Intermediate Algebra

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Subsection 1

Solving by Graphing and by Substitution

Systems of Equations and Solutions

- A **system** of equations is any collection of two or more equations;
- A **solution** of a system of equations in two variables x and y is a pair (a, b) that satisfies all equations in the system;
- The **solution set** is the set of all pairs satisfying the system;
- Example: Is $(2, 5)$ a solution of $\left\{ \begin{array}{rcl} x + y & = & 7 \\ 3x - 5y & = & -19 \end{array} \right\}$;

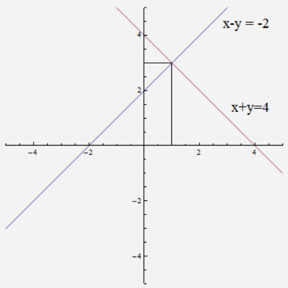
  To check, we must plug-in $x = 2$ and $y = 5$ and verify that both equations of the system are satisfied:

  We have $2 + 5 = 7$, which is true, and $3 \cdot 2 - 5 \cdot 5 = -19$, which is also true!

  Therefore, $(2, 5)$ is a solution of the given system;

Solving a System by Graphing

- The graph of a linear equation in two variables is a line in the plane;
- Thus, a given system of two linear equations in two variables represents a pair of straight lines in the plane;
- A solution of the system is a point that lies on both lines, i.e., it is the point of intersection, if such a point exists;
- The graphical method of solving a system consists of graphing the straight lines and discovering their points of intersection, if any!
- Example: Solve the system $\left\{\begin{array}{rcl} x-y & = & -2 \\ x+y & = & 4 \end{array}\right\}$;
  We graph the two lines: Their point of intersection is $(1,3)$.



Another Example

Solve the system $\left\{ \begin{array}{rcl} 2(y+2) & = & x \\ x-2y & = & 4 \end{array} \right\}$;

We simplify to have a nicer form to graph: $\left\{ \begin{array}{rcl} x-2y & = & 4 \\ x-2y & = & 4 \end{array} \right\}$;

The two equations in the system coincide, i.e., the graphs are identical lines;

Therefore, there are infinitely may intersection points: for any y, $x = 2y+4$;

So the solution set may be expressed as

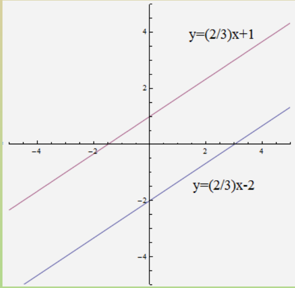
$$(2y+4, y), \ \ y \text{ any real number;}$$

An Inconsistent System

Solve the system $\left\{ \begin{array}{rcl} 2x - 3y & = & 6 \\ 3y - 2x & = & 3 \end{array} \right\}$;

We simplify to have a nicer form to graph: $\left\{ \begin{array}{rcl} y & = & \frac{2}{3}x - 2 \\ y & = & \frac{2}{3}x + 1 \end{array} \right\}$;

The two equations in the system represent two different parallel lines,



i.e., the graphs do not have any points of intersection; Therefore, there are no solutions for the given system!

The Substitution Method

- Suppose we would like to solve the system

$$\left\{ \begin{array}{lcl} y & = & 2x-3 \\ y & = & x+5 \end{array} \right\}$$

- Solve one of the two equations for one of two variables, say the second for y:

$$y = x + 5;$$

- Substitute this value in the other equation:

$$x + 5 = 2x - 3;$$

- Solve this equation for x:

$$x = 8;$$

- Now plug-in into the equation we solved first: $y = x + 5 = 8 + 5 = 13$;
- The system has the unique solution $(x, y) = (8, 13)$;

Another Example

- Suppose we would like to solve the system

$$\left\{ \begin{array}{rcl} 2x+3y & = & 8 \\ y+2x & = & 6 \end{array} \right\}$$

- Solve one of the two equations for one of two variables, say the second for y:

$$y = 6 - 2x;$$

- Substitute this value in the other equation:

$$2x + 3(6 - 2x) = 8;$$

- Solve this equation for x:

$$2x + 18 - 6x = 8 \quad \Rightarrow \quad -4x = -10 \quad \Rightarrow \quad x = \frac{5}{2};$$

- Now plug-in into the equation we solved first: $y = 6 - 2 \cdot \frac{5}{2} = 1$;
- The system has the unique solution $(x, y) = (\frac{5}{2}, 1)$;

A Dependent System (Infinite Solutions)

- Suppose we would like to solve the system

$$\left\{ \begin{array}{rcl} 2x+3y & = & 5+x+4y \\ y & = & x-5 \end{array} \right\}$$

- Solve one of the two equations for one of two variables, say the second for y:

$$y = x - 5;$$

- Substitute this value in the other equation:

$$2x + 3(x-5) = 5 + x + 4(x-5);$$

- Solve this equation for x:

$$2x + 3x - 15 = 5 + x + 4x - 20 \quad \Rightarrow \quad 5x - 15 = 5x - 15;$$

- The system reduces to just the equation $y = x - 5$;
- Thus, it has infinitely many solutions $(x, x-5)$, for any real x;

An Inconsistent System (No Solutions)

- Suppose we would like to solve the system

$$\left\{\begin{array}{rcl} x-2y & = & 3 \\ 2x-4y & = & 7 \end{array}\right\}$$

- Solve one of the two equations for one of two variables, say the first for x:

$$x = 2y + 3;$$

- Substitute this value in the other equation:

$$2(2y+3) - 4y = 7;$$

- Solve this equation for x:

$$4y + 6 - 4y = 7 \quad \Rightarrow \quad 6 = 7;$$

- Thus, we obtained a false equation;
- So this system is inconsistent (has no solutions);

Application: Perimeter of a Rectangle

The length of a rectangular swimming pool is twice its width. If the perimeter is 120 feet, what are its length and its width?

Suppose that l is the length and w is the width;
The length being twice the width may be expressed as $l = 2w$;
The perimeter being 120 feet gives $2l + 2w = 120$;
Therefore, we get

$$\begin{aligned} 2(2w) + 2w &= 120 \\ 4w + 2w &= 120 \\ 6w &= 120 \\ w &= 20; \end{aligned}$$

Hence, the pool has length 40 feet and width 20 feet;

Application: An Investment Problem

Last year Sonya made \$2160 income on a \$20,000 investment. This year she wants to split the \$20,000 between two funds; one is expected to have a 10% return and the other, more risky, a 12% return. If she wants to have again a total return of \$2160, how much should she invest in each fund?

Suppose that \$x is to be invested in the 10% fund and \$y in the 12% fund; Then, since the total to be invested in \$20,000, we must have

$$x + y = 20000;$$

Moreover, the return from the first investment will be $0.1x$ and that from the second $0.12y$; Therefore, since the total return is to be \$2160, we must have

$$0.1x + 0.12y = 2160;$$

Solve the first equation for y:

$$y = 20000 - x;$$

Example (Cont'd)

We obtained:

$$x + y = 20000 \quad \text{and} \quad 0.1x + 0.12y = 2160;$$

Also $y = 20000 - x$;
Substitute into the second equation and solve for x:

$$\begin{aligned} 0.1x + 0.12(20000 - x) &= 2160 \\ 0.1x + 2400 - 0.12x &= 2160 \\ -0.02x &= -240 \\ x &= 12,000 \end{aligned}$$

Thus, Sonya needs to invest \$12,000 in the first fund and \$8,000 in the second fund for a total return of \$2160;

Subsection 2

The Addition Method

The Addition Method

- Apply carefully the following steps:
  1. Rewrite both equations in form $Ax + By = C$;
  2. Multiply one or both equations by the appropriate integer to obtain opposite coefficients on one of the variables;
  3. Add the equations to eliminate one variable;
  4. Solve the equation for the remaining variable;
  5. Substitute the value in one of the original equations and solve for the other variable;
- Example: Solve $\left\{\begin{array}{rcl} 2x & = & y+12 \\ x+3y & = & 13 \end{array}\right\}$

$$\left\{\begin{array}{rcl} 2x-y & = & 12 \\ x+3y & = & 13 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} 6x-3y & = & 36 \\ x+3y & = & 13 \end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{rcl} 7x & = & 49 \\ x+3y & = & 13 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} x & = & 7 \\ x+3y & = & 13 \end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{rcl} x & = & 7 \\ 7+3y & = & 13 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} x & = & 7 \\ y & = & 2 \end{array}\right\}$$

Example I

Solve $\left\{\begin{array}{rcl} 2x-3y & = & -13 \\ 5x-12y & = & -46 \end{array}\right\}$

$$\left\{\begin{array}{rcl} 2x-3y & = & -13 \\ 5x-12y & = & -46 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} -8x+12y & = & 52 \\ 5x-12y & = & -46 \end{array}\right\} \Rightarrow$$
$$\left\{\begin{array}{rcl} -3x & = & 6 \\ 5x-12y & = & -46 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} x & = & -2 \\ 5x-12y & = & -46 \end{array}\right\} \Rightarrow$$
$$\left\{\begin{array}{rcl} x & = & -2 \\ 5\cdot(-2)-12y & = & -46 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} x & = & -2 \\ -12y & = & -36 \end{array}\right\} \Rightarrow$$
$$\left\{\begin{array}{rcl} x & = & -2 \\ y & = & 3 \end{array}\right\}$$

Example II

Solve $\left\{\begin{array}{rcl} -2x+3y & = & 6 \\ 3x-5y & = & -11 \end{array}\right\}$

$$\left\{\begin{array}{rcl} -2x+3y & = & 6 \\ 3x-5y & = & -11 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} -6x+9y & = & 18 \\ 6x-10y & = & -22 \end{array}\right\} \Rightarrow$$
$$\left\{\begin{array}{rcl} -y & = & -4 \\ 6x-10y & = & -22 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} y & = & 4 \\ 6x-10y & = & -22 \end{array}\right\} \Rightarrow$$
$$\left\{\begin{array}{rcl} y & = & 4 \\ 6x-10\cdot 4 & = & -22 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} y & = & 4 \\ 6x & = & 18 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} x & = & 3 \\ y & = & 4 \end{array}\right\}$$

Example Involving Fractions

Solve $\left\{\begin{array}{rcl} \frac{1}{2}x-\frac{2}{3}y & = & 7 \\ \frac{2}{3}x-\frac{3}{4}y & = & 11 \end{array}\right\}$

$$\left\{\begin{array}{rcl} \frac{1}{2}x-\frac{2}{3}y & = & 7 \\ \frac{2}{3}x-\frac{3}{4}y & = & 11 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} 3x-4y & = & 42 \\ 8x-9y & = & 132 \end{array}\right\} \Rightarrow$$
$$\left\{\begin{array}{rcl} -24x+32y & = & -336 \\ 24x-27y & = & 396 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} 5y & = & 60 \\ 3x-4y & = & 42 \end{array}\right\} \Rightarrow$$
$$\left\{\begin{array}{rcl} y & = & 12 \\ 3x-4y & = & 42 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} y & = & 12 \\ 3x-4\cdot 12 & = & 42 \end{array}\right\} \Rightarrow$$
$$\left\{\begin{array}{rcl} y & = & 12 \\ 3x & = & 90 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} x & = & 30 \\ y & = & 12 \end{array}\right\}$$

Example Involving Decimals

Solve $\left\{\begin{array}{rcl} 0.05x+0.7y & = & 40 \\ x+0.4y & = & 120 \end{array}\right\}$

$$\left\{\begin{array}{rcl} 0.05x+0.7y & = & 40 \\ x+0.4y & = & 120 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} 5x+70y & = & 4000 \\ 10x+4y & = & 1200 \end{array}\right\} \Rightarrow$$
$$\left\{\begin{array}{rcl} -10x-140y & = & -8000 \\ 10x+4y & = & 1200 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} -136y & = & -6800 \\ 10x+4y & = & 1200 \end{array}\right\} \Rightarrow$$
$$\left\{\begin{array}{rcl} y & = & 50 \\ 10x+4y & = & 1200 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} y & = & 50 \\ 10x+4\cdot 50 & = & 1200 \end{array}\right\} \Rightarrow$$
$$\left\{\begin{array}{rcl} y & = & 50 \\ 10x & = & 1000 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} x & = & 100 \\ y & = & 50 \end{array}\right\}$$

Application: Flautas and Burritos at "La Posta" de Mesilla

Suppose that at "La Posta" the total price for four flauta dinners and three burrito dinners is \$57 and the total price for three flauta dinners and two burrito dinners is \$41. What is the price of each type of dinner?

Suppose that each flauta dinner costs \$x and each burrito dinner costs \$y; Then, the data yield the following system: $\left\{\begin{array}{rcl} 4x+3y & = & 57 \\ 3x+2y & = & 41 \end{array}\right\}$; We solve this system as before

$$\left\{\begin{array}{rcl} 4x+3y & = & 57 \\ 3x+2y & = & 41 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} -12x-9y & = & -171 \\ 12x+8y & = & 164 \end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{rcl} -y & = & -7 \\ 3x+2y & = & 41 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} y & = & 7 \\ 3x+2y & = & 41 \end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{rcl} y & = & 7 \\ 3x+2\cdot 7 & = & 41 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} y & = & 7 \\ 3x & = & 27 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} x & = & 9 \\ y & = & 7 \end{array}\right\}$$

Thus, each flauta dinner costs \$9 and each burrito dinner costs \$7; Bon Appetit!

Application: Mixing Cooking Oil

Canola oil is 7% saturated fat and corn oil is 14% saturated fat. A blend sold by "Spartan" is 11% saturated fat. How many gallons of each type should be mixed to get 280 gallons of this blend?

Suppose we need to mix x gallons of canola and y gallons of corn oil; Then, the data yield the following system:

$$\left\{\begin{array}{rcl} x+y & = & 280 \\ 0.07x+0.14y & = & 0.11\cdot 280 \end{array}\right\}; \text{ We solve this system as before}$$

$$\left\{\begin{array}{rcl} x+y & = & 280 \\ 0.07x+0.14y & = & 30.8 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} x+y & = & 280 \\ 7x+14y & = & 3080 \end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{rcl} -7x-7y & = & -1960 \\ 7x+14y & = & 3080 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} 7y & = & 1120 \\ x+y & = & 280 \end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{rcl} y & = & 160 \\ x+y & = & 280 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} y & = & 160 \\ x+160 & = & 280 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} x & = & 120 \\ y & = & 160 \end{array}\right\}$$

Thus, we must mix 120 gallons of canola with 160 gallons of corn;

Subsection 3

Systems of Linear Equations in Three Variables

Example I

Solve the system of three linear equations in three unknowns

$$\left\{\begin{array}{rcl} x+y-z & = & -1 \\ 2x-2y+3z & = & 8 \\ 2x-y+2z & = & 9 \end{array}\right\};$$

Take the first two equations and eliminate x:

$$\left\{\begin{array}{rcl} x+y-z & = & -1 \\ 2x-2y+3z & = & 8 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} -2x-2y+2z & = & 2 \\ 2x-2y+3z & = & 8 \end{array}\right\} \Rightarrow$$

$-4y+5z=10$. Take the first and third equations and eliminate x:

$$\left\{\begin{array}{rcl} x+y-z & = & -1 \\ 2x-y+2z & = & 9 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} -2x-2y+2z & = & 2 \\ 2x-y+2z & = & 9 \end{array}\right\} \Rightarrow$$

$-3y+4z=11$. Take the system $\left\{\begin{array}{rcl} -4y+5z & = & 10 \\ -3y+4z & = & 11 \end{array}\right\}$ and eliminate y:

$$\left\{\begin{array}{rcl} -4y+5z & = & 10 \\ -3y+4z & = & 11 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} -12y+15z & = & 30 \\ 12y-16z & = & -44 \end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{rcl} -z & = & -14 \\ -3y+4z & = & 11 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} z & = & 14 \\ y & = & 15 \end{array}\right\}$$ So, $(x,y,z)=(-2,15,14)$;

Example II

Solve the system of three linear equations in three unknowns

$$\left\{\begin{array}{rcl} x+y & = & 4 \\ 2x-3z & = & 14 \\ 2y+z & = & 2 \end{array}\right\};$$

Take the first two equations and eliminate x:

$$\left\{\begin{array}{rcl} x+y & = & 4 \\ 2x-3z & = & 14 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} -2x-2y & = & -8 \\ 2x-3z & = & 14 \end{array}\right\} \Rightarrow -2y-3z=6.$$

Take this with the third equation and eliminate y:

$$\left\{\begin{array}{rcl} 2y+z & = & 2 \\ -2y-3z & = & 6 \end{array}\right\} \Rightarrow -2z=8 \Rightarrow z=-4.$$ So $y=3$ and $x=1$, i.e., $(x,y,z)=(1,3,-4)$;

Application: Tim's Rentals

Tim took in a total of \$1240 from renting three condos. For updates, he had to pay 10% of the rent of the one-bedroom, 20% of the rent of the two-bedroom and 30% of the rent of the three-bedroom. If the three bedroom rents for twice as much as the one-bedroom and the total update bill was \$276, how much does Tim charge for each condo?

Assume x, y and z are the rents for the 1-,2- and 3-bedroom condos;

Then: $\left\{\begin{array}{rcl} x+y+z & = & 1240 \\ 0.1x+0.2y+0.3z & = & 276 \\ z & = & 2x \end{array}\right\}$; Substitute $z=2x$:

$\left\{\begin{array}{rcl} 3x+y & = & 1240 \\ 0.7x+0.2y & = & 276 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} 3x+y & = & 1240 \\ 7x+2y & = & 2760 \end{array}\right\} \Rightarrow$

$\left\{\begin{array}{rcl} -6x-2y & = & -2480 \\ 7x+2y & = & 2760 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} x & = & 280 \\ 3x+y & = & 1240 \end{array}\right\} \Rightarrow$

$\left\{\begin{array}{rcl} x & = & 280 \\ y & = & 400 \end{array}\right\}$ So $(x,y,z)=(280,400,560)$;

Subsection 4

The Matrix Method

Matrices

- A **matrix** is an array of numbers enclosed in brackets;
- The **rows** run horizontally; the **columns** run vertically;
- If a matrix has m rows and n columns, we say that its **size** is $m \times n$;
- Each of the numbers in the matrix is called an **element** or an **entry** of the matrix;
- Example: $\begin{bmatrix} -1 & 2 \\ 5 & \sqrt{2} \\ 0 & 3 \end{bmatrix}$ is a 3×2 matrix; $\begin{bmatrix} -1 & 3 & 6 \end{bmatrix}$ is a 1×3 matrix;
- A matrix with the same number of rows and columns is called a **square matrix**; E.g., $\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}$ is a 2×2 square matrix;

The Augmented Matrix

- Consider a system of linear equation, like, e.g., $\left\{ \begin{array}{rcl} x-2y & = & -5 \\ 3x+y & = & 6 \end{array} \right\}$;
- The matrix $\left[\begin{array}{rr} 1 & -2 \\ 3 & 1 \end{array} \right]$ is called the **matrix of the coefficients**;
- If we also attach the column of the constants, we get the **augmented matrix** of the system: $\left[\begin{array}{rr|r} 1 & -2 & -5 \\ 3 & 1 & 6 \end{array} \right]$;
- Two systems of equations are **equivalent** if they have the same solution sets;
- Two augmented matrices are **equivalent** if they correspond to equivalent systems of linear equations;

Some More Examples

- Write the augmented matrix of the following system:
  - $\left\{\begin{array}{rcl} 3x-5y & = & 11 \\ 7x-2y & = & 21 \end{array}\right\}$; The augmented matrix is $\left[\begin{array}{cc|c} 3 & -5 & 11 \\ 7 & -2 & 21 \end{array}\right]$;
  - $\left\{\begin{array}{rcl} x+y-z & = & 5 \\ 2x+z & = & 3 \\ 2x-y+4z=0 \end{array}\right\}$; The augmented matrix is $\left[\begin{array}{rrr|r} 1 & 1 & -1 & 5 \\ 2 & 0 & 1 & 3 \\ 2 & -1 & 4 & 0 \end{array}\right]$;
- Write the system represented by the following augmented matrix:
  - $\left[\begin{array}{rr|r} 1 & 4 & -2 \\ 1 & -3 & 7 \end{array}\right]$; We get $\left\{\begin{array}{rcl} x+4y & = & -2 \\ x-3y & = & 7 \end{array}\right\}$;
  - $\left[\begin{array}{rrr|r} 2 & 3 & 4 & 6 \\ -1 & 0 & 5 & -2 \\ 1 & -2 & 0 & 1 \end{array}\right]$; We get $\left\{\begin{array}{rcl} 2x+3y+4z & = & 6 \\ -x+5z & = & -2 \\ x-2y & = & 1 \end{array}\right\}$;

The Gauss-Jordan Method

- The following **row operations** on an augmented matrix give an equivalent augmented matrix, i.e., resulting in a system with exactly the same solution set as the original:
  1. Interchange two rows of the matrix;
  2. Multiply every element in a row by a nonzero real number;
  3. Add to a row a multiple of another row;
- Example: We practice with one operation of each kind:
  - Consider $\left[\begin{array}{rrr|r} 2 & 3 & 4 & 6 \\ -1 & 0 & 5 & -2 \\ 1 & -2 & 0 & 1 \end{array}\right]$; Perform $R_1 \leftrightarrow R_3$; The result is $\left[\begin{array}{rrr|r} 1 & -2 & 0 & 1 \\ -1 & 0 & 5 & -2 \\ 2 & 3 & 4 & 6 \end{array}\right]$;
  - Consider $\left[\begin{array}{rr|r} 1 & 0 & 5 \\ 0 & 3 & -6 \end{array}\right]$; Perform $R_2 \leftarrow \frac{1}{3}R_2$; We get $\left[\begin{array}{rr|r} 1 & 0 & 5 \\ 0 & 1 & -2 \end{array}\right]$;
  - Consider $\left[\begin{array}{rr|r} 1 & 1 & 4 \\ 2 & 3 & 6 \end{array}\right]$; Perform $R_2 \leftarrow R_2 - 2R_1$; Get $\left[\begin{array}{rr|r} 1 & 1 & 4 \\ 0 & 1 & -2 \end{array}\right]$;

Gauss-Jordan Elimination for 2×2 Systems

Example: Solve the system $\left\{ \begin{array}{rcl} x - 3y & = & 11 \\ 2x + y & = & 1 \end{array} \right\}$;

$$\left[\begin{array}{rr|r} 1 & -3 & 11 \\ 2 & 1 & 1 \end{array} \right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{rr|r} 1 & -3 & 11 \\ 0 & 7 & -21 \end{array} \right] \xrightarrow{R_3 \leftarrow \frac{1}{7} R_3}$$

$$\left[\begin{array}{rr|r} 1 & -3 & 11 \\ 0 & 1 & -3 \end{array} \right] \xrightarrow{R_1 \leftarrow R_1 + 3R_2} \left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & -3 \end{array} \right]$$

Thus, the system has the solution $(x, y) = (2, -3)$;

Example: Solve the system $\left\{ \begin{array}{rcl} 3x - 2y & = & 4 \\ 2x + y & = & 5 \end{array} \right\}$;

$$\left[\begin{array}{rr|r} 3 & -2 & 4 \\ 2 & 1 & 5 \end{array} \right] \xrightarrow{R_1 \leftarrow \frac{1}{3} R_1} \left[\begin{array}{rr|r} 1 & -\frac{2}{3} & \frac{4}{3} \\ 2 & 1 & 5 \end{array} \right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{rr|r} 1 & -\frac{2}{3} & \frac{4}{3} \\ 0 & \frac{7}{3} & \frac{7}{3} \end{array} \right] \xrightarrow{R_2 \leftarrow \frac{3}{7} R_2}$$

$$\left[\begin{array}{rr|r} 1 & -\frac{2}{3} & \frac{4}{3} \\ 0 & 1 & 1 \end{array} \right] \xrightarrow{R_1 \leftarrow R_1 + \frac{2}{3} R_2} \left[\begin{array}{rr|r} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array} \right]$$

Thus, the system has the solution $(x, y) = (2, 1)$;

Gauss-Jordan Elimination for 3×3 Systems

Example: Solve the system $\left\{ \begin{array}{rcl} 2x - y + z & = & -3 \\ x + y - z & = & 6 \\ 3x - y - z & = & 4 \end{array} \right\}$;

$$\left[\begin{array}{rrr|r} 2 & -1 & 1 & -3 \\ 1 & 1 & -1 & 6 \\ 3 & -1 & -1 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} 1 & 1 & -1 & 6 \\ 2 & -1 & 1 & -3 \\ 3 & -1 & -1 & 4 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 - 3R_1]{R_2 \leftarrow R_2 - 2R_1}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 6 \\ 0 & -3 & 3 & -15 \\ 0 & -4 & 2 & -14 \end{array}\right] \xrightarrow{R_2 \leftarrow -\frac{1}{3}R_2} \left[\begin{array}{rrr|r} 1 & 1 & -1 & 6 \\ 0 & 1 & -1 & 5 \\ 0 & -4 & 2 & -14 \end{array}\right] \xrightarrow[R_3 \leftarrow R_3 + 4R_2]{R_1 \leftarrow R_1 - R_2}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & -2 & 6 \end{array}\right] \xrightarrow{R_3 \leftarrow -\frac{1}{2}R_3} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 1 & -3 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 + R_3}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

Thus, the solution is $(x, y, z) = (1, 2, -3)$;