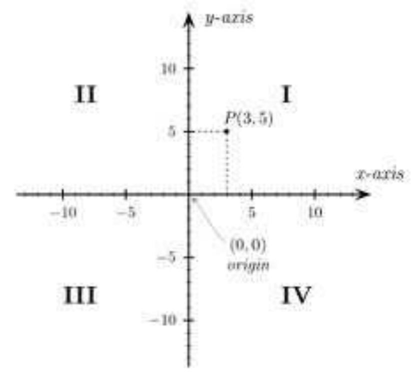
LSSU Math 102

Subsection 1

# Graphing Lines in the Plane

# The Cartesian Coordinate System

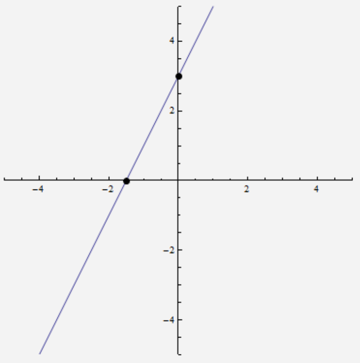
- Basic Features:
  - The origin;
  - *x*- and *y*-axes;
  - The four Quadrants;



# Graphing Lines

- It suffices to plot two points!
- Example: Use the $x$- and $y$-intercepts to draw the line $y = 2x + 3$;

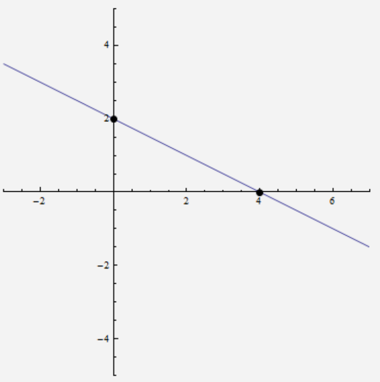
| $x$ | $y$ |
|---|---|
| $0$ | $3$ |
| $-\frac{3}{2}$ | $0$ |



# Another Example

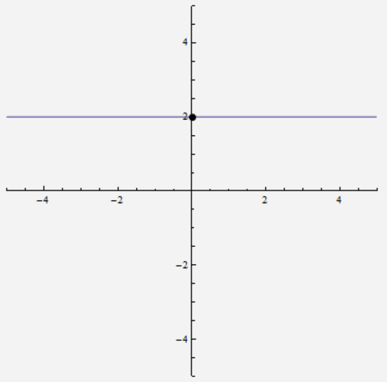
- Use the $x$- and $y$-intercepts to draw the line $x + 2y = 4$;

| $x$ | $y$ |
|---|---|
| 0 | 2 |
| 4 | 0 |

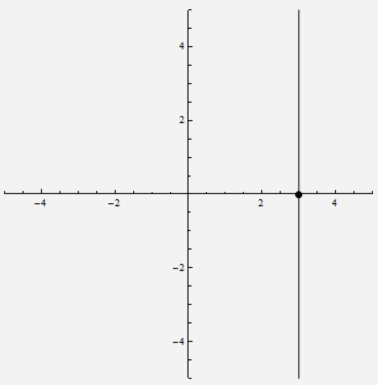


# Graphing Horizontal and Vertical Lines

$y = 2$

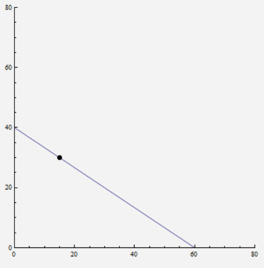



$x = 3$



# An Application

- A store manager is ordering shirts at \$ 20 each and jackets at \$ 30 each. The total cost of the order is going to be \$ 1,200. Write an equation for the total cost and graph it. If she orders 15 shirts how many jackets can she order?
  Suppose $s$ is the number of shirts and $j$ the number of jackets; The cost of the shirts is $20s$ and the cost of the jackets is $30j$; Thus, the total cost is $20s + 30j = 1200$; Solving for $j$, we get $j = -\frac{2}{3}s + 40$; The graph of $j$ versus $s$ is given below;



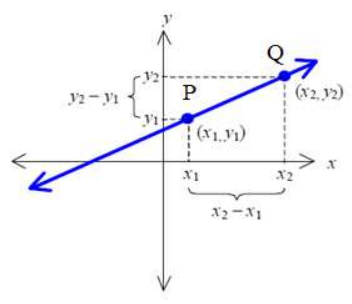

Finally, if she orders 15 shirts, she may order $j = -\frac{2}{3} \cdot 15 + 40 = 30$ jackets;

Subsection 2

## Slope of a Line

# Definition of Slope

- Suppose a straight line passes through two points $P(x_1, y_1)$ and $Q(x_2, y_2)$;




The change $\Delta x = x_2 - x_1$ in the $x$-coordinate is termed the **run** and the change $\Delta y = y_2 - y_1$ in the $y$-coordinate the **rise**;

- Then, the **slope** $m$ of the straight line is defined by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1};$$

- The slope is a measure of how steeply the line rises (if $m > 0$) or descends (if $m < 0$);

# Computing Slopes

- Compute the slope of the line passing through the points $P(2,5)$ and $Q(6,3)$;

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-5}{6-2} = \frac{-2}{4} = -\frac{1}{2};$$

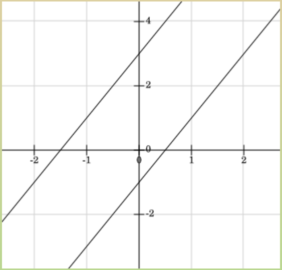
- Compute the slope of the line passing through the points $P(-2,3)$ and $Q(-10,-1)$;

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-3}{-10-(-2)} = \frac{-4}{-8} = \frac{1}{2};$$

- Compute the slope of the horizontal line with $y$-intercept 5;
  m=0
- How about the slope of the vertical line with $x$-intercept $-2$?
  The slope is undefined!

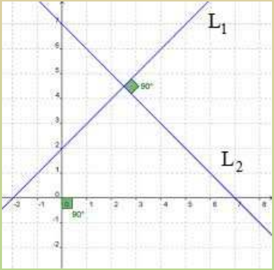
# Parallel and Perpendicular Lines

Two lines $L_1$ and $L_2$ are **parallel**, written $L_1 \parallel L_2$, if they have no points in common;



If $L_1$ has slope $m_1$ and $L_2$ has slope $m_2$, we have $L_1 \parallel L_2$ if and only if $m_1 = m_2$;

Two lines $L_1$ and $L_2$ are **perpendicular**, written $L_1 \perp L_2$, if they intersect at a right (90°) angle;




If $L_1$ has slope $m_1$ and $L_2$ has slope $m_2$, we have $L_1 \perp L_2$ if and only if $m_1 = -\frac{1}{m_2}$;

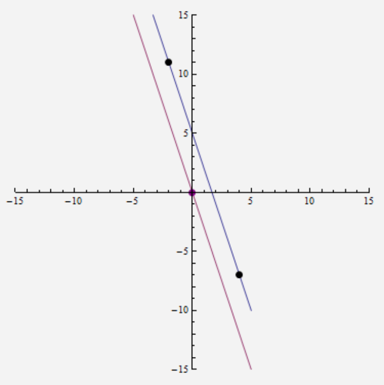
# Example I

Find the slope of the line $l$ that passes through the origin and that is parallel to the line $l'$ passing through the points $(-2, 11)$ and $(4, -7)$.

Line $l'$ has slope

$$\begin{aligned} m' &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-7 - 11}{4 - (-2)} \\ &= \frac{-18}{6} = -3; \end{aligned}$$

Since $l \parallel l'$, we must have $m = m' = -3$;

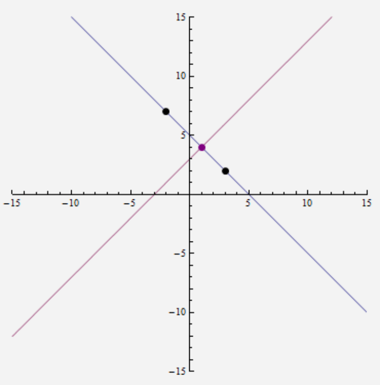


# Example II

Find the slope of the line $l$ that passes through the point $(1,4)$ and that is perpendicular to the line $l'$ passing through the points $(-2,7)$ and $(3,2)$.

Line $l'$ has slope

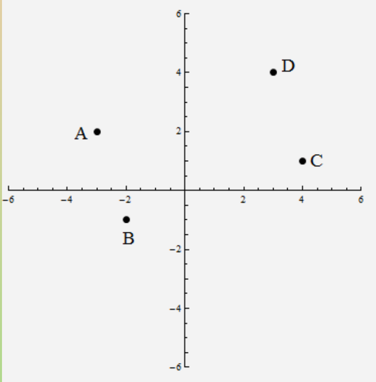
$$\begin{aligned} m' &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{2-7}{3-(-2)} \\ &= \frac{-5}{5} = -1; \end{aligned}$$




Since $l \perp l'$, we must have $m = -\frac{1}{m'} = 1$;

# A More Intricate Example

Determine whether $(-3,2),(-2,-1),(4,1)$ and $(3,4)$ are the vertices of a rectangle;




We compute the slopes of the various line segments: Slopes are as follows:

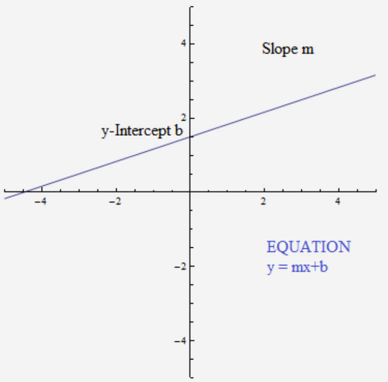
$$\begin{aligned} m_{AB} &= \frac{-1-2}{-2-(-3)} = -3 \\ m_{BC} &= \frac{1-(-1)}{4-(-2)} = \frac{1}{3} \\ m_{CD} &= \frac{4-1}{3-4} = -3 \\ m_{AD} &= \frac{4-2}{3-(-3)} = \frac{1}{3} \end{aligned}$$

Now it is clear that $AB \perp BC \perp CD \perp AD$; Thus the quadrilateral is indeed a rectangle;

Subsection 3

# Three Forms for the Equation of a Line

# The Slope-Intercept Form




The equation of a line with slope $m$ and $y$-intercept $(0, b)$ is

$$\boxed{y = mx + b.}$$

Example: Find an equation for the line passing through the point $(0, -3)$ and having slope 5;

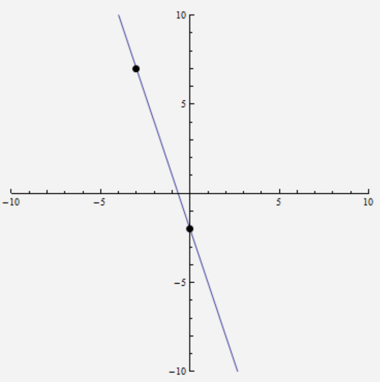
We have $y = mx + b = 5x - 3$;

# Example I

- What is the slope and the $y$-intercept of the line with equation
$$3x - 2y = 5?$$
We solve for $y$ to convert the equation of the line in the slope-intercept form:
$$\begin{array}{rcl} 3x - 2y & = & 5 \\ -2y & = & -3x + 5 \\ y & = & \frac{3}{2}x - \frac{5}{2}; \end{array}$$
Now we compare with $y = mx + b$; We have slope $m = \frac{3}{2}$ and $y$-intercept $b = -\frac{5}{2}$;

# Example II

- Find an equation for the line passing through the points $(-3, 7)$ and $(0, -2)$;




First compute the slope:

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2-7}{0-(-3)} \\ &= -3; \end{aligned}$$

The $y$-intercept is $b = -2$; Therefore the equation is

$$y = -3x - 2;$$

# The Standard Form

- The equation of a line in **standard form** is

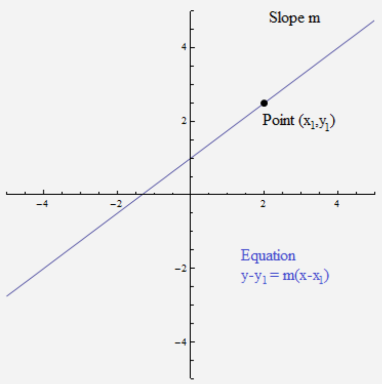
$$\boxed{Ax + By = C,}$$

where $A, B, C$ are reals, with $A$ and $B$ not both zero;

- Example: Write the equation $y = \frac{1}{2}x - \frac{3}{4}$ in the standard form using only *integers* and a *positive coefficient for* $x$;

$$\begin{array}{rcl} y & = & \frac{1}{2}x - \frac{3}{4} \\ 4y & = & 2x - 3 \\ 3 & = & 2x - 4y \\ 2x - 4y & = & 3; \end{array}$$

# The Point-Slope Form




The equation of a line with slope $m$ and passing through a point $(x_1, y_1)$ is

$$\boxed{y - y_1 = m(x - x_1).}$$

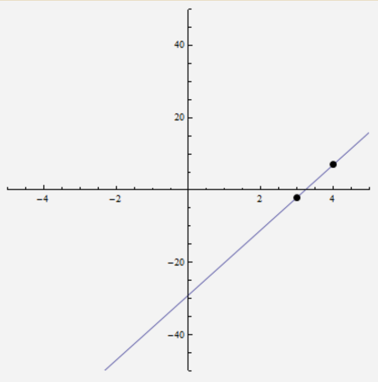
Example: Find an equation for the line passing through the point $(-3, 7)$ and having slope $-2$;
We have $m = -2$ and $(x_1, y_1) = (-3, 7)$; Therefore the equation is $y - 7 = -2(x + 3)$;

# Example I

- Find an equation for the line passing through $(3,-2)$ and $(4,7)$; Then put it in the slope-intercept form.

We first compute the slope:

$$m = \frac{7-(-2)}{4-3} = 9;$$




Now using the point-slope form with $m = 9$ and $(x_1, y_1) = (3,-2)$ we get

$$y + 2 = 9(x - 3);$$

Thus the slope-intercept form is $y = 9x - 29$;

# Example II

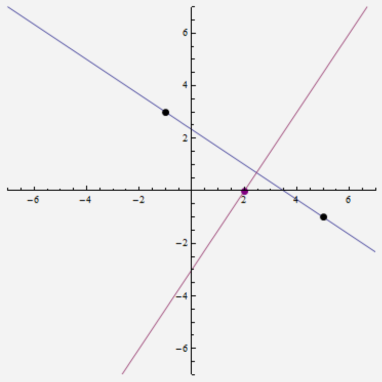
- Find an equation for the line $l$ passing through $(2, 0)$ that is perpendicular to the line $l'$ through $(5, -1)$ and $(-1, 3)$;

We first compute the slope of the second line:

$$m' = \frac{3-(-1)}{-1-5} = -\frac{2}{3};$$

Since $l \perp l'$, we get $m = \frac{3}{2}$; Now using the point-slope form with $m = \frac{3}{2}$ and $(x_1, y_1) = (2, 0)$ we get

$$y = \frac{3}{2}(x-2) = \frac{3}{2}x - 3;$$



# Example III

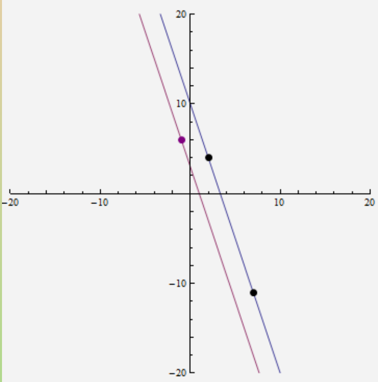
- Find an equation for the line $l$ passing through $(-1, 6)$ that is parallel to the line $l'$ through $(2, 4)$ and $(7, -11)$;

We first compute the slope of $l'$:

$$m' = \frac{-11-4}{7-2} = -3;$$

Since $l \parallel l'$, we get $m = -3$; Now using the point-slope form with $m = -3$ and $(x_1, y_1) = (-1, 6)$ we get

$$y - 6 = -3(x+1);$$



# Example IV

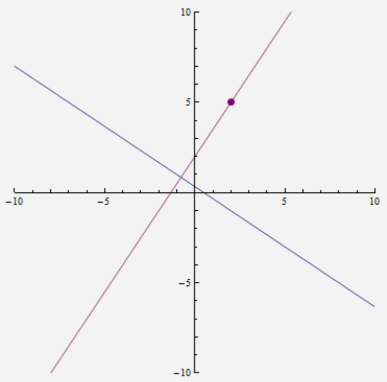
- Find an equation for the line $l$ passing through $(2,5)$ that is perpendicular to the line $l'$ with equation $2x+3y=1$;

We first compute the slope of $l'$:

$$\begin{aligned} 2x+3y &= 1 \\ 3y &= -2x+1 \\ y &= -\tfrac{2}{3}x+\tfrac{1}{3}; \end{aligned}$$

Thus, $m' = -\frac{2}{3}$; Since $l \perp l'$, we get $m=\frac{3}{2}$; Now using the point-slope form with $m=\frac{3}{2}$ and $(x_1,y_1)=(2,5)$ we get

$$y-5=\frac{3}{2}(x-2);$$



# An Application: Fahrenheit versus Celsius

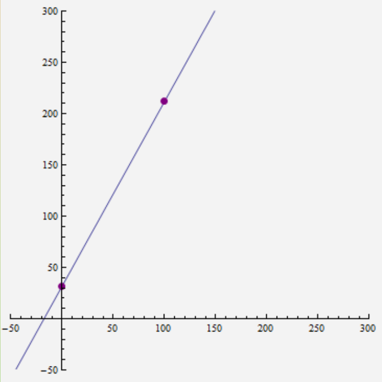
Fahrenheit temperature $F$ is a linear function of Celsius temperature $C$; Water freezes at $0^\circ$C or $32^\circ$F and boils at $100^\circ$C or $212^\circ$F. Find the linear function;

We first compute the slope of the line passing through $(0, 32)$ and $(100, 212)$:

$$\begin{aligned} m &= \frac{212 - 32}{100 - 0} \\ &= \frac{180}{100} = \frac{9}{5}; \end{aligned}$$

Now using the slope-intercept form with $m = \frac{9}{5}$ and $b = 32$ we get

$$F = \frac{9}{5}C + 32;$$



Subsection 4

## Linear Inequalities and their Graphs

# Linear Inequalities and Graphing Strategy

- A **linear inequality in two variables** is of the form

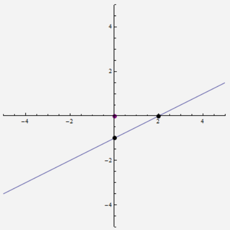
$$Ax + By \leq C,$$

  where $A, B, C$ are real constants, with $A, B$ not both zero, and possibly with $<, \geq$ or $>$ used in place of $\leq$;
- Strategy for Graphing the Solution Set:
  - Graph the line $Ax + By = C$;
  - Pick a point $(a, b)$ not on the line;
  - Plug-in $x = a$ and $y = b$ and check whether the inequality is verified;
    - If yes, the solution set is the half-plane containing $(a, b)$;
    - If not, the solution set is the half-plane not containing $(a, b)$;

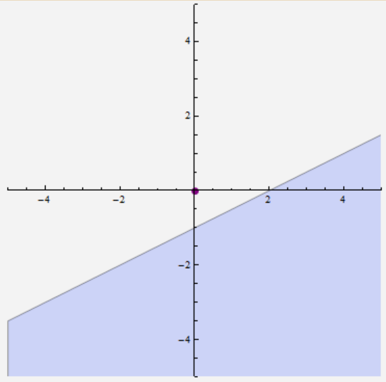
# Example I

Graph the linear inequality $y < \frac{1}{2}x - 1$;

First, graph the line $y = \frac{1}{2}x - 1$;



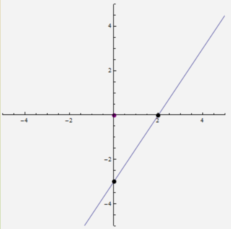

Pick a test point not on the line, say $(0,0)$;
We have $0 \not< \frac{1}{2} \cdot 0 - 1$;
Therefore, the solution set is the half-plane not containing $(0,0)$;



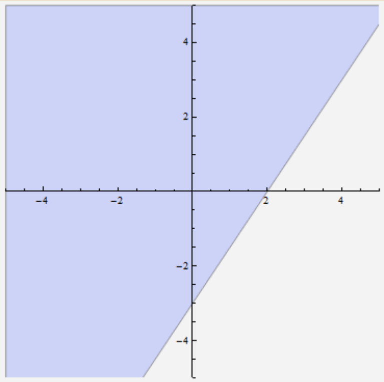
# Example II

Graph the linear inequality $3x - 2y < 6$;

First, graph the line $3x - 2y = 6$;



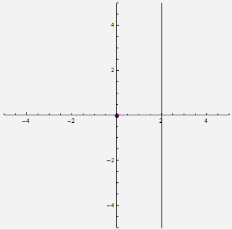

Pick a test point not on the line, say $(0, 0)$;
We have $3 \cdot 0 - 2 \cdot 0 < 6$;
Therefore, the solution set is the half-plane containing $(0, 0)$;



# Example III

Graph the linear inequality $x \leq 2$;

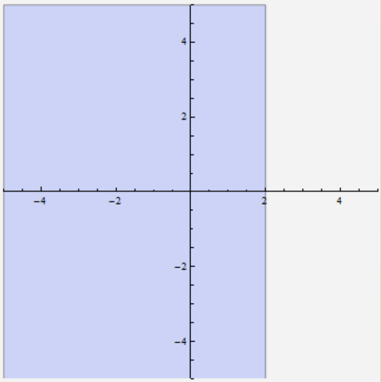
First, graph the line $x = 2$;




Pick a test point not on the line, say $(0,0)$;

We have $0 \leq 2$;

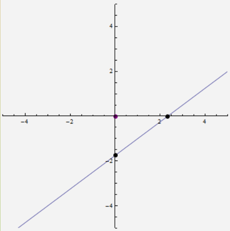
Therefore, the solution set is the half-plane containing $(0,0)$;



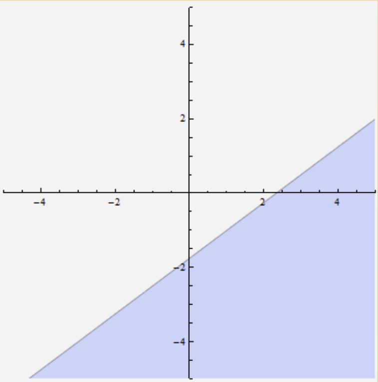
# Example IV

Graph the linear inequality $3x - 4y > 7$;

First, graph the line $3x - 4y = 7$;




Pick a test point not on the line, say $(0,0)$;
We have $3 \cdot 0 - 4 \cdot 0 \not> 7$;
Therefore, the solution set is the half-plane not containing $(0,0)$;

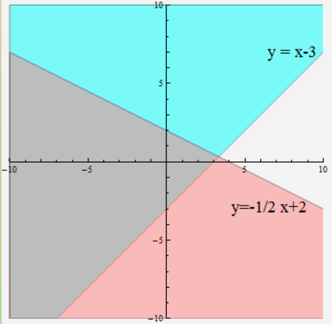


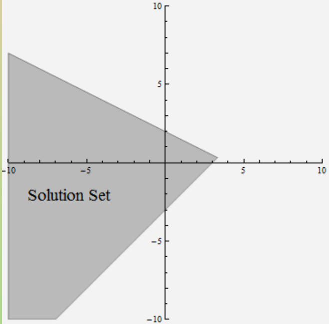
# A Compound Inequality With AND

Graph the solution set of the compound inequality

$$y > x - 3 \quad \text{and} \quad y < -\frac{1}{2}x + 2;$$

We first graph each of the components on the same system of axes:



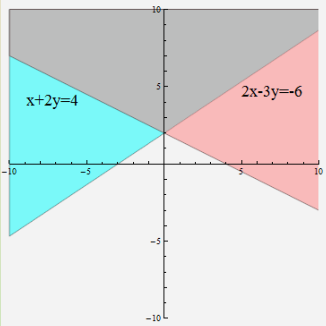

Because of the "and" we take as the solution set the set of all points belonging to both of individual solution sets:

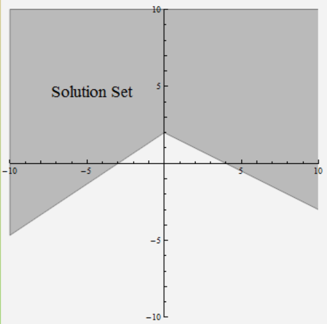
# A Compound Inequality With OR

Graph the solution set of the compound inequality

$$2x - 3y \leq -6 \quad \text{or} \quad x + 2y \geq 4;$$

We first graph each of the components on the same system of axes:



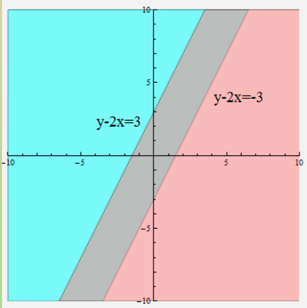

Because of the "or" we take as the solution set the set of all points belonging to at least one of the individual solution sets:

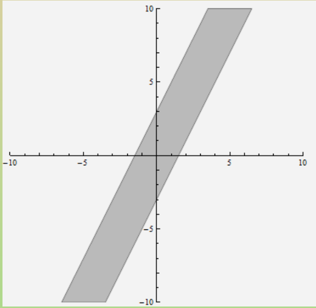
# Absolute Value Inequalities I

Graph the solution set of the absolute value inequality

$$|y - 2x| \leq 3;$$

Recall the given inequality is equivalent to $-3 \leq y - 2x \leq 3$; We first graph each of the components $-3 \leq y - 2x$ and $y - 2x \leq 3$ on the same system of axes:



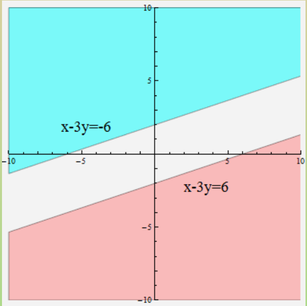

Because of the "and" we take as the solution set the set of all points belonging to both individual solution sets:

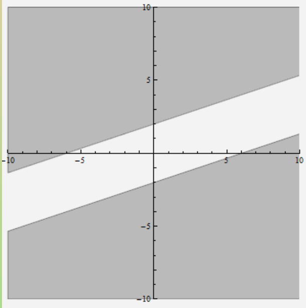
# Absolute Value Inequalities II

Graph the solution set of the absolute value inequality

$$|x - 3y| \geq 6;$$

Recall the given inequality is equivalent to $x - 3y \leq -6$ or $x - 3y \geq 6$; We first graph each of the components $x - 3y \leq -6$ and $x - 3y \geq 6$ on the same system of axes:




Because of the "or" we take as the solution set the set of all points belonging to at least one of the individual solution sets: