## Intermediate Algebra

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## LSSU Math 102

## (1) Exponents and Polynomials

- Integral Exponents and Scientific Notation
- The Power Rules
- Polynomials
- Multiplying Binomials
- Factoring Polynomials
- Factoring $a x^{2}+b x+c$
- Factoring Strategy
- Solving Equations by Factoring


## Subsection 1

## Integral Exponents and Scientific Notation

## Positive and Negative Exponents

- Recall that, if $n$ is a positive integer and $a$ is a real number, then

$$
a^{n}=\underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text { factors }}
$$

- If $a \neq 0$, then

$$
a^{-n}=\frac{1}{a^{n}} ;
$$

- The following rules are applicable when manipulating negative exponents:

$$
\begin{array}{ll}
a^{-1}=\frac{1}{a} & \frac{1}{a^{-n}}=a^{n} \\
a^{-n}=\frac{1}{a^{n}} & \left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^{n}
\end{array}
$$

## Some Examples

Evaluate each of the following expressions:

- $3^{-1}=\frac{1}{3}$;
- $\frac{1}{5^{-3}}=5^{3}=125$;
- $3^{-2}=\frac{1}{3^{2}}=\frac{1}{9}$;
- $(-3)^{-2}=\frac{1}{(-3)^{2}}=\frac{1}{9}$;
- $-3^{-2}=-\frac{1}{3^{2}}=-\frac{1}{9}$;
- $\left(\frac{3}{4}\right)^{-3}=\left(\frac{4}{3}\right)^{3}=\frac{64}{27}$;


## Product Rule for Exponents and Zero Exponent

- If $a \neq 0$ and $m, n$ are integers,

$$
a^{m} \cdot a^{n}=a^{m+n} ;
$$

- Examples: Simplify and write answers with positive exponents:
- $3^{4} \cdot 3^{6}=3^{4+6}=3^{10}$;
- $4 x^{-3} \cdot 5 x=4 \cdot 5 x^{-3} x^{1}=20 x^{-3+1}=20 x^{-2}=\frac{20}{x^{2}}$;
- $-2 y^{-3}\left(-5 y^{-4}\right)=(-2)(-5) y^{-3} y^{-4}=10 y^{-7}=\frac{10}{y^{7}}$;
- If $a \neq 0$, then $a^{0}=1$;
- Examples: Simplify and write answers with positive exponents:

$$
\begin{aligned}
& -3^{0}=-1 \\
& \left(\frac{1}{4}-\frac{3}{2}\right)^{0}=1 \\
& -2 a^{5} b^{-6} \cdot 3 a^{-5} b^{2}=(-2) \cdot 3 \cdot a^{5-5} b^{-6+2}=-6 a^{0} b^{-4}= \\
& -6 \cdot 1 \cdot b^{-4}=-\frac{6}{b^{4}}
\end{aligned}
$$

## Changing the Sign of an Exponent

- Write each expression without negative exponents and simplify:
- $\frac{5 a^{-3}}{a^{2} \cdot 2^{-2}}=\frac{5 \cdot 2^{2}}{a^{2} a^{3}}=\frac{5 \cdot 4}{a^{2+3}}=\frac{20}{a^{5}}$;
- $\frac{-2 x^{-3}}{y^{-2} z^{3}}=\frac{-2 y^{2}}{x^{3} z^{3}}$;
- Caution!! Do not flip expressions that are not factors!
- $\frac{2^{-1}+3^{-1}}{4^{-1}} \neq \frac{4}{2+3}!!$

The expression on the right is equal to $\frac{4}{5}$; But the expression on the left is

$$
\frac{2^{-1}+3^{-1}}{4^{-1}}=\frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{4}}=\frac{\frac{5}{6}}{\frac{1}{4}}=\frac{5}{6} \cdot \frac{4}{1}=\frac{10}{3}
$$

and this does not equal $\frac{4}{5}$ !

## Quotient Rule for Exponents

- If $m, n$ are integers and $a \neq 0$,

$$
\frac{a^{m}}{a^{n}}=a^{m-n}
$$

- Examples: Simplify and write answers with positive exponents:
- $\frac{2^{11}}{2^{5}}=2^{11-5}=2^{6}=64$;
$\frac{m^{5}}{m^{-3}}=m^{5-(-3)}=m^{8}$;
$\frac{y^{-4}}{y^{-2}}=y^{-4-(-2)}=y^{-2}=\frac{1}{y^{2}}$
- Examples: Simplify and write answers with positive exponents:

$$
\begin{aligned}
& \frac{2 x^{-7}}{x^{-7}}=2 x^{-7-(-7)}=2 x^{0}=2 \\
& \frac{x\left(10 x^{-4}\right)}{6 x^{-2}}=\frac{10 x^{-3}}{6 x^{-2}}=\frac{5}{3} x^{-1}=\frac{5}{3 x} \\
& \frac{x^{-1} x^{-3} y^{5}}{x^{-2} y^{2}}=\frac{x^{-4} y^{5}}{x^{-2} y^{2}}=x^{-2} y^{3}=\frac{y^{3}}{x^{2}}
\end{aligned}
$$

## Scientific Notation

- A number is in scientific notation if it is written in the form $a \times 10^{n}$, where $1 \leq a<10$ and $n \neq 0$ is an integer;
- Converting to Standard Notation: We use the exponent of 10 to move the decimal point as many places to the left if the exponent is negative or to the right if the exponent is positive;
- Examples: Write each number using standard notation:
- $5.79 \times 10^{4}=57900$;
- $4.92 \times 10^{-3}=0.00492$;
- Converting to Scientific Notation: Count the number of places to move the decimal point so that it follows the first nonzero digit of the number; Use $10^{n}$ if original number is larger than 10 and $10^{-n}$ if original number is smaller than 10;
- Examples: Write each number using scientific notation:
- $354,000,000=3.54 \times 10^{8}$;
- $0.000072=7.2 \times 10^{-5}$;


## Using Scientific Notation and Laws of Exponents

- Evaluate each expression without using a calculator and express your answer in scientific notation;
- $\left(3 \times 10^{4}\right)\left(2.5 \times 10^{-9}\right)=3 \cdot 2.5 \times 10^{4+(-9)}=7.5 \times 10^{-5}$;
- $\frac{7 \times 10^{17}}{2 \times 10^{9}}=3.5 \times 10^{17-9}=3.5 \times 10^{8}$;
- $\frac{(10,000)(0.000025)}{0.000005}=\frac{\left(10^{4}\right)\left(2.5 \times 10^{-5}\right)}{5 \times 10^{-6}}=\frac{2.5 \times 10^{-1}}{5 \times 10^{-6}}=$ $0.5 \times 10^{5}=5 \times 10^{4} ;$


## Subsection 2

## The Power Rules

## Raising an Exponential Expression to a Power

- If $m, n$ are integers and $a \neq 0$,

$$
\left(a^{m}\right)^{n}=a^{m n}
$$

- Examples: Simplify and write the answer with positive exponents:
- $\left(2^{3}\right)^{5}=2^{3.5}=2^{15}$;
- $\left(x^{3}\right)^{-7}=x^{3 \cdot(-7)}=x^{-21}=\frac{1}{x^{21}}$;
- $3\left(x^{-3}\right)^{-7} x^{-9}=3 x^{21} x^{-9}=3 x^{12}$;
- $\frac{\left(x^{2}\right)^{-3}}{\left(x^{-4}\right)^{2}}=\frac{x^{-6}}{x^{-8}}=x^{-6-(-8)}=x^{2}$;


## Raising a Product to a Power

- If $a, b \neq 0$ and $n$ is an integer,

$$
(a b)^{n}=a^{n} \cdot b^{n} ;
$$

- Examples: Simplify and write the answer with positive exponents:
- $(-3 x)^{4}=(-3)^{4} x^{4}=81 x^{4}$;
- $\left(-2 x^{2}\right)^{3}=(-2)^{3}\left(x^{2}\right)^{3}=-8 x^{6}$;
- $\left(3 x^{-2} y^{3}\right)^{-2}=3^{-2}\left(x^{-2}\right)^{-2}\left(y^{3}\right)^{-2}=\frac{1}{9} x^{4} y^{-6}=\frac{x^{4}}{9 y^{6}}$;


## Raising a Quotient to a Power

- If $a, b \neq 0$ and $n$ is an integer,

$$
\left(\frac{a}{b}\right)^{n}=\frac{a^{n}}{b^{n}}
$$

- Examples: Simplify and write the answer with positive exponents:
- $\left(\frac{x}{2}\right)^{3}=\frac{x^{3}}{2^{3}}=\frac{x^{3}}{8}$;
- $\left(-\frac{2 x^{3}}{3 y^{2}}\right)^{3}=\frac{\left(-2 x^{3}\right)^{3}}{\left(3 y^{2}\right)^{3}}=\frac{(-2)^{3}\left(x^{3}\right)^{3}}{3^{3}\left(y^{2}\right)^{3}}=\frac{-8 x^{9}}{27 y^{6}}$;
- $\left(\frac{x^{-2}}{2^{3}}\right)^{-1}=\frac{\left(x^{-2}\right)^{-1}}{8^{-1}}=\frac{x^{2}}{8^{-1}}=8 x^{2}$;
- $\left(-\frac{3}{4 x^{3}}\right)^{-2}=\frac{(-3)^{-2}}{\left(4 x^{3}\right)^{-2}}=\frac{(-3)^{-2}}{4^{-2}\left(x^{3}\right)^{-2}}=\frac{4^{2}\left(x^{3}\right)^{2}}{(-3)^{2}}=\frac{16 x^{6}}{9}$;


## Negative Powers of Fractions and Variable Exponents

- Simplify and write the answer with positive exponents:
- $\left(\frac{3}{4}\right)^{-3}=\left(\frac{4}{3}\right)^{3}=\frac{4^{3}}{3^{3}}=\frac{64}{27}$;
- $\left(\frac{x^{2}}{5}\right)^{-2}=\left(\frac{5}{x^{2}}\right)^{2}=\frac{5^{2}}{\left(x^{2}\right)^{2}}=\frac{25}{x^{4}}$;
- $\left(-\frac{2 x^{3}}{3}\right)^{-2}=\left(-\frac{3}{2 x^{3}}\right)^{2}=\frac{(-3)^{2}}{2^{2}\left(x^{3}\right)^{2}}=\frac{9}{4 x^{6}}$;
- Simplify:
- $3^{4 x} \cdot 3^{5 x}=3^{4 x+5 x}=3^{9 x}$;
- $\left(5^{2 x}\right)^{3 x}=5^{2 x \cdot 3 x}=5^{6 x^{2}}$;
- $\left(\frac{2^{n}}{3^{m}}\right)^{5 n}=\frac{\left(2^{n}\right)^{5 n}}{\left(3^{m}\right)^{5 n}}=\frac{2^{n \cdot 5 n}}{3^{m \cdot 5 n}}=\frac{2^{5 n^{2}}}{3^{5 m n}}$;


## Summarizing the Rules for Exponents

## Rules for Integral Exponents

Suppose $m, n$ are integers and $a, b \neq 0$ reals.
(1) $a^{-n}=\frac{1}{a^{n}}$;
(2) $\frac{1}{a^{-n}}=a^{n}$;
(3) $a^{0}=1$;
(9) $a^{m} a^{n}=a^{m+n}$;
(3) $\frac{a^{m}}{a^{n}}=a^{m-n}$;
(6) $\left(a^{m}\right)^{n}=a^{m n}$;
(1) $(a b)^{n}=a^{n} b^{n}$;
(8) $\left(\frac{a}{b}\right)^{n}=\frac{a^{n}}{b^{n}}$;

## Applications

- The amount $A$ of an investment of $P$ dollars with interest rate $r$ compounded annually for $n$ years is given by

$$
A=P(1+r)^{n} ;
$$

- Example: Suppose an investment of $\$ 1,000$ grows for 50 years at $10 \%$ annually. How much is the final amount?
We compute

$$
A=P(1+r)^{n}=1000(1+0.1)^{50}=1000 \cdot 1.1^{50}(\approx 117,390.85)
$$

- Example: Suppose your father would like to invest today an amount at $5 \%$ annual interest rate so that in 10 years he will have available $\$ 20,000$; How much should be invested now?
Suppose, we need to invest $P$ to get $A=20000$; Then

$$
P=\frac{A}{(1+r)^{n}}=\frac{20000}{1.05^{10}}=20000 \cdot 1.05^{-10}(\approx 12,278.27)
$$

## Subsection 3

## Polynomials

## Polynomials

- A term is a single number or the product of a number and one or more variables raised to powers; e.g., $3 x^{3},-15 x^{2}, 7 x$ and -2 ;
- A polynomial is a single term or a sum of terms in which the powers of the variables are positive integers; e.g., $3 x^{3}-15 x^{2}+7 x-2$; The numbers $3,-15,7$ and -2 are the coefficients of the terms;
- Note that the powers must be positive integers; Thus, e.g., $3 x^{-2}+4 x^{2}$ is not a polynomial;
- When the polynomial is written with the exponents in decreasing order from left to right, the coefficient of the first term, i.e., of the term with the highest exponent, is called the leading coefficient;
- The terms monomial, binomial, trinomial etc. are used for polynomials consisting of one, two, three, etc., terms;
- The degree of a polynomial is the highest power of the variable in the polynomial;
- Examples: $\frac{1}{3} x^{2}-5 x^{3}+7$ has degree 3 and leading coefficient -5 ; $13 x^{37}+5 x^{9}-x^{2}$ has degree 37 and leading coefficient 13 ;


## Evaluating Polynomials and Polynomial Functions

- Find the value of the polynomial $x^{3}-3 x+5$, when $x=2$;

$$
2^{3}-3 \cdot 2+5=8-6+5=7
$$

- If $P(x)=x^{3}-3 x+5$, find $P(5)$;

$$
\begin{aligned}
P(5) & =5^{3}-3 \cdot 5+5 \\
& =125-15+5 \\
& =115 ;
\end{aligned}
$$

## Addition and Subtraction of Polynomials

- The gist is that we are combining like terms, i.e., terms with the same power of the variable(s);
- Examples: Add or subtract:
- $\left(x^{2}-5 x-7\right)+\left(7 x^{2}-4 x+10\right)=8 x^{2}-9 x+3 ;$
- $\left(3 x^{3}-5 x^{2}-7\right)+\left(4 x^{2}-2 x+3\right)=3 x^{3}-x^{2}-2 x-4$;
- $\left(x^{2}-7 x-2\right)-\left(5 x^{2}+6 x-4\right)=x^{2}-7 x-2-5 x^{2}-6 x+4=-4 x^{2}-13 x+2$;
- $\left(6 y^{3} z-5 y z+7\right)-\left(4 y^{2} z-3 y z-9\right)=$ $6 y^{3} z-5 y z+7-4 y^{2} z+3 y z+9=6 y^{3} z-4 y^{2} z-2 y z+16$;


## Multiplication of Polynomials

- First, multiply by a monomial:
- $3 x\left(x^{3}-5\right)=3 x^{4}-15 x$;
- $2 a b^{2} \cdot 3 a^{2} b=6 a^{3} b^{3}$;
- $(-1)(5-x)=-5+x=x-5$;
- $\left(x^{3}-5 x+2\right)(-3 x)=-3 x^{4}+15 x^{2}-6 x$;
- Next use distributivity to multiply two polynomials together:
- $(x+2)\left(x^{2}+3 x-5\right)=(x+2) x^{2}+(x+2) 3 x+(x+2)(-5)=$ $x^{3}+2 x^{2}+3 x^{2}+6 x-5 x-10=x^{3}+5 x^{2}+x-10 ;$
- $(x+y)(z+4)=(x+y) z+(x+y) 4=x z+y z+4 x+4 y$;
- $(x-3)(2 x+5)=(x-3) 2 x+(x-3) 5=2 x^{2}-6 x+5 x-15=2 x^{2}-x-15$;


## Subsection 4

## Multiplying Binomials

## The FOIL Method

- Note that $(a+b)(c+d)=$

First Terms Outer Terms Inner Terms Last Terms

- That is why this method of multiplying is called the FOIL Method;
- Examples: Apply the FOIL method to multiply:
- $(x-3)(x+4)=x^{2}+4 x-3 x-12=x^{2}+x-12$;
- $(2 x-3)(3 x+4)=6 x^{2}+8 x-9 x-12=6 x^{2}-x-12$;
- $\left(2 x^{3}+5\right)\left(2 x^{3}-5\right)=4 x^{6}-10 x^{3}+10 x^{3}-25=4 x^{6}-25$;
- $(a+b)(a-3)=a^{2}-3 a+a b-3 b$;


## Square of a Binomial

- Note that

$$
(a+b)^{2}=(a+b)(a+b)=a^{2}+a b+b a+b^{2}=a^{2}+2 a b+b^{2} ;
$$

- Similarly, $(a-b)^{2}=a^{2}-2 a b+b^{2}$;
- Therefore, to square a binomial, we apply

$$
(a \pm b)^{2}=a^{2} \pm 2 a b+b^{2}
$$

- Examples:
- $(x+5)^{2}=x^{2}+2 \cdot x \cdot 5+5^{2}=x^{2}+10 x+25 ;$
- $(2 w+3)^{2}=(2 w)^{2}+2 \cdot 2 w \cdot 3+3^{2}=4 w^{2}+12 w+9$;
- $\left(2 y^{4}+3\right)^{2}=\left(2 y^{4}\right)^{2}+2 \cdot 2 y^{4} \cdot 3+3^{2}=4 y^{8}+12 y^{4}+9$;
- $(x-6)^{2}=x^{2}-2 \cdot x \cdot 6+6^{2}=x^{2}-12 x+36$;
- $(3 w-5 y)^{2}=(3 w)^{2}-2 \cdot 3 w \cdot 5 y+(5 y)^{2}=9 w^{2}-30 w y+25 y^{2}$;
- $\left(3-5 a^{3}\right)^{2}=3^{2}-2 \cdot 3 \cdot 5 a^{3}+\left(5 a^{3}\right)^{2}=9-30 a^{3}+25 a^{6}$;


## Product of a Sum and a Difference

- Note that $(a+b)(a-b)=a^{2}-a b+a b-b^{2}=a^{2}-b^{2}$;
- So, a rule for computing the product of a sum and a difference is

$$
(a+b)(a-b)=a^{2}-b^{2}
$$

- Examples:

$$
\begin{aligned}
& (x+3)(x-3)=x^{2}-3^{2}=x^{2}-9 ; \\
& \left(a^{3}+8\right)\left(a^{3}-8\right)=\left(a^{3}\right)^{2}-8^{2}=a^{6}-64 ; \\
& \left(3 x^{2}-y^{3}\right)\left(3 x^{2}+y^{3}\right)=\left(3 x^{2}\right)^{2}-\left(y^{3}\right)^{2}=9 x^{4}-y^{6} ; \\
& {[(x+y)+3][(x+y)-3]=(x+y)^{2}-3^{2}=x^{2}+2 x y+y^{2}-9 ;} \\
& {[(m-n)+5]^{2}=(m-n)^{2}+2 \cdot(m-n) \cdot 5+5^{2}=} \\
& m^{2}-2 m n+n^{2}+10 m-10 n+25 ;
\end{aligned}
$$

## Polynomial Functions

Suppose that the width of a rectangular box is $x$ inches. If its length is 2 inches greater than the width and its height is 4 inches greater than the width, write a function $V(x)$ providing the volume of the box in terms of its width $x$;

If the width is $x$ inches, then

$$
\text { length }=x+2 \text { inches and height }=x+4 \text { inches; }
$$

Therefore,

$$
\begin{aligned}
V(x) & =\text { width } \times \text { length } \times \text { height } \\
& =x(x+2)(x+4) \\
& =x\left(x^{2}+4 x+2 x+8\right) \\
& =x\left(x^{2}+6 x+8\right) \\
& =x^{3}+6 x^{2}+8 x \mathrm{in}^{3} .
\end{aligned}
$$

## Subsection 5

## Factoring Polynomials

## Factoring out the Greatest Common Factor

- Consider the terms $8 x^{2} y$ and $20 x y^{3}$; If we factor them completely, we get $2^{3} x^{2} y$ and $2^{2} \cdot 5 x y^{3}$; Thus, the greatest common factor of these terms is $2^{2} x y=4 x y$;
- Consider, similarly, the terms $30 a^{2}, 45 a^{3} b^{2}$ and $75 a^{4} b$; If we factor them completely, we get $2 \cdot 3 \cdot 5 a^{2}, 3^{2} \cdot 5 a^{3} b^{2}$ and $3 \cdot 5^{2} a^{4} b$; Thus, the greatest common factor of these terms is $3 \cdot 5 a^{2}=15 a^{2}$;
- Factor out the GCF:
- $5 x^{4}-10 x^{3}+15 x^{2}=5 x^{2}\left(x^{2}-2 x+3\right)$;
- $8 x y^{2}+20 x^{2} y=4 x y(2 y+5 x)$;
- $60 x^{5}+24 x^{3}+36 x^{2}=12 x^{2}\left(5 x^{3}+2 x+3\right)$;
- Factor out the opposite of the GCF:
- $5 x-5 y=-5(-x+y)$;
- $-x^{2}-3=-\left(x^{2}+3\right)$;
- $-x^{3}+3 x^{2}-5 x=-x\left(x^{2}-3 x+5\right)$;
- Factoring out a common binomial:
- $(x+3) w+(x+3) a=(x+3)(w+a)$;
- $x(x-9)-4(x-9)=(x-4)(x-9)$;


## Factoring by Grouping

- This method of factoring a four-term polynomial calls for the following steps:
(1) Factor the GCF from the first group of two terms;
(2) Factor the GCF from the second group of two terms;
(3) Factor out the common binomial;
- Examples: Factor each four-term polynomial by grouping:
- $2 x+2 y+a x+a y=2(x+y)+a(x+y)=(2+a)(x+y)$;
- $w a-w b+a-b=w(a-b)+(a-b)=(w+1)(a-b)$;
- $4 a m-4 a n-b m+b n=4 a(m-n)-b(m-n)=(4 a-b)(m-n)$;


## Factoring the Difference of Two Squares

- When we see an expression of the form $a^{2}-b^{2}$, i.e., a difference of two squares, we should recall that

$$
a^{2}-b^{2}=(a+b)(a-b)
$$

This will enable us to factor such a difference very quickly!

- Examples: Factor the polynomials:
- $y^{2}-36=y^{2}-6^{2}=(y+6)(y-6)$;
- $9 x^{2}-1=(3 x)^{2}-1^{2}=(3 x+1)(3 x-1)$;
- $4 x^{2}-y^{2}=(2 x)^{2}-y^{2}=(2 x+y)(2 x-y)$;


## Factoring Perfect Square Trinomials

- This factoring method relies on the identities

$$
a^{2} \pm 2 a b+b^{2}=(a \pm b)^{2}
$$

- To recognize the left-hand size note that
- The first and last terms are of the form $a^{2}$ and $b^{2}$;
- The middle term is $\pm 2$ times the product of $a$ and $b$;
- Examples: Factor each polynomial:
- $x^{2}-8 x+16=x^{2}-2 \cdot x \cdot 4+4^{2}=(x-4)^{2}$;
- $a^{2}+14 a+49=a^{2}+2 \cdot a \cdot 7+7^{2}=(a+7)^{2}$;
- $4 x^{2}+12 x+9=(2 x)^{2}+2 \cdot 2 x \cdot 3+3^{2}=(2 x+3)^{2}$;


## Factoring a Polynomial Completely

- A prime polynomial is one that cannot be factored;
- A polynomial is factored completely if it is written as a product of prime polynomials;
- To factor a given polynomial completely, we start by factoring out the GCF and continue factoring until all factors obtained are prime factors;
- Examples: Factor each polynomial completely:

$$
\begin{aligned}
& 5 x^{2}-20=5\left(x^{2}-4\right)=5\left(x^{2}-2^{2}\right)=5(x+2)(x-2) ; \\
& 3 a^{3}-30 a^{2}+75 a=3 a\left(a^{2}-10 a+25\right)=3 a\left(a^{2}-2 \cdot a \cdot 5+5^{2}\right)=3 a(a-5)^{2} ; \\
& -2 b^{3}-16 b^{2}-32 b=-2 b\left(b^{2}+8 b+16\right)= \\
& -2 b\left(b^{2}+2 \cdot b \cdot 4+4^{2}\right)=-2 b(b+4)^{2}
\end{aligned}
$$

## Subsection 6

## Factoring $a x^{2}+b x+c$

## Factoring $a x^{2}+b x+c$, with $a=1$

- To factor $x^{2}+b x+c$
(1) Find integers $m$ and $n$ with sum equal to $b$ and product equal to $c$;
(2) Replace bx by $m x+n x$;
(3) Factor the four-term polynomial by grouping;
- Examples:
- $x^{2}+9 x+18=x^{2}+3 x+6 x+18=x(x+3)+6(x+3)=(x+6)(x+3)$;
- $x^{2}-2 x-24=x^{2}-6 x+4 x-24=x(x-6)+4(x-6)=(x+4)(x-6)$;
- Note that we actually end with the factors $x+m$ and $x+n$, where $m+n=b$ and $m n=c$; So we may actually skip grouping and simply write $(x+m)(x+n)$;
- Examples: Factor each polynomial:
- $x^{2}+4 x+3=(x+1)(x+3)$;
- $x^{2}+3 x-10=(x+5)(x-2)$;
- $a^{2}-5 a+6=(a-3)(a-2)$;


## Factoring $a x^{2}+b x+c$, with $a \neq 1$

- This is done following the steps of the ac Method:
(1) Find two integers $m$ and $n$ with sum $b$ and product $a c$;
(2) Replace $b x$ by $m x+n x$;
(3) Factor the four-term polynomial by grouping;
- Examples: Factor each trinomial:
- $2 x^{2}+9 x+4=2 x^{2}+x+8 x+4=x(2 x+1)+4(2 x+1)=(x+4)(2 x+1)$;
- $2 x^{2}+5 x-12=2 x^{2}+8 x-3 x-12=2 x(x+4)-3(x+4)=(2 x-3)(x+4)$;
- $6 w^{2}-w-15=6 w^{2}+9 w-10 w-15=3 w(2 w+3)-5(2 w+3)=$ $(3 w-5)(2 w+3)$;


## Using Trial and Error I

- Examples: Factor the following quadratic polynomial by trial and error:
- $2 x^{2}+5 x-3$

Note that $2 x^{2}$ factors only as $2 x \cdot x$; Moreover, 3 factors only as $1 \cdot 3$; Thus, the only available possibilities are

$$
\left(\begin{array}{ll}
2 x & 1
\end{array}\right)\left(\begin{array}{ll}
x & 3
\end{array}\right) \text { and }\left(\begin{array}{lll}
2 x & 3
\end{array}\right)\left(\begin{array}{ll}
x & 1
\end{array}\right)
$$

The fact that the constant term is negative implies that one of the missing signs must be + and the other -; Therefore, by slowly experimenting with the available possibilities, we obtain the correct form:

$$
\begin{aligned}
& (2 x+1)(x-3)=2 x^{2}-5 x-3 \\
& (2 x+3)(x-1)=2 x^{2}+x-3 \\
& (2 x-1)(x+3)=2 x^{2}+5 x-3
\end{aligned}
$$

## Using Trial and Error II

- Examples: Factor the following quadratic polynomial by trial and error:
- $3 x^{2}-11 x+6$

Note that $3 x^{2}$ factors only as $3 x \cdot x$; Moreover, 6 factors only either as $1 \cdot 6$ or as $2 \cdot 3$; Thus, there are four possibilities:
$\left(\begin{array}{ll}3 x & 1\end{array}\right)\left(\begin{array}{ll}x & 6\end{array}\right)$
$\left(\begin{array}{ll}3 x & 2\end{array}\right)(x$
3)
$\left(\begin{array}{ll}3 x & 6\end{array}\right)\left(\begin{array}{ll}x & 1\end{array}\right)$
$\left(\begin{array}{ll}3 x & 3\end{array}\right)\left(\begin{array}{ll}x & 2\end{array}\right)$

The fact that the last term is positive and the middle negative implies that both missing signs must be -; By trial and error we arrive at the correct form:

$$
(3 x-2)(x-3)=3 x^{2}-11 x+6
$$

## Factoring by Substitution

- Examples: Factor each polynomial:

$$
\begin{aligned}
& \text { - } x^{4}-9=\left(x^{2}\right)^{2}-9 \overbrace{=}^{y=x^{2}} y^{2}-9=(y+3)(y-3)=\left(x^{2}+3\right)\left(x^{2}-3\right) ; \\
& \text { - } y^{8}-14 y^{4}+49=\left(y^{4}\right)^{2}-14 y^{4}+49 \overbrace{=}^{4} z^{2}-14 z+49= \\
& z^{2}-2 \cdot z \cdot 7+7^{2}=(z-7)^{2}=\left(y^{4}-7\right)^{2} ; \\
& \text { - } x^{2 m}-y^{2}=\left(x^{m}\right)^{2}-y^{2} \overbrace{=x^{m}}^{=} z^{2}-y^{2}=(z+y)(z-y)=\left(x^{m}+y\right)\left(x^{m}-y\right) ; \\
& \text { - } z^{2 n+1}-6 z^{n+1}+9 z=z\left(z^{2 n}-6 z^{n}+9\right)=z\left[\left(z^{n}\right)^{2}-6 z^{n}+9\right] \overbrace{=}^{n} z\left(w^{2}-\right. \\
& 6 w+9)=z\left(w^{2}-2 \cdot w \cdot 3+3^{2}\right)=z(w-3)^{2}=z\left(z^{n}-3\right)^{2} ;
\end{aligned}
$$

## More Examples on Substitution

- Examples: Factor completely each polynomial:

$$
\begin{aligned}
& -x^{8}-2 x^{4}-15=\left(x^{4}\right)^{2}-2 x^{4}-15 \overbrace{=}^{y=x^{4}} y^{2}-2 y-15= \\
& (y-5)(y+3)=\left(x^{4}-5\right)\left(x^{4}+3\right) ; \\
& -18 y^{7}+21 y^{4}+15 y=-3 y\left(6 y^{6}-7 y^{3}-5\right)= \\
& -3 y\left[6\left(y^{3}\right)^{2}-7 y^{3}-5\right] \overbrace{=}^{z=y^{3}}-3 y\left(6 z^{2}-7 z-5\right)= \\
& -3 y\left(6 z^{2}+3 z-10 z-5\right)=-3 y[3 z(2 z+1)-5(2 z+1)]= \\
& -3 y(3 z-5)(2 z+1)=-3 y\left(3 y^{3}-5\right)\left(2 y^{3}+1\right) ; \\
& \\
& 2 u^{2 m}-5 u^{m}-3=2\left(u^{m}\right)^{2}-5 u^{m}-3 \overbrace{=}^{z=u^{m}} 2 z^{2}-5 z-3=2 z^{2}-6 z+z-3= \\
& 2 z(z-3)+(z-3)=(2 z+1)(z-3)=\left(2 u^{m}+1\right)\left(u^{m}-3\right) ;
\end{aligned}
$$

## Subsection 7

## Factoring Strategy

## Prime Polynomials

- Recall that a prime polynomial is one that cannot be factored;
- Example: Is the polynomial $x^{2}+3 x+4$ prime?

According to our method for factoring trinomials of the form $x^{2}+b x+c$, we must find two integers $m$ and $n$, with sum $b=3$ and product $c=4$; The only possibilities for product 4 are the pairs $(1,4),(-1,-4),(2,2)$ and ( $-2,-2$ ); The corresponding sums, however, are $5,-5,4$ and -4 ; Thus, none of them works out as the required pair; This shows that $x^{2}+3 x+4$ cannot be factored, i.e., it is a prime polynomial;

- In general, even though the difference of two squares $a^{2}-b^{2}=(a+b)(a-b)$ is not prime, the sum of two squares $a^{2}+b^{2}$ is prime, unless $a^{2}$ and $b^{2}$ have a common factor as, e.g., in $4 x^{2}+16=4\left(x^{2}+4\right) ;$


## Factoring a Difference/Sum of Two Cubes

- Neither the difference nor the sum of two cubes $a^{3} \pm b^{3}$ is a prime binomial; In fact, we have

$$
\begin{aligned}
a^{3}-b^{3} & =(a-b)\left(a^{2}+a b+b^{2}\right) \\
a^{3}+b^{3} & =(a+b)\left(a^{2}-a b+b^{2}\right)
\end{aligned}
$$

- Examples: Factor each polynomial:

$$
\begin{aligned}
& x^{3}-8=x^{3}-2^{3}=(x-2)\left(x^{2}+x \cdot 2+2^{2}\right)=(x-2)\left(x^{2}+2 x+4\right) \\
& y^{3}+1=y^{3}+1^{3}=(y+1)\left(y^{2}-y \cdot 1+1^{2}\right)=(y+1)\left(y^{2}-y+1\right) \\
& 8 x^{3}-27=(2 x)^{3}-3^{3}=(2 x-3)\left((2 x)^{2}+(2 x) \cdot 3+3^{2}\right)= \\
& (2 x-3)\left(4 x^{2}+6 x+9\right) ;
\end{aligned}
$$

## Factoring Polynomials Completely

- Examples: Factor completely:

$$
\begin{aligned}
& x^{4}+x^{2}-2=\left(x^{2}\right)^{2}+x^{2}-2 \overbrace{=}^{y=x^{2}} y^{2}+y-2=(y+2)(y-1)= \\
& \left(x^{2}+2\right)\left(x^{2}-1\right)=\left(x^{2}+2\right)\left(x^{2}-1^{2}\right)=\left(x^{2}+2\right)(x+1)(x-1) ; \\
& 3 x^{6}-3=3\left(x^{6}-1\right)=3\left(\left(x^{3}\right)^{2}-1^{2}\right) \overbrace{=}^{y=x^{3}}=3\left(y^{2}-1^{2}\right)= \\
& 3(y+1)(y-1)=3\left(x^{3}+1\right)\left(x^{3}-1\right)=3\left(x^{3}+1^{3}\right)\left(x^{3}-1^{3}\right)= \\
& 3(x+1)\left(x^{2}-x+1\right)(x-1)\left(x^{2}+x+1\right) ; \\
& \left(y^{2}-1\right)^{2}-11\left(y^{2}-1\right)+24 \overbrace{z=y^{2}-1}^{=} z^{2}-11 z+24=(z-3)(z-8)= \\
& \left(y^{2}-1-3\right)\left(y^{2}-1-8\right)=\left(y^{2}-4\right)\left(y^{2}-9\right)=\left(y^{2}-2^{2}\right)\left(y^{2}-3^{2}\right)= \\
& (y+2)(y-2)(y+3)(y-3) ; \\
& x^{2}-3 w-3 x+x w=x^{2}+x w-3 x-3 w=x(x+w)-3(x+w)= \\
& (x-3)(x+w) ; \\
& x^{2}-6 x+9-y^{2}=\left(x^{2}-2 \cdot x \cdot 3+3^{2}\right)-y^{2}=(x-3)^{2}-y^{2}= \\
& ((x-3)+y)((x-3)-y)=(x+y-3)(x-y-3) ;
\end{aligned}
$$

## Strategy for Factoring Polynomials

## Steps for Factoring Polynomials

(1) First, factor out the greatest common factor of the terms;
(2) For binomials look out for special cases:

- difference of two squares;
- difference/sum of two cubes;
(3) For trinomials check for perfect squares;
(3) For trinomials that are not perfect squares, use grouping or trial and error;
(3) For polynomials of high degrees, use substitution to get a polynomial of degree 2 or 3 and revert to one of the previous steps;
(3) For tetranomials try factoring by grouping;


## Few More Factoring Examples

- $3 x^{3}-3 x^{2}-18 x=3 x\left(x^{2}-x-6\right)=3 x(x-3)(x+2)$;
- $10 x^{2}-160=10\left(x^{2}-16\right)=10\left(x^{2}-4^{2}\right)=10(x+4)(x-4)$;
- $16 a^{2} b-80 a b+100 b=4 b\left(4 a^{2}-20 a+25\right)=$ $4 b\left[(2 a)^{2}-2 \cdot(2 a) \cdot 5+5^{2}\right]=4 b(2 a-5)^{2}$;
- $a w+m w+a z+m z=(a+m) w+(a+m) z=(a+m)(w+z)$;
- $a^{4} b+125 a b=a b\left(a^{3}+5^{3}\right)=a b(a+5)\left(a^{2}-a \cdot 5+5^{2}\right)=$ $a b(a+5)\left(a^{2}-5 a+25\right)$;
- $12 x^{2} y-26 x y-30 y=2 y\left(6 x^{2}-13 x-15\right)=2 y\left(6 x^{2}-18 x+5 x-15\right)=$ $2 y[6 x(x-3)+5(x-3)]=2 y(6 x+5)(x-3)$;


## Subsection 8

## Solving Equations by Factoring

## The Zero Factor Property

- To solve an equation of the form $a b=0$, we may reduce it to the compound equation

$$
a=0 \quad \text { or } \quad b=0
$$

- Example: Solve the equation $x^{2}+x-12=0$; We start by factoring the nonzero side

$$
(x+4)(x-3)=0
$$

Now, we apply the zero-factor property:

$$
x+4=0 \quad \text { or } \quad x-3=0
$$

These are easier to solve:

$$
x=-4 \text { or } x=3
$$

## Solving Quadratic Equations

- If $a, b, c$ are reals, with $a \neq 0$, then

$$
a x^{2}+b x+c=0
$$

is called a quadratic equation;

- To solve a quadratic equation, we follow the steps:
(1) Write the equation with 0 on one side;
(2) Factor the nonzero side completely;
(3) Use the zero factor property to get simpler equations;
(3) Solve the simpler equations;
- Example: Solve $10 x^{2}=5 x$;

$$
\begin{aligned}
& 10 x^{2}=5 x \Rightarrow 10 x^{2}-5 x=0 \Rightarrow 5 x(2 x-1)=0 \\
& \Rightarrow(5 x=0 \quad \text { or } \quad 2 x-1=0) \Rightarrow\left(x=0 \quad \text { or } \quad x=\frac{1}{2}\right)
\end{aligned}
$$

## More Examples

- Solve $3 x-6 x^{2}=-9$;

$$
\begin{aligned}
& 3 x-6 x^{2}=-9 \Rightarrow 6 x^{2}-3 x-9=0 \Rightarrow 3\left(2 x^{2}-x-3\right)=0 \\
& \Rightarrow 3(x+1)(2 x-3)=0 \Rightarrow(x+1=0 \quad \text { or } 2 x-3=0) \\
& \Rightarrow\left(x=-1 \quad \text { or } \quad x=\frac{3}{2}\right)
\end{aligned}
$$

- Solve $(x-4)(x+1)=14$;

$$
\begin{aligned}
& (x-4)(x+1)=14 \Rightarrow x^{2}-3 x-4=14 \Rightarrow x^{2}-3 x-18=0 \\
& \Rightarrow(x-6)(x+3)=0 \Rightarrow(x-6=0 \text { or } x+3=0) \\
& \Rightarrow(x=6 \text { or } x=-3)
\end{aligned}
$$

## A Cubic Equation

- Solve $2 x^{3}-3 x^{2}-8 x+12=0$;

$$
\begin{aligned}
& 2 x^{3}-3 x^{2}-8 x+12=0 \\
\Rightarrow & x^{2}(2 x-3)-4(2 x-3)=0 \\
\Rightarrow & \left(x^{2}-4\right)(2 x-3)=0 \\
\Rightarrow & (x+2)(x-2)(2 x-3)=0 \\
\Rightarrow & (x+2=0 \text { or } x-2=0 \text { or } 2 x-3=0) \\
\Rightarrow & \left(x=-2 \text { or } x=2 \text { or } x=\frac{3}{2}\right)
\end{aligned}
$$

## An Absolute Value Equation

- Solve $\left|x^{2}-2 x-16\right|=8$;

$$
\begin{aligned}
& \left|x^{2}-2 x-16\right|=8 \\
\Rightarrow & x^{2}-2 x-16=-8 \text { or } x^{2}-2 x-16=8 \\
\Rightarrow & x^{2}-2 x-8=0 \text { or } x^{2}-2 x-24=0 \\
\Rightarrow & (x+2)(x-4)=0 \text { or }(x+4)(x-6)=0 \\
\Rightarrow & x+2=0 \text { or } x-4=0 \text { or } x+4=0 \text { or } x-6=0 \\
\Rightarrow & x=-2 \text { or } x=4 \text { or } x=-4 \text { or } x=6
\end{aligned}
$$

## Application: Area of Dining Room

- Isabel's dining room is 2 feet longer than it is wide and has area 168 $\mathrm{ft}^{2}$; What are its dimensions?

- We must have $x(x+2)=168$;

Thus, $x^{2}+2 x-168=0$;
This yields $(x+14)(x-12)=0$;
Therefore $x+14=0$ or $x-12=0$, whence $x=-14$ or $x=12$;
Therefore, Isabel's dining room is $14 \mathrm{ft} . \times 12 \mathrm{ft}$.

## Application: The Pythagorean Theorem

- Francisco used 14 meters of fencing to enclose a rectangular piece of land; Each of the diagonals of the region is 5 meters; What are the dimensions of the rectangular plot?

- Suppose the land has length $x$ and width $y$; The statement suggests that $2 x+2 y=14$ and that $x^{2}+y^{2}=5^{2}$;
Solving the first for $y$ we get $y=7-x$;
Therefore, $x^{2}+(7-x)^{2}=25$;
This yields $x^{2}+49-14 x+x^{2}=25$, i.e., $2 x^{2}-14 x+24=0$;
Divide both sides by $2 x^{2}-7 x+12=0$; factor $(x-3)(x-4)=0$;
use zero -factor property $x-3=0$ or $x-4=0$
and find the solutions $x=3$ or $x=4$;
Therefore, the plot is 4 meters $\times 3$ meters.

