Intermediate Algebra

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Subsection 1

Properties of Rational Expressions

Rational Expressions

- A **rational expression** is the quotient of two polynomials with the denominator not equal to zero;
- Examples:
$$\frac{2}{3}, \quad 3x-7, \quad \frac{x-3}{5x^2-4}, \quad \frac{x-3}{x+5};$$
- The **domain** of an expression involving a variable is the set of all real numbers that can be substituted in for the variable;
- Examples: Find the domain of each rational expression:
  - $\frac{x-2}{x+9}$; We must have $x+9 \neq 0$; We solve $x+9=0$ to get $x=-9$; Thus the domain is all reals except -9; This can be written in set notation $\mathbb{R}-\{-9\}$ and in interval notation $(-\infty,-9)\cup(-9,\infty)$;
  - $\frac{x-3}{2x^2-2}$; We must have $2x^2-2 \neq 0$; We solve $2x^2-2=0$; factor to get $2(x^2-1)=0$, whence $2(x+1)(x-1)=0$; The solutions are $x=-1$ or $x=1$; Thus the domain is all reals except ± 1; This can be written in set notation $\mathbb{R}-\{-1,1\}$ or $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$;

Reducing to Lowest Terms

- Reducing a rational number **to lowest terms** means canceling out all common factors occurring in numerator and denominator;
- For example, rewriting $\frac{10}{14} = \frac{2 \cdot 5}{2 \cdot 7} = \frac{5}{7}$ reduces $\frac{10}{14}$ to lowest terms;
- Reduction to lowest terms is based on

Basic Principle of Rational Numbers

If $\frac{a}{b}$ is a rational number and $c \neq 0$ a real, then $\dfrac{a \cdot c}{b \cdot c} = \dfrac{a}{b}$.

- Examples: Reduce each rational expression to lowest terms:
  - $\dfrac{18}{42} = \dfrac{2 \cdot 3^2}{2 \cdot 3 \cdot 7} = \dfrac{3}{7}$;
  - $\dfrac{-2a^7 b}{a^2 b^3} = \dfrac{-2a^5}{b^2}$;

Reduction Strategy

Strategy for Reducing Rational Expressions

1. Factor completely top and bottom and look for common factors;
2. Divide out (simplify) common factors;
3. The quotient rule for exponents $\frac{a^m}{a^n} = a^{m-n}$ is used to divide out common factors;
4. Sometimes, a negative sign has to be used with the greatest common factor to obtain identical factors;

- Examples: Reduce to lowest terms:
  - $\dfrac{2x^2-18}{x^2+x-6} = \dfrac{2(x^2-9)}{(x+3)(x-2)} = \dfrac{2(x+3)(x-3)}{(x+3)(x-2)} = \dfrac{2(x-3)}{x-2}$;
  - $\dfrac{w-2}{2-w} = \dfrac{-(2-w)}{2-w} = -1$;
  - $\dfrac{2a^3-16}{16-4a^2} = \dfrac{2(a^3-8)}{-4(a^2-4)} = \dfrac{2(a-2)(a^2+2a+4)}{-4(a+2)(a-2)} = -\dfrac{a^2+2a+4}{2(a+2)}$;

Building Up the Denominator

- To add or subtract fractions we need to take common denominators;
- To convert a fraction to an equivalent one with a given denominator, we use the reverse process of that of reducing a fraction to lowest terms;
- We call this process **building up the denominator**;
- Examples: Convert into an equivalent rational expression with the indicated denominator:
  - $\frac{2}{7} = \frac{?}{42}$
    $\frac{2}{7} = \frac{2 \cdot 6}{7 \cdot 6} = \frac{12}{42}$;
  - $\frac{5}{3a^2b} = \frac{?}{9a^3b^4}$
    $\frac{5}{3a^2b} = \frac{5 \cdot 3ab^3}{3a^2b \cdot 3ab^3} = \frac{15ab^3}{9a^3b^4}$;

Building Up the Denominator: More Examples

- Convert each rational expression into an equivalent one with the indicated denominator:
  - $\frac{5}{2a-2b} = \frac{?}{6b-6a}$

    Notice that $6b - 6a = -3(2a-2b)$;
    $\frac{5}{2a-2b} = \frac{(-3)\cdot 5}{(-3)(2a-2b)} = \frac{-15}{6b-6a}$;
  - $\frac{x+2}{x+3} = \frac{?}{x^2+7x+12}$

    Notice that $x^2+7x+12 = (x+4)(x+3)$;
    $\frac{x+2}{x+3} = \frac{(x+4)(x+2)}{(x+4)(x+3)} = \frac{x^2+6x+8}{x^2+7x+12}$;

Rational Functions

- If y is given as a rational expression in terms of x, as, for example, in $y = \dfrac{3x-1}{x^2-4}$, we say that y is a **rational function** of x;
- The **domain** of a rational function is the same as that of the rational expression used to define the function;
- Example: Find the domain of $R(x) = \dfrac{3x-1}{x^2-4}$; Then, compute $R(3)$ and $R(2)$;

  We set $x^2-4=0$; Factor $(x+2)(x-2)=0$; Use zero-factor property to solve for x: $x=-2$ or $x=2$; Thus, the domain of R is $\mathbb{R}-\{-2,2\}$;
  $$R(3) = \frac{3\cdot 3-1}{3^2-4} = \frac{8}{5};$$

  On the other hand $R(2)$ is undefined, since 2 is not in the domain of $R(x)$;

Subsection 2

Multiplication and Division

Multiplying Rational Expressions

Multiplication Rule

If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$.

- Examples: Compute the products:
  - $\frac{3a^8b^3}{6b} \cdot \frac{10a}{a^2b^6} = \frac{2 \cdot 3 \cdot 5a^9b^3}{2 \cdot 3a^2b^7} = \frac{5a^7}{b^4}$;
  - $\frac{x^2+7x+12}{x^2+3x} \cdot \frac{x^2}{x^2-16} = \frac{(x+3)(x+4)x^2}{x(x+3)(x+4)(x-4)} = \frac{x}{x-4}$;
  - $(a^2-1) \cdot \frac{6}{2a^2+4a+2} = \frac{(a+1)(a-1)6}{2(a+1)^2} = \frac{3(a-1)}{a+1}$;
  - $\frac{a^3-b^3}{b-a} \cdot \frac{6}{2a^2+2ab+2b^2} = \frac{(a-b)(a^2+ab+b^2)6}{-(a-b)2(a^2+ab+b^2)} = \frac{3}{-1} = -3$;

Dividing Rational Expressions

Division Rule

If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, with $\frac{c}{d} \neq 0$, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$.

- Examples: Perform the operations:
  - $\frac{10}{3x} \div \frac{6}{5x} = \frac{10}{3x} \cdot \frac{5x}{6} = \frac{50x}{18x} = \frac{25}{9}$;
  - $\frac{5a^2b^8}{c^3} \div (4ab^3c) = \frac{5a^2b^8}{c^3} \cdot \frac{1}{4ab^3c} = \frac{5a^2b^8}{4ab^3c^4} = \frac{5ab^5}{4c^4}$;
  - $\frac{a^2-4}{a^2+a-2} \div \frac{2a-4}{3a-3} = \frac{(a+2)(a-2)}{(a+2)(a-1)} \cdot \frac{3(a-1)}{2(a-2)} =$
    $\frac{(a+2)(a-2)3(a-1)}{(a+2)(a-1)2(a-2)} = \frac{3}{2}$;
  - $\frac{25-x^2}{x^2+x} \div \frac{x-5}{x^2-1} = \frac{-(x+5)(x-5)}{x(x+1)} \cdot \frac{(x+1)(x-1)}{x-5} =$
    $\frac{-(x+5)(x-5)(x+1)(x-1)}{x(x+1)(x-5)} = \frac{-(x+5)(x-1)}{x}$;

More Examples

- Perform the operations:
  - $\dfrac{\frac{a+b}{3}}{\frac{1}{2}} = \dfrac{a+b}{3} \cdot \dfrac{2}{1} = \dfrac{2(a+b)}{3}$;
  - $\dfrac{\frac{x^2-4}{2}}{\frac{x-2}{3}} = \dfrac{(x+2)(x-2)}{2} \cdot \dfrac{3}{x-2} = \dfrac{3(x+2)}{2}$;
  - $\dfrac{\frac{m^2+1}{5}}{3} = \dfrac{m^2+1}{5} \cdot \dfrac{1}{3} = \dfrac{m^2+1}{15}$;

Subsection 3

Addition and Subtraction

Adding/Subtracting With Common Denominators

Adding/Subtracting With Identical Denominators

If $b \neq 0$, then

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \quad \text{and} \quad \frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}.$$

- Examples: Perform the operations:
  - $\frac{3}{2x} + \frac{5}{2x} = \frac{3+5}{2x} = \frac{8}{2x} = \frac{4}{x}$;
  - $\frac{5x-3}{x-1} + \frac{5-7x}{x-1} = \frac{5x-3+5-7x}{x-1} = \frac{-2x+2}{x-1} = \frac{-2(x-1)}{x-1} = -2$;
  - $\frac{x^2+4x+7}{x^2-1} - \frac{x^2-2x+1}{x^2-1} = \frac{(x^2+4x+7)-(x^2-2x+1)}{x^2-1} = \frac{x^2+4x+7-x^2+2x-1}{x^2-1} = \frac{6x+6}{x^2-1} = \frac{6(x+1)}{(x+1)(x-1)} = \frac{6}{x-1}$;

Least Common Denominator

- To add fractions with non-identical denominators, we build up the denominators to the **least common denominator**, i.e., the **least common multiple** of the two denominators;
- For instance $\frac{1}{4}+\frac{1}{6}=\frac{3}{12}+\frac{2}{12}=\frac{3+2}{12}=\frac{5}{12}$;
- Examples: Find the least common multiple of the given polynomials:
  - $4x^2y$ and $6y$;
    We factor completely: 2^2x^2y and $2\cdot 3y$; We write all factors and choose the highest power appearing in any of the polynomials: $2^2\cdot 3x^2y=12x^2y$; This is the least common multiple of the two polynomials;
  - a^2bc, ab^3c^2 and a^3bc;
    The same principle gives least common multiple $a^3b^3c^2$;
  - x^2+5x+6 and x^2+6x+9;
    We factor completely: $(x+2)(x+3)$ and $(x+3)^2$; Thus, the least common multiple is $(x+2)(x+3)^2$;

Adding/Subtracting With Different Denominators

- Examples:
  - $\dfrac{3}{a^2b}+\dfrac{5}{ab^3}=\dfrac{3b^2}{a^2b^3}+\dfrac{5a}{a^2b^3}=\dfrac{3b^2+5a}{a^2b^3}$;
  - $\dfrac{x+1}{6}-\dfrac{2x-3}{4}=\dfrac{2(x+1)}{12}-\dfrac{3(2x-3)}{12}=\dfrac{2(x+1)-3(2x-3)}{12}=\dfrac{2x+2-6x+9}{12}=\dfrac{-4x+11}{12}$;
  - $\dfrac{1}{x^2-1}+\dfrac{2}{x^2+x}=\dfrac{1}{(x+1)(x-1)}+\dfrac{2}{x(x+1)}=\dfrac{x}{x(x+1)(x-1)}+\dfrac{2(x-1)}{x(x+1)(x-1)}=\dfrac{x+2(x-1)}{x(x+1)(x-1)}=\dfrac{3x-2}{x(x+1)(x-1)}$;
  - $\dfrac{5}{a-2}-\dfrac{3}{2-a}=\dfrac{5}{a-2}-\dfrac{3}{-(a-2)}=\dfrac{5}{a-2}+\dfrac{3}{a-2}=\dfrac{5+3}{a-2}=\dfrac{8}{a-2}$;

Reducing Before Finding the Least Common Multiple

- Sometimes reducing the fractions to be added or subtracted before taking common denominators simplifies the work;
- For instance, to add $\frac{2}{12}+\frac{5}{15}$, we could add directly: $\frac{2}{12}+\frac{5}{15}=\frac{10}{60}+\frac{20}{60}=\frac{30}{60}=\frac{1}{2}$, or, to avoid working with such high numbers, we could first simplify: $\frac{2}{12}+\frac{5}{15}=\frac{1}{6}+\frac{1}{3}=\frac{1}{6}+\frac{2}{6}=\frac{3}{6}=\frac{1}{2}$;
- Examples: Perform the operations:
  - $\dfrac{2xy}{4x}+\dfrac{x^2}{xy}=\dfrac{y}{2}+\dfrac{x}{y}=\dfrac{y^2}{2y}+\dfrac{2x}{2y}=\dfrac{y^2+2x}{2y}$;
  - $\dfrac{8x-8}{4x^2-4}-\dfrac{9x}{3x^2-3x-6}=\dfrac{8(x-1)}{4(x^2-1)}-\dfrac{9x}{3(x^2-x-2)}=$ $\dfrac{8(x-1)}{4(x+1)(x-1)}-\dfrac{9x}{3(x-2)(x+1)}=\dfrac{2}{x+1}-\dfrac{3x}{(x-2)(x+1)}=$ $\dfrac{2(x-2)}{(x-2)(x+1)}-\dfrac{3x}{(x-2)(x+1)}=\dfrac{2(x-2)-3x}{(x-2)(x+1)}=\dfrac{-x-4}{(x-2)(x+1)}$;

Shortcuts

- In simple cases, we can speed things up by using

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm cb}{bd};$$

- Examples: Compute the sums/differences:
  - $\frac{1}{2} + \frac{1}{3} = \frac{3+2}{6} = \frac{5}{6}$;
  - $\frac{1}{a} - \frac{1}{x} = \frac{x-a}{ax}$;
  - $\frac{a}{5} + \frac{a}{3} = \frac{3a+5a}{15} = \frac{8a}{15}$;
  - $x - \frac{2}{3} = \frac{3x-2}{3}$;

Application: Grading Papers

- Brian takes an average of x minutes to grade a calculus midterm paper, whereas Kimberly takes on average 3 minutes longer. Write a rational function $P(x)$ that gives the number of papers that they can grade together in 2 hours. Find $P(5)$;

  The number of papers graded by Brian in 2 hours is $\frac{1}{x}\frac{\text{papers}}{\text{minute}} \cdot 120 \text{ minutes} = \frac{120}{x}$ papers; On the other hand, the number of papers graded by Kimberly in 2 hours is $\frac{1}{x+3}\frac{\text{papers}}{\text{minute}} \cdot 120 \text{ minutes} = \frac{120}{x+3}$ papers; Thus, working together for 2 hours, they can manage $P(x) = \frac{120}{x} + \frac{120}{x+3} = \frac{120(x+3)}{x(x+3)} + \frac{120x}{x(x+3)} = \frac{120x+360+120x}{x(x+3)} = \frac{240x+360}{x(x+3)}$ papers
  Finally, we have $P(5) = \frac{240 \cdot 5 + 360}{5(5+3)} = \frac{1560}{40} = 39$;

Subsection 4

Complex Fractions

Simplifying Complex Fractions

- Simplify $\dfrac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{4}+\frac{1}{5}}$;
- **Method A**: Work separately with numerator and denominator and then divide.

$$\frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{4}+\frac{1}{5}} = \frac{\frac{3}{6}+\frac{2}{6}}{\frac{5}{20}+\frac{4}{20}} = \frac{\frac{5}{6}}{\frac{9}{20}} = \frac{5}{6}\cdot\frac{20}{9} = \frac{100}{54} = \frac{50}{27};$$

- **Method B**: Multiply both numerator and denominator by the least common multiple of all denominators appearing.

$$\frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{4}+\frac{1}{5}} = \frac{60(\frac{1}{2}+\frac{1}{3})}{60(\frac{1}{4}+\frac{1}{5})} = \frac{30+20}{15+12} = \frac{50}{27};$$

Complex Fractions With Variables

- Simplify $\dfrac{3-\frac{2}{x}}{\frac{1}{x^2}-\frac{1}{4}}$;

$$\frac{3-\frac{2}{x}}{\frac{1}{x^2}-\frac{1}{4}} = \frac{4x^2(3-\frac{2}{x})}{4x^2(\frac{1}{x^2}-\frac{1}{4})} = \frac{12x^2-8x}{4-x^2} = \frac{4x(3x-2)}{(2+x)(2-x)};$$

- Simplify $\dfrac{\frac{x+2}{x^2-9}}{\frac{x}{x^2-6x+9}+\frac{4}{x-3}}$;

$$\frac{\frac{x+2}{x^2-9}}{\frac{x}{x^2-6x+9}+\frac{4}{x-3}} = \frac{\frac{x+2}{(x+3)(x-3)}}{\frac{x}{(x-3)^2}+\frac{4}{x-3}} = \frac{(x+3)(x-3)^2\frac{x+2}{(x+3)(x-3)}}{(x+3)(x-3)^2(\frac{x}{(x-3)^2}+\frac{4}{x-3})} =$$
$$\frac{(x-3)(x+2)}{x(x+3)+4(x+3)(x-3)} = \frac{(x-3)(x+2)}{(x+3)[x+4(x-3)]} =$$
$$\frac{(x+2)(x-3)}{(x+3)(5x-12)};$$

Simplifying Expressions with Negative Exponents

- Simplify the complex fractions:
  - $\dfrac{3a^{-1}-2^{-1}}{1-b^{-1}} = \dfrac{\frac{3}{a}-\frac{1}{2}}{1-\frac{1}{b}} = \dfrac{2ab(\frac{3}{a}-\frac{1}{2})}{2ab(1-\frac{1}{b})} = \dfrac{6b-ab}{2ab-2a} = \dfrac{b(6-a)}{2a(b-1)}$;
  - $\dfrac{a^{-1}+b^{-2}}{ab^{-2}+ba^{-3}} = \dfrac{\frac{1}{a}+\frac{1}{b^2}}{\frac{a}{b^2}+\frac{b}{a^3}} = \dfrac{a^3b^2(\frac{1}{a}+\frac{1}{b^2})}{a^3b^2(\frac{a}{b^2}+\frac{b}{a^3})} = \dfrac{a^2b^2+a^3}{a^4+b^3}$;
  - $p+p^{-1}q^{-2} = p+\frac{1}{pq^2} = \frac{p^2q^2}{pq^2}+\frac{1}{pq^2} = \frac{p^2q^2+1}{pq^2}$;

Application: Riding a Bus to School

- "Washington" and "Lincoln" both have the same number of students; One-half of the students at Washington ride the buses to school, whereas two-thirds of those at Lincoln do. One-sixth of the students at Washington are female and one-third at Lincoln are female. If all female students ride the buses, then what percentage of the students riding the buses are female?

  Let x be the number of students at Washington; Then, the number at Lincoln is also x; The number of students riding the buses are

  $$\underbrace{\frac{1}{2}x}_{\text{Washington}} + \underbrace{\frac{2}{3}x}_{\text{Lincoln}};$$

  Moreover, the number of female students, which is equal to the number of female students riding the buses, is

  $$\underbrace{\frac{1}{6}x}_{\text{Washington}} + \underbrace{\frac{1}{3}x}_{\text{Lincoln}};$$

Riding to School (Cont'd)

- Therefore, the fraction of the students riding the buses that are female is $\dfrac{\overbrace{\dfrac{1}{6}x+\dfrac{1}{3}x}^{\text{Female}}}{\underbrace{\dfrac{1}{2}x+\dfrac{2}{3}x}_{\text{Total}}} = \dfrac{6(\frac{1}{6}x+\frac{1}{3}x)}{6(\frac{1}{2}x+\frac{2}{3}x)} = \dfrac{x+2x}{3x+4x} = \dfrac{3x}{7x} = \dfrac{3}{7}$; or approximately 43%;

Subsection 5

Division of Polynomials

Dividing a Polynomial by a Monomial

- To divide a monomial by a monomial, we cancel common factors:

$$6x^3 \div (3x) = \frac{6x^3}{3x} = 2x^2;$$
$$-12x^5 \div (2x^3) = \frac{-12x^5}{2x^3} = -6x^2;$$

- To divide a polynomial by a monomial, we distribute:
  - $(6x^3 + 15x^2 - 12x) \div (3x) = \frac{6x^3 + 15x^2 - 12x}{3x} =$
    $\frac{6x^3}{3x} + \frac{15x^2}{3x} - \frac{12x}{3x} = 2x^2 + 5x - 4;$
  - $(-20x^6 + 8x^4 - 4x^2) \div (4x^2) = \frac{-20x^6 + 8x^4 - 4x^2}{4x^2} =$
    $\frac{-20x^6}{4x^2} + \frac{8x^4}{4x^2} - \frac{4x^2}{4x^2} = -5x^4 + 2x^2 - 1;$
- Keep in mind that the denominator or **divisor** does not always go evenly in the numerator or **dividend**; So we may have a **remainder**;

A Couple of Divisions With a Remainder

- Find the quotient and the remainder:
  - $\dfrac{6x-1}{2x} = \dfrac{6x}{2x} - \dfrac{1}{2x} = 3 - \dfrac{1}{2x}$;

    Thus, we have quotient $Q = 3$ and remainder $R = -1$;
  - $\dfrac{x^3 - 4x^2 + 5x - 3}{2x^2} = \dfrac{x^3}{2x^2} - \dfrac{4x^2}{2x^2} + \dfrac{5x-3}{2x^2} = \frac{1}{2}x - 2 + \dfrac{5x-3}{2x^2}$;

    Thus, we have quotient $Q = \frac{1}{2}x - 2$ and remainder $R = 5x - 3$;

Dividing a Polynomial by a Binomial: Long Division I

Find the quotient and remainder of the division $(x^2+3x-9) \div (x-2)$;

$$
\begin{array}{rlll}
 & x & +5 & \\
\cline{2-4}
x-2) & x^2 & +3x & -9 \\
 & x^2 & -2x & \\
\cline{2-3}
 & & 5x & -9 \\
 & & 5x & -10 \\
\cline{3-4}
 & & & 1
\end{array}
$$

Thus, we have quotient $Q = x+5$ and remainder $R = 1$;
Using these, we may write

$$\frac{x^2+3x-9}{x-2} = x+5+\frac{1}{x-2},$$

or, equivalently, by multiplying by $x-2$,

$$x^2+3x-9 = (x-2)(x+5)+1;$$

Long Division II

Find the quotient and remainder of the division $(3x^4 - 5x - 2) \div (x^2 - 3x)$;

$$
\begin{array}{r|llllll}
 & 3x^2 & +9x & +27 & & \\
\cline{2-6}
x^2 - 3x) & 3x^4 & +0x^3 & +0x^2 & -5x & -2 \\
 & 3x^4 & -9x^3 & & & \\
\cline{2-3}
 & & 9x^3 & +0x^2 & & \\
 & & 9x^3 & -27x^2 & & \\
\cline{3-4}
 & & & 27x^2 & -5x & \\
 & & & 27x^2 & -81x & \\
\cline{4-5}
 & & & & 76x & -2
\end{array}
$$

Thus, we have quotient $Q = 3x^2 + 9x + 27$ and remainder $R = 76x - 2$;
Using these, we may write $\dfrac{3x^4 - 5x - 2}{x^2 - 3x} = 3x^2 + 9x + 27 + \dfrac{76x - 2}{x^2 - 3x}$, or, equivalently, by multiplying by $x^2 - 3x$,
$3x^4 - 5x - 2 = (x^2 - 3x)(3x^2 + 9x + 27) + 76x - 2$;

Long Division III

Find the quotient and remainder of the division $(4x^3 - x - 9) \div (2x - 3)$;

$$\begin{array}{rrrrr}
 & 2x^2 & +3x & +4 & \\
\cline{2-5}
2x-3) & 4x^3 & +0x^2 & -x & -9 \\
 & 4x^3 & -6x^2 & & \\
\cline{2-3}
 & & 6x^2 & -x & \\
 & & 6x^2 & -9x & \\
\cline{3-4}
 & & & 8x & -9 \\
 & & & 8x & -12 \\
\cline{4-5}
 & & & & 3
\end{array}$$

Thus, we have quotient $Q = 2x^2 + 3x + 4$ and remainder $R = 3$;

Using these, we may write $\dfrac{4x^3 - x - 9}{2x - 3} = 2x^2 + 3x + 4 + \dfrac{3}{2x - 3}$, or, equivalently, by multiplying by $2x - 3$,

$4x^3 - x - 9 = (2x - 3)(2x^2 + 3x + 4) + 3$;

Synthetic Division

Synthetic Division

1. List the coefficients of the dividend; include zeros for missing terms;
2. For dividing by $x - c$, place c to the left;
3. Bring first coefficient down;
4. Multiply by c and add for each column;
5. Read quotient Q and remainder R from bottom row.

- Divide $(x^3 - 5x^2 + 4x - 3) \div (x - 2)$;

$$\begin{array}{r|rrrr} 2 & 1 & -5 & 4 & -3 \\ & & 2 & -6 & -4 \\ \hline & 1 & -3 & -2 & -7 \end{array}$$

The quotient is $Q = x^2 - 3x - 2$ and the remainder is $R = -7$;

Synthetic Division: One More Example

- Divide $(2x^4 - 5x^2 + 6x - 9) \div (x + 2)$;

$$\begin{array}{r|rrrrr} -2 & 2 & 0 & -5 & 6 & -9 \\ & & -4 & 8 & -6 & 0 \\ \hline & 2 & -4 & 3 & 0 & -9 \end{array}$$

  The quotient is $Q = 2x^3 - 4x^2 + 3x$ and the remainder is $R = -9$;
  So we may write $\dfrac{2x^4 - 5x^2 + 6x - 9}{x + 2} = 2x^3 - 4x^2 + 3x - \dfrac{9}{x + 2}$;

Division and Factoring

- Is $x-1$ a factor of $6x^3-5x^2-4x+3$?

  To find out, we divide $(6x^3-5x^2-4x+3)\div(x-1)$ and check whether $R=0$;

$$\begin{array}{r|rrrr} 1 & 6 & -5 & -4 & 3 \\ & & 6 & 1 & -3 \\ \hline & 6 & 1 & -3 & 0 \end{array}$$

  Since $R=0$, $x-1$ is a factor of $6x^3-5x^2-4x+3$; In fact, $6x^3-5x^2-4x+3=(x-1)(6x^2+x-3)$.

The Remainder Theorem

The Remainder Theorem

If the polynomial $P(x)$ is divided by $x - c$, the remainder is equal to the value $P(c)$.

- Use synthetic division to compute $P(2)$ if $P(x) = 4x^3 - 5x^2 + 6x - 7$;

  If we divide $(4x^3 - 5x^2 + 6x - 7) \div (x - 2)$, then, according to the Remainder Theorem, the remainder will be equal to $P(2)$!

$$\begin{array}{r|rrrr} 2 & 4 & -5 & 6 & -7 \\ & & 8 & 6 & 24 \\ \hline & 4 & 3 & 12 & 17 \end{array}$$

  Therefore $P(2) = 17$;
  Can anyone see why this happens? Write $P(x) = (x - c)Q(x) + R$;
  Plug in c for x: $P(c) = (c - c)Q(c) + R$, i.e., $P(c) = R$!

Subsection 6

Solving Equations With Rational Expressions

Multiplying by the Least Common Denominator

- Solve the equation $\frac{1}{x}+\frac{1}{4}=\frac{1}{6}$;
  The least common denominator of the three fractions is $12x$; We multiply both sides by $12x$:
  $$12x(\frac{1}{x}+\frac{1}{4})=12x\cdot\frac{1}{6}\Rightarrow 12+3x=2x\Rightarrow x=-12;$$
  Check that this is indeed a valid solution!
- Solve the equation $\frac{10}{x}+\frac{14}{x+2}=4$;
  The least common denominator of the three fractions is $x(x+2)$; We multiply both sides by $x(x+2)$:
  $x(x+2)(\frac{10}{x}+\frac{14}{x+2})=4x(x+2)\Rightarrow 10(x+2)+14x=4x(x+2)\Rightarrow 10x+20+14x=4x^2+8x\Rightarrow 4x^2-16x-20=0\Rightarrow x^2-4x-5=0\Rightarrow(x-5)(x+1)=0\Rightarrow x=-1$ or $x=5$; Check that both are valid solutions!

Discarding Extraneous Solutions I

- Solve the equation $\frac{3}{x}+\frac{6}{x-2}=\frac{12}{x^2-2x}$;

  Factor all denominators $\frac{3}{x}+\frac{6}{x-2}=\frac{12}{x(x-2)}$; The least common denominator is $x(x-2)$; Multiply both sides by $x(x-2)$: $x(x-2)(\frac{3}{x}+\frac{6}{x-2})=x(x-2)\cdot\frac{12}{x(x-2)}\Rightarrow 3(x-2)+6x=12\Rightarrow$ $3x-6+6x=12\Rightarrow 9x=18\Rightarrow x=2$; Note that this is not a valid solution! Therefore, the given equation has no solutions!

Discarding Extraneous Solutions II

- Solve the equation $x+4-\frac{x}{x-4}=\frac{-4}{x-4}$;

  The least common denominator is $x-4$; Multiply both sides by $x-4$: $(x-4)(x+4-\frac{x}{x-4})=(x-4)\cdot\frac{-4}{x-4}\Rightarrow(x-4)(x+4)-x=-4\Rightarrow x^2-16-x=-4\Rightarrow x^2-x-12=0\Rightarrow(x-4)(x+3)=0\Rightarrow x=-3$ or $x=4$; Note that 4 is not a valid solution! Therefore, the only admissible solution is $x=-3$!

Proportions

- A **proportion** is an equation of the form $\frac{a}{b} = \frac{c}{d}$;
- The numbers a and d are called **extremes** and the numbers b and c are called **means**;

Extremes-Means Property

If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$. (Product of extremes equals product of means.)

- Example: Solve $\frac{20}{x} = \frac{30}{x+20}$;

  Use extremes-means: $20(x+20) = 30x$; Distribute $20x + 400 = 30x$; Now we get $10x = 400$, whence $x = 40$;

More Proportions

- Example: Solve $\frac{2}{x} = \frac{x+3}{5}$;

  Use extremes-means: $2 \cdot 5 = x(x+3)$; Distribute $10 = x^2 + 3x$; Make one side zero: $x^2 + 3x - 10 = 0$; Factor to get $(x+5)(x-2) = 0$; Use the zero-factor property: $x = -5$ or $x = 2$; Check that both solutions work!

Application: Men to Women in Poker Tournament

- The ratio of men to women in a poker tournament is 13 to 2; If there are 8 women, how many men are in the tournament?

  Let x be the number of men participating; We set up the given data as a proportion:
  $$\frac{13}{2} = \frac{x}{8};$$
  Use extremes-means to solve:
  $$13 \cdot 8 = 2x,$$
  whence $x = 52$;

Application: Men to Women at a Football Game

- The ratio of men to women attending a football game was 4 to 3; If there were 12,000 more men than women attending, how many men and how many women were in the audience?

  Let x be the number of women attending; Then, the number of men attending is $x + 12000$; We set up the given data as a proportion:

  $$\frac{4}{3} = \frac{x + 12000}{x};$$

  Use extremes-means to solve:

  $$4x = 3(x + 12000),$$

  Distribute to get $4x = 3x + 36000$; Thus, $x = 36000$; Therefore, the number of men attending was 48000 and the number of women attending was 36000.

Subsection 7

Applications

Formulas I

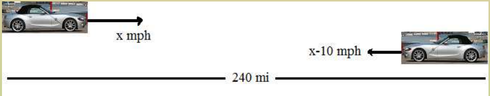
- Solve the given formula for y:
  - $\frac{1}{2y} + \frac{1}{x} = \frac{a}{3}$;
    The least common denominator is $6xy$; Multiplying both sides by $6xy$, we get
    $$6xy(\frac{1}{2y} + \frac{1}{x}) = 6xy \cdot \frac{a}{3};$$
    Distribute to obtain
    $$3x + 6y = 2axy;$$
    Since we would like to solve for y, we take everything containing y on one side and everything else on the other:
    $$2axy - 6y = 3x;$$
    Factor y out: $y(2ax - 6) = 3x$ and, finally, divide to solve for y:
    $$y = \frac{3x}{2ax - 6};$$

Formulas II

- Solve the given formula for y:
  - $\dfrac{a-b}{2y} = \dfrac{6}{a+b}$;
    This is a proportion; so we use the extremes-means to solve:
    $$(a+b)(a-b) = 6 \cdot 2y;$$
    This gives $a^2 - b^2 = 12y$ and, diving by 12,
    $$y = \frac{a^2 - b^2}{12};$$

Uniform Motion

- Ken drove from SSM south to Alma, a distance of 240 miles to pick up his wife. With his overcautious wife nagging him to drive more slowly, Ken was forced to average 10 miles per hour less on the return trip. If the return trip took him 20 minutes longer, what was his average speed during his trip from SSM to Alma?



Recall that distance = time × speed; Suppose that driving at x mph from SSM to Alma, it took t hours to cover 240 miles; Therefore, for the return trip, the speed was $x - 10$ mph and it took $t + \frac{1}{3}$ hours to cover the same distance; Now we set up the equations for each trip:

$$x \cdot t = 240 \qquad (x - 10) \cdot (t + \frac{1}{3}) = 240;$$

Uniform Motion (Cont'd)

The equations for the trips:

$$x \cdot t = 240 \qquad (x-10)\cdot(t+\frac{1}{3}) = 240;$$

Solve the first for $t = \frac{240}{x}$; Substitute into the second:

$$(x-10)(\frac{240}{x}+\frac{1}{3}) = 240;$$

We proceed to solve this equation for x: $240 + \frac{x}{3} - \frac{2400}{x} - \frac{10}{3} = 240 \Rightarrow \frac{x}{3} - \frac{2400}{x} - \frac{10}{3} = 0 \Rightarrow 3x(\frac{x}{3} - \frac{2400}{x} - \frac{10}{3}) = 0 \Rightarrow x^2 - 7200 - 10x = 0 \Rightarrow x^2 - 10x - 7200 = 0 \Rightarrow (x+80)(x-90) = 0 \Rightarrow x = -80$ or $x = 90$;
Therefore, Ken drove an average 90 mph on his way to Alma;

Work Problems

- Whitney can do inventory by herself in 8 hours. Colleen can do the same job by herself in 6 hours. Assuming they do not interfere with each other's productivity, how long would it take them to do the inventory working together?

  Assume that it takes them x hours to do the inventory together;
  Since it takes Whitney 8 hours and Colleen 6 hours to do the job alone, the rate of Whitney is $\frac{1}{8}$ inventories per hour and that of Colleen $\frac{1}{6}$ inventories per hour;
  Thus, their joint rate is $\frac{1}{8}+\frac{1}{6}$ inventories per hour;
  Since rate multiplied by time yields the number of inventories completed, we must have

  $$x(\frac{1}{8}+\frac{1}{6})=1;$$

  Therefore, $x\cdot\frac{6+8}{48}=1\Rightarrow x\cdot\frac{7}{24}=1\Rightarrow x=\frac{24}{7}$ hours.

Purchasing of Supplies

- Mark and Patrick bought 50 pounds of meat consisting of hamburger and steak; Steak costs twice as much per pound as hamburger. If they bought \$ 30 worth of hamburger and \$ 90 worth of steak, how many pounds of each did they buy?

  Suppose that they bought x pounds of hamburger and y pounds of steak; Suppose also that hamburger costs $\$c$ per pound and, hence, that steak costs $\$2c$ per pound; Then, we have the equations

  $$x + y = 50, \qquad x \cdot c = 30, \qquad y \cdot 2c = 90;$$

  Solve the first equation for $y = 50 - x$; Solve the second equation for $c = \frac{30}{x}$; Plug in these values into the last equation and solve for x: $(50 - x) \cdot (2 \cdot \frac{30}{x}) = 90 \Rightarrow (50 - x)\frac{60}{x} = 90 \Rightarrow \frac{3000}{x} - 60 = 90 \Rightarrow \frac{3000}{x} = 150 \Rightarrow 150x = 3000 \Rightarrow x = 20$; Thus, the guys bought 20 pounds of hamburger and 30 pounds of steak;