Intermediate Algebra

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LSSU Math 102

1. Radicals and Rational Exponents
   - Radicals
   - Rational Exponents
   - Adding, Subtracting and Multiplying Radicals
   - Quotients, Powers and Rationalizing Denominators
   - Solving Equations With Radicals and Exponents

Subsection 1

Radicals

Roots

- If $a = b^n$ for some positive integer n, then b is an ***n*-th root of** a; In particular, if $a = b^2$, then b is a **square root** of a and if $a = b^3$, then b is a **cube root** of a;
- If n is even and $a > 0$, then there are two real n-th roots of a, called the **even roots**; The positive one is called the **principal root**;
  - The principal square root of 9 is 3; the principal fourth root of 16 is 2;
- If n is odd and a is any real (not necessarily positive), then there is only one n-th root of a, called the **odd root**;

The Symbol $\sqrt[n]{a}$

- If n is a positive even integer and $a > 0$, then $\sqrt[n]{a}$ denotes the *principal n-th root* of a;
- If n is a positive odd integer, then $\sqrt[n]{a}$ is the n-th root of a;
- For n any integer, $\sqrt[n]{0} = 0$;

- In $\sqrt[n]{a}$, n is the **index of the radical** and a is called the **radicand**;

Evaluating Radical Expressions

- Find the following roots:
  - $\sqrt{25} = 5$; (Because $5^2 = 25$;)
  - $\sqrt[3]{-27} = -3$; (Because $(-3)^3 = -27$;)
  - $\sqrt[6]{64} = 2$; (Because $2^6 = 64$;)
  - $-\sqrt{4} = -2$; (Because $2^2 = 4$;)

Roots and Variables

Perfect Powers

- Perfect Squares: $x^2, x^4, x^6, \ldots$, i.e., exponent divisible by 2;
- Perfect Cubes: $x^3, x^6, x^9, \ldots$, i.e., exponent divisible by 3;
- Perfect Fourth Powers: $x^4, x^8, x^{12}, \ldots$, i.e., exponent divisible by 4;

- To find the square root of a perfect square, we divide the exponent by 2:
  - Assuming $x \geq 0$, $\sqrt{x^2} = x$, $\sqrt{x^4} = x^2$, $\sqrt{x^6} = x^3$, etc.;
  - If we are not sure $x \geq 0$, we must add an absolute value, e.g., $\sqrt{x^2} = |x|$ and $\sqrt{x^6} = |x^3|$;
- To find the cube root of a perfect cube, divide the exponent by 3:
  - $\sqrt[3]{x^3} = x$, $\sqrt[3]{x^6} = x^2$, $\sqrt[3]{x^9} = x^3$, etc.;

Roots of Exponential Expressions

- Find each of the following roots, assuming that the variables can represent any real number (not necessarily positive):
  - $\sqrt{a^2} = |a|$; (Because $(a)^2 = a^2$;)
  - $\sqrt{x^{22}} = |x^{11}|$; (Because $(x^{11})^2 = x^{22}$;)
  - $\sqrt[4]{w^{40}} = |w^{10}| = w^{10}$; (Because $(w^{10})^4 = w^{40}$;)
  - $\sqrt[3]{t^{18}} = t^6$; (Because $(t^6)^3 = t^{18}$;)
  - $\sqrt[5]{s^{30}} = s^6$; (Because $(s^6)^5 = s^{30}$;)

Product Rule for Radicals

Product Rule for Radicals

The n-th root of a product is equal to the product of the n-th roots:

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b};$$

- Example: Simplify each radical (assume all variables represent nonnegative reals):
  - $\sqrt{4y} = \sqrt{4}\sqrt{y} = 2\sqrt{y}$;
  - $\sqrt{3y^8} = \sqrt{3}\sqrt{y^8} = \sqrt{3}y^4$;
  - $\sqrt[3]{125w^2} = \sqrt[3]{125}\sqrt[3]{w^2} = 5\sqrt[3]{w^2}$;
  - $\sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$;
  - $\sqrt[3]{54} = \sqrt[3]{27 \cdot 2} = \sqrt[3]{27}\sqrt[3]{2} = 3\sqrt[3]{2}$;
  - $\sqrt[4]{80} = \sqrt[4]{16 \cdot 5} = \sqrt[4]{16}\sqrt[4]{5} = 2\sqrt[4]{5}$;
  - $\sqrt[5]{64} = \sqrt[5]{32 \cdot 2} = \sqrt[5]{32}\sqrt[5]{2} = 2\sqrt[5]{2}$;

Some Additional Examples

- Simplify each radical; Again assume that all variables represent nonnegative reals;
  - $\sqrt{20x^3} = \sqrt{4 \cdot x^2 \cdot 5x} = \sqrt{4}\sqrt{x^2}\sqrt{5x} = 2x\sqrt{5x}$;
  - $\sqrt[3]{40a^8} = \sqrt[3]{8 \cdot a^6 \cdot 5a^2} = \sqrt[3]{8}\sqrt[3]{a^6}\sqrt[3]{5a^2} = 2a^2\sqrt[3]{5a^2}$;
  - $\sqrt[4]{48a^4b^{11}} = \sqrt[4]{16 \cdot a^4 \cdot b^8 \cdot 3b^3} = \sqrt[4]{16}\sqrt[4]{a^4}\sqrt[4]{b^8}\sqrt[4]{3b^3} = 2ab^2\sqrt[4]{3b^3}$;
  - $\sqrt[5]{w^{12}} = \sqrt[5]{w^{10} \cdot w^2} = \sqrt[5]{w^{10}}\sqrt[5]{w^2} = w^2\sqrt[5]{w^2}$;

Quotient Rule for Radicals

Quotient Rule for Radicals

The n-th root of a quotient is equal to the quotient of the n-th roots:

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}};$$

- Example: Simplify each radical, assuming that all variables represent positive real numbers:
  - $\sqrt{\frac{25}{9}} = \frac{\sqrt{25}}{\sqrt{9}} = \frac{5}{3}$;
  - $\frac{\sqrt{15}}{\sqrt{3}} = \sqrt{\frac{15}{3}} = \sqrt{5}$;
  - $\sqrt[3]{\frac{b}{125}} = \frac{\sqrt[3]{b}}{\sqrt[3]{125}} = \frac{\sqrt[3]{b}}{5}$;
  - $\sqrt[3]{\frac{x^{21}}{y^6}} = \frac{\sqrt[3]{x^{21}}}{\sqrt[3]{y^6}} = \frac{x^7}{y^2}$;

Using Both the Product and the Quotient Rule

- Simplify each radical (all variables represent positive real numbers):
  - $\sqrt{\dfrac{50}{49}} = \dfrac{\sqrt{25 \cdot 2}}{\sqrt{49}} = \dfrac{\sqrt{25}\sqrt{2}}{\sqrt{49}} = \dfrac{5\sqrt{2}}{7}$;
  - $\sqrt[3]{\dfrac{x^5}{8}} = \dfrac{\sqrt[3]{x^3 \cdot x^2}}{\sqrt[3]{8}} = \dfrac{\sqrt[3]{x^3}\sqrt[3]{x^2}}{\sqrt[3]{8}} = \dfrac{x\sqrt[3]{x^2}}{2}$;
  - $\sqrt[4]{\dfrac{a^5}{b^8}} = \dfrac{\sqrt[4]{a^4 \cdot a}}{\sqrt[4]{b^8}} = \dfrac{\sqrt[4]{a^4}\sqrt[4]{a}}{\sqrt[4]{b^8}} = \dfrac{a\sqrt[4]{a}}{b^2}$;

Domains of Radical Functions

- Find the domain of each function:
    - $f(x) = \sqrt{x-5}$;
      We must have $x - 5 \geq 0$; Therefore $x \geq 5$; So $\mathrm{Dom}(f) = [5, \infty)$;
    - $f(x) = \sqrt[3]{x+7}$;
      Since the root is an odd-indexed root, there are no restrictions on the values of x: $\mathrm{Dom}(f) = \mathbb{R} = (-\infty, \infty)$;
    - $f(x) = \sqrt[4]{2x+6}$;
      We must have $2x + 6 \geq 0$; Therefore $2x \geq -6$, i.e., $x \geq -3$; So $\mathrm{Dom}(f) = [-3, \infty)$;

Subsection 2

Rational Exponents

Rational Exponents I

Definition of $a^{1/n}$

If n is a positive integer, then

$$a^{1/n} = \sqrt[n]{a};$$

- Example: Rewrite radical expressions as exponential expressions and exponential expressions as radical expressions:
  - $\sqrt[3]{35} = 35^{1/3}$;
  - $\sqrt[4]{xy} = (xy)^{1/4}$;
  - $5^{1/2} = \sqrt{5}$;
  - $a^{1/5} = \sqrt[5]{a}$;
- Example: Evaluate each expression:
  - $4^{1/2} = \sqrt{4} = 2$;
  - $(-8)^{1/3} = \sqrt[3]{-8} = -2$;
  - $81^{1/4} = \sqrt[4]{81} = 3$;
  - $(-9)^{1/2} = \sqrt{-9} =$ undefined;
  - $-9^{1/2} = -\sqrt{9} = -3$;

Rational Exponents II

Definition of $a^{m/n}$

If m, n are positive integers, and $a^{1/n}$ is defined,

$$a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m;$$

Evaluation of $a^{m/n}$ in Either Order

If m, n are positive integers, and $a^{1/n}$ is defined,

$$a^{m/n} = (\sqrt[n]{a})^m = \sqrt[n]{a^m};$$

In other words taking first root and then power or first power and then root leads to the same result;

Definition of $a^{-m/n}$

If m, n are positive integers, $a \neq 0$ and $a^{1/n}$ is defined, then

$$a^{-m/n} = \frac{1}{a^{m/n}} = \frac{1}{(\sqrt[n]{a})^m};$$

Examples

- Example: Rewrite radical expressions as exponential expressions and exponential expressions as radical expressions:
  - $\sqrt[3]{x^2} = x^{2/3}$;
  - $\dfrac{1}{\sqrt[4]{m^3}} = \dfrac{1}{m^{3/4}} = m^{-3/4}$;
  - $5^{2/3} = \sqrt[3]{5^2}$;
  - $a^{-2/5} = \dfrac{1}{a^{2/5}} = \dfrac{1}{\sqrt[5]{a^2}}$;
- Example: Evaluate each expression:
  - $27^{2/3} = (\sqrt[3]{27})^2 = 3^2 = 9$;
  - $4^{-3/2} = \dfrac{1}{4^{3/2}} = \dfrac{1}{(\sqrt{4})^3} = \dfrac{1}{2^3} = \dfrac{1}{8}$;
  - $81^{-3/4} = \dfrac{1}{81^{3/4}} = \dfrac{1}{(\sqrt[4]{81})^3} = \dfrac{1}{3^3} = \dfrac{1}{27}$;
  - $(-8)^{-5/3} = \dfrac{1}{(-8)^{5/3}} = \dfrac{1}{(\sqrt[3]{-8})^5} = \dfrac{1}{(-2)^5} = \dfrac{1}{-32} = -\dfrac{1}{32}$;

Rules for Rational Exponents

If a, b are nonzero real numbers and r, s rational numbers,

1. Product Rule: $a^r a^s = a^{r+s}$;
2. Quotient Rule: $\dfrac{a^r}{a^s} = a^{r-s}$;
3. Power of a Power: $(a^r)^s = a^{rs}$;
4. Power of a Product: $(ab)^r = a^r b^r$;
5. Power of a Quotient: $\left(\dfrac{a}{b}\right)^r = \dfrac{a^r}{b^r}$;

Using Rules of Exponents

- Example: Simplify each expression:
  - $27^{1/6} \cdot 27^{1/2} = 27^{\frac{1}{6}+\frac{1}{2}} = 27^{\frac{1}{6}+\frac{3}{6}} = 27^{4/6} = 27^{2/3} = \sqrt[3]{27}^2 = 3^2 = 9$;
  - $\frac{5^{3/4}}{5^{1/4}} = 5^{\frac{3}{4}-\frac{1}{4}} = 5^{2/4} = 5^{1/2} = \sqrt{5}$;
- Example: Simplify each expression:
  - $3^{1/2} \cdot 12^{1/2} = (3 \cdot 12)^{1/2} = 36^{1/2} = \sqrt{36} = 6$;
  - $(3^{10})^{1/2} = 3^{10 \cdot (1/2)} = 3^5 = 243$;
  - $\left(\frac{2^6}{3^9}\right)^{-1/3} = \frac{(2^6)^{-1/3}}{(3^9)^{-1/3}} = \frac{2^{6 \cdot (-1/3)}}{3^{9 \cdot (-1/3)}} = \frac{2^{-2}}{3^{-3}} = \frac{3^3}{2^2} = \frac{27}{4}$;

Simplifying Expressions Involving Variables

- Example: Simplify each expression (assume all variables represent positive real numbers):
  - $(x^8y^4)^{1/4} = (x^8)^{1/4}(y^4)^{1/4} = x^{8\cdot(1/4)}y^{4\cdot(1/4)} = x^2y$;
  - $\left(\frac{x^9}{8}\right)^{1/3} = \frac{(x^9)^{1/3}}{8^{1/3}} = \frac{x^{9\cdot(1/3)}}{\sqrt[3]{8}} = \frac{x^3}{2}$;
- Example: Simplify each expression, writing all answers with positive exponents (assume all variables represent positive real numbers):
  - $x^{2/3}x^{4/3} = x^{\frac{2}{3}+\frac{4}{3}} = x^{6/3} = x^2$;
  - $\frac{a^{1/2}}{a^{1/4}} = a^{\frac{1}{2}-\frac{1}{4}} = a^{1/4} = \sqrt[4]{a}$;
  - $(x^{1/2}y^{-3})^{1/2} = (x^{1/2})^{1/2}(y^{-3})^{1/2} = x^{1/4}y^{-3/2} = \frac{x^{1/4}}{y^{3/2}}$;
  - $\left(\frac{x^2}{y^{1/3}}\right)^{-1/2} = \frac{(x^2)^{-1/2}}{(y^{1/3})^{-1/2}} = \frac{x^{-1}}{y^{-1/6}} = \frac{y^{1/6}}{x}$;

Subsection 3

Adding, Subtracting and Multiplying Radicals

Adding/Subtracting Radicals

- Example: Simplify the following expressions:
  - $3\sqrt{5}+4\sqrt{5}=7\sqrt{5}$;
  - $\sqrt[4]{w}-6\sqrt[4]{w}=-5\sqrt[4]{w}$;
  - $\sqrt{3}+\sqrt{5}-4\sqrt{3}+6\sqrt{5}=(\sqrt{3}-4\sqrt{3})+(\sqrt{5}+6\sqrt{5})=-3\sqrt{3}+7\sqrt{5}$;
  - $3\sqrt[3]{6x}+2\sqrt[3]{x}+\sqrt[3]{6x}+\sqrt[3]{x}=(3\sqrt[3]{6x}+\sqrt[3]{6x})+(2\sqrt[3]{x}+\sqrt[3]{x})=4\sqrt[3]{6x}+3\sqrt[3]{x}$;
- Example: Perform the operations (first simplify and then combine like radicals):
  - $\sqrt{8}+\sqrt{18}=\sqrt{4\cdot 2}+\sqrt{9\cdot 2}=\sqrt{4}\sqrt{2}+\sqrt{9}\sqrt{2}=2\sqrt{2}+3\sqrt{2}=5\sqrt{2}$;
  - $\sqrt{2x^3}-\sqrt{4x^2}+5\sqrt{18x^3}=\sqrt{x^2\cdot 2x}-\sqrt{(2x)^2}+5\sqrt{(3x)^2\cdot 2x}=$ $\sqrt{x^2}\sqrt{2x}-2x+5\sqrt{(3x)^2}\sqrt{2x}=x\sqrt{2x}-2x+5\cdot 3x\sqrt{2x}=$ $x\sqrt{2x}-2x+15x\sqrt{2x}=16x\sqrt{2x}-2x$;
  - $\sqrt[3]{16x^4y^3}-\sqrt[3]{54x^4y^3}=\sqrt[3]{(2xy)^3\cdot 2x}-\sqrt[3]{(3xy)^3\cdot 2x}=$ $\sqrt[3]{(2xy)^3}\sqrt[3]{2x}-\sqrt[3]{(3xy)^3}\sqrt[3]{2x}=2xy\sqrt[3]{2x}-3xy\sqrt[3]{2x}=-xy\sqrt[3]{2x}$;

Multiplying Radicals

- Here we exploit the rule $\sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$ to multiply radicals with the same index;
- Example: Multiply and simplify the following expressions:
  - $5\sqrt{6} \cdot 4\sqrt{3} = 20\sqrt{6 \cdot 3} = 20\sqrt{18} = 20\sqrt{9 \cdot 2} = 20\sqrt{9}\sqrt{2} = 20 \cdot 3\sqrt{2} = 60\sqrt{2}$;
  - $\sqrt{3a^2} \cdot \sqrt{6a} = \sqrt{3a^2 \cdot 6a} = \sqrt{(3a)^2 \cdot 2a} = \sqrt{(3a)^2}\sqrt{2a} = 3a\sqrt{2a}$;
  - $\sqrt[3]{4}\sqrt[3]{4} = \sqrt[3]{4 \cdot 4} = \sqrt[3]{2^3 \cdot 2} = \sqrt[3]{2^3}\sqrt[3]{2} = 2\sqrt[3]{2}$;
  - $\sqrt[4]{\dfrac{x^3}{2}} \cdot \sqrt[4]{\dfrac{x^2}{8}} = \sqrt[4]{\dfrac{x^3}{2} \cdot \dfrac{x^2}{8}} = \sqrt[4]{\left(\dfrac{x}{2}\right)^4 \cdot x} = \sqrt[4]{\left(\dfrac{x}{2}\right)^4}\sqrt[4]{x} = \dfrac{x}{2}\sqrt[4]{x}$;
- Example: Multiply and simplify:
  - $3\sqrt{2}(4\sqrt{2}-\sqrt{3}) = 3\sqrt{2}\cdot 4\sqrt{2} - 3\sqrt{2}\sqrt{3} = 12 \cdot \sqrt{2}^2 - 3\sqrt{2 \cdot 3} = 24 - 3\sqrt{6}$;
  - $\sqrt[3]{a}(\sqrt[3]{a} - \sqrt[3]{a^2}) = \sqrt[3]{a}\sqrt[3]{a} - \sqrt[3]{a}\sqrt[3]{a^2} = \sqrt[3]{a \cdot a} - \sqrt[3]{a \cdot a^2} = \sqrt[3]{a^2} - a$;
  - $(2\sqrt{3}+\sqrt{5})(3\sqrt{3}-2\sqrt{5}) = 2\sqrt{3}\cdot 3\sqrt{3} - 2\sqrt{3}\cdot 2\sqrt{5} + \sqrt{5}\cdot 3\sqrt{3} - \sqrt{5}\cdot 2\sqrt{5} = 6\sqrt{3^2} - 4\sqrt{3 \cdot 5} + 3\sqrt{3 \cdot 5} - 2\sqrt{5^2} = 18 - 4\sqrt{15} + 3\sqrt{15} - 10 = 8 - \sqrt{15}$;
  - $(3 + \sqrt{x-9})^2 = 3^2 + 2 \cdot 3 \cdot \sqrt{x-9} + (\sqrt{x-9})^2 = 9 + 6\sqrt{x-9} + x - 9 = x + 6\sqrt{x-9}$;

Conjugates

- Recall the special product rule $(a+b)(a-b)=a^2-b^2$;
- Because of this rule, when two expression containing roots and differing only in sign, such as $5+2\sqrt{3}$ and $5-2\sqrt{3}$ are multiplied together, we get an expression not containing a root:

$$(5+2\sqrt{3})(5-2\sqrt{3})=5^2-(2\sqrt{3})^2=25-4\cdot 3=13;$$

- Expressions of this form are called **conjugates**;
- Example: Compute the products:
  - $(2+3\sqrt{5})(2-3\sqrt{5})=2^2-(3\sqrt{5})^2=4-9\cdot 5=4-45=-41;$
  - $(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})=(\sqrt{3})^2-(\sqrt{2})^2=3-2=1;$
  - $(\sqrt{2x}-\sqrt{y})(\sqrt{2x}+\sqrt{y})=(\sqrt{2x})^2-(\sqrt{y})^2=2x-y;$

Multiplying Radicals with Different Indices

- Here, apart from the product rule for exponents: $a^m \cdot a^n = a^{m+n}$, we will also need the product rule for radicals $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$ and the power of power rule $(a^m)^n = a^{mn}$;
- Example: Write each product as a single radical expression:
  - $\sqrt[3]{2} \cdot \sqrt[4]{2} = 2^{1/3} \cdot 2^{1/4} = 2^{\frac{1}{3}+\frac{1}{4}} = 2^{\frac{4}{12}+\frac{3}{12}} = 2^{7/12} = \sqrt[12]{2^7} = \sqrt[12]{128}$;
  - $\sqrt[3]{2} \cdot \sqrt{3} = 2^{1/3} \cdot 3^{1/2} = 2^{2/6} \cdot 3^{3/6} = (2^2)^{1/6} \cdot (3^3)^{1/6} = (4 \cdot 27)^{1/6} = \sqrt[6]{108}$;

Subsection 4

Quotients, Powers and Rationalizing Denominators

Rationalizing the Denominator

- **Rationalizing** a denominator means rewriting a given fraction so as not to contain any radicals in the denominator;
- Example: Rewrite each expression with a rational denominator:
  - $\frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3}\cdot\sqrt{5}}{\sqrt{5}\cdot\sqrt{5}} = \frac{\sqrt{3\cdot 5}}{\sqrt{5}^2} = \frac{\sqrt{15}}{5}$;
  - $\frac{3}{\sqrt[3]{2}} = \frac{3\sqrt[3]{2}^2}{\sqrt[3]{2}\cdot\sqrt[3]{2}^2} = \frac{3\sqrt[3]{2^2}}{\sqrt[3]{2}^3} = \frac{3\sqrt[3]{4}}{2}$;

Simplifying Radicals

- We agree to consider a radical expression of index n **simplified** if
  1. no perfect n-th powers as factors of the radicand;
  2. no fractions inside the radical;
  3. no radicals in the denominator;
- Example: Simplify (all conditions above need apply at the end):
  - $\frac{\sqrt{10}}{\sqrt{6}} = \frac{\sqrt{2\cdot 5}}{\sqrt{2\cdot 3}} = \frac{\sqrt{2}\sqrt{5}}{\sqrt{2}\sqrt{3}} = \frac{\sqrt{5}}{\sqrt{3}} = \frac{\sqrt{5}\sqrt{3}}{\sqrt{3}^2} = \frac{\sqrt{15}}{3}$;
  - $\sqrt{\frac{x^3}{y^5}} = \frac{\sqrt{x^2 x}}{\sqrt{y^4 y}} = \frac{\sqrt{x^2}\sqrt{x}}{\sqrt{y^4}\sqrt{y}} = \frac{x\sqrt{x}}{y^2\sqrt{y}} = \frac{x\sqrt{x}\sqrt{y}}{y^2\sqrt{y}^2} = \frac{x\sqrt{xy}}{y^3}$;
  - $\sqrt[3]{\frac{x}{y}} = \frac{\sqrt[3]{x}}{\sqrt[3]{y}} = \frac{\sqrt[3]{x}\sqrt[3]{y}^2}{\sqrt[3]{y}^3} = \frac{\sqrt[3]{xy^2}}{y}$;

Dividing Radicals

- To divide two radical expressions, we rewrite the quotient as a ratio and then simplify, for instance, $\sqrt[n]{a} \div \sqrt[n]{b} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$;
- Example: Divide and simplify:
  - $\sqrt{10} \div \sqrt{5} = \frac{\sqrt{5 \cdot 2}}{\sqrt{5}} = \frac{\sqrt{5}\sqrt{2}}{\sqrt{5}} = \sqrt{2}$;
  - $(3\sqrt{2}) \div (2\sqrt{3}) = \frac{3\sqrt{2}}{2\sqrt{3}} = \frac{3\sqrt{2}\sqrt{3}}{2\sqrt{3}^2} = \frac{3\sqrt{2 \cdot 3}}{2 \cdot 3} = \frac{\sqrt{6}}{2}$;
  - $\sqrt[3]{10x^2} \div \sqrt[3]{5x} = \frac{\sqrt[3]{10x^2}}{\sqrt[3]{5x}} = \sqrt[3]{\frac{10x^2}{5x}} = \sqrt[3]{2x}$;
- Example: Divide and simplify:
  - $\sqrt{12} \div \sqrt{72x} = \frac{\sqrt{12}}{\sqrt{72x}} = \frac{\sqrt{4 \cdot 3}}{\sqrt{36 \cdot 2x}} = \frac{2\sqrt{3}}{6\sqrt{2x}} = \frac{\sqrt{3}\sqrt{2x}}{3\sqrt{2x}^2} = \frac{\sqrt{6x}}{6x}$;
  - $\sqrt[4]{16a} \div \sqrt[4]{a^5} = \frac{\sqrt[4]{16a}}{\sqrt[4]{a^4 a}} = \frac{2\sqrt[4]{a}}{a\sqrt[4]{a}} = \frac{2}{a}$;

Dividing Radicals: One More Example

- Example: Simplify:
  - $\dfrac{4-\sqrt{12}}{4}=\dfrac{4-\sqrt{4\cdot 3}}{4}=\dfrac{4-2\sqrt{3}}{4}=\dfrac{2(2-\sqrt{3})}{4}=\dfrac{2-\sqrt{3}}{2}$;
  - $\dfrac{-6+\sqrt{20}}{-2}=\dfrac{-6+\sqrt{4\cdot 5}}{-2}=\dfrac{-6+2\sqrt{5}}{-2}=\dfrac{-2(3-\sqrt{5})}{-2}=3-\sqrt{5}$;

Rationalizing Denominators Using Conjugates

- Recall that multiplying a sum including radicals by the conjugate results in an expression without radicals;
- This property may be used to simplify fractions;
- Example: Write in simplified form:
  - $\frac{2+\sqrt{3}}{4-\sqrt{3}} = \frac{(2+\sqrt{3})(4+\sqrt{3})}{(4-\sqrt{3})(4+\sqrt{3})} = \frac{8+2\sqrt{3}+4\sqrt{3}+\sqrt{3}^2}{4^2-\sqrt{3}^2} = \frac{8+2\sqrt{3}+4\sqrt{3}+3}{16-3} = \frac{11+6\sqrt{3}}{13}$;
  - $\frac{\sqrt{5}}{\sqrt{6}+\sqrt{2}} = \frac{\sqrt{5}(\sqrt{6}-\sqrt{2})}{(\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})} = \frac{\sqrt{5}\sqrt{6}-\sqrt{5}\sqrt{2}}{\sqrt{6}^2-\sqrt{2}^2} = \frac{\sqrt{30}-\sqrt{10}}{6-2} = \frac{\sqrt{30}-\sqrt{10}}{4}$;

Powers of Radical Expressions

- Recall, once more, the following rules:
  - $(ab)^n = a^n b^n$;
  - $(a^m)^n = a^{mn}$;
  - $\sqrt[n]{a}^m = \sqrt[n]{a^m}$;
- Example: Simplify the following expressions:
  - $(5\sqrt{2})^3 = 5^3\sqrt{2}^3 = 125 \cdot 2\sqrt{2} = 250\sqrt{2}$;
  - $(2\sqrt{x^3})^4 = 2^4\sqrt{x^3}^4 = 16\sqrt{x^{12}} = 16x^6$;
  - $(3w\sqrt[3]{2w})^3 = 3^3w^3\sqrt[3]{2w}^3 = 27w^3 \cdot 2w = 54w^4$;
  - $(2t\sqrt[4]{3t})^3 = 2^3t^3\sqrt[4]{3t}^3 = 8t^3\sqrt[4]{27t^3}$;

Subsection 5

Solving Equations With Radicals and Exponents

The Odd-Root Property

Odd-Root Property

If n is odd, then for any k,

$$x^n = k \quad \text{implies} \quad x = \sqrt[n]{k}.$$

- Example: Solve each equation:
  - $x^3 = 27$
    According to the Odd-Root Property, we get
    $$x^3 = 27 \Rightarrow x = \sqrt[3]{27} \Rightarrow x = 3;$$
  - $x^5 + 32 = 0$
    $$x^5 + 32 = 0 \Rightarrow x^5 = -32 \Rightarrow x = \sqrt[5]{-32} \Rightarrow x = -2;$$
  - $(x-2)^3 = 24$
    $(x-2)^3 = 24 \Rightarrow x - 2 = \sqrt[3]{24} \Rightarrow x = 2 + \sqrt[3]{8 \cdot 3} \Rightarrow x = 2 + \sqrt[3]{8}\sqrt[3]{3} \Rightarrow$
    $x = 2 + 2\sqrt[3]{3};$

The Even-Root Property

Even-Root Property

If n is even, then

- If $k > 0$, then $x^n = k$ is equivalent to $x = \pm\sqrt[n]{k}$;
- If $k = 0$, then $x^n = k(= 0)$ is equivalent to $x = 0$;
- If $k < 0$, then $x^n = k$ has no real solution;

- Example: Solve each equation:
  - $x^2 = 10$
    $x^2 = 10 \Rightarrow x = \pm\sqrt{10}$;
  - $w^8 = 0$
    $w^8 = 0 \Rightarrow w = 0$;
  - $x^4 = -4$
    This equation has no real solutions!

The Even-Root Property: More Examples

- Solve each of the following equations:
  - $(x-3)^2 = 4$
    $(x-3)^2 = 4 \Rightarrow x-3 = \pm\sqrt{4} \Rightarrow x = 3 \pm 2 \Rightarrow x = 1$ or $x = 5$;
  - $2(x-5)^2 - 7 = 0$
    $$2(x-5)^2 - 7 = 0 \Rightarrow 2(x-5)^2 = 7 \Rightarrow (x-5)^2 = \frac{7}{2} \Rightarrow x-5 = \pm\sqrt{\frac{7}{2}} \Rightarrow$$
    $$x = 5 \pm \frac{\sqrt{7}}{\sqrt{2}} \Rightarrow x = 5 \pm \frac{\sqrt{7}\sqrt{2}}{\sqrt{2}^2} \Rightarrow x = 5 \pm \frac{\sqrt{14}}{2} \Rightarrow x = \frac{10 \pm \sqrt{14}}{2};$$
  - $x^4 - 1 = 80$
    $x^4 - 1 = 80 \Rightarrow x^4 = 81 \Rightarrow x = \pm\sqrt[4]{81} \Rightarrow x = \pm 3$;

Equations Involving Radicals

- Sometimes in solving equations involving radicals, one obtains solutions that are not solutions of the original equation;
- This makes it necessary to check all solutions and discard all those that do not satisfy the original equation;
- Example: Solve the following equations:
  - $\sqrt{2x-3}-5=0$
    $\sqrt{2x-3}-5=0 \Rightarrow \sqrt{2x-3}=5 \Rightarrow (\sqrt{2x-3})^2=5^2 \Rightarrow 2x-3=25 \Rightarrow 2x=28 \Rightarrow x=14$; This is an admissible solution!
  - $\sqrt[3]{3x+5}=\sqrt[3]{x-1}$
    $\sqrt[3]{3x+5}=\sqrt[3]{x-1} \Rightarrow (\sqrt[3]{3x+5})^3=(\sqrt[3]{x-1})^3 \Rightarrow 3x+5=x-1 \Rightarrow 2x=-6 \Rightarrow x=-3$; This is an admissible solution!
  - $\sqrt{3x+18}=x$
    $\sqrt{3x+18}=x \Rightarrow (\sqrt{3x+18})^2=x^2 \Rightarrow 3x+18=x^2 \Rightarrow x^2-3x-18=0 \Rightarrow (x+3)(x-6)=0 \Rightarrow x=-3$ or $x=6$; Test whether they satisfy the original equation: $x=-3$ does not work; Only $x=6$ is an admissible solution!

Equations Involving Radicals: Squaring Twice

- Sometimes to solve an equation involving radicals, we need to square twice;
- Example: Solve the radical equation $\sqrt{5x-1} - \sqrt{x+2} = 1$;

$$\begin{aligned}
&\sqrt{5x-1} - \sqrt{x+2} = 1 \Rightarrow \sqrt{5x-1} = 1 + \sqrt{x+2} \\
&\Rightarrow (\sqrt{5x-1})^2 = (1+\sqrt{x+2})^2 \\
&\Rightarrow 5x - 1 = 1^2 + 2\sqrt{x+2} + (\sqrt{x+2})^2 \\
&\Rightarrow 5x - 1 = 1 + 2\sqrt{x+2} + x + 2 \\
&\Rightarrow 4x - 4 = 2\sqrt{x+2} \Rightarrow 2x - 2 = \sqrt{x+2} \\
&\Rightarrow (2x-2)^2 = (\sqrt{x+2})^2 \Rightarrow 4x^2 - 2 \cdot 2x \cdot 2 + 2^2 = x + 2 \\
&\Rightarrow 4x^2 - 8x + 4 = x + 2 \Rightarrow 4x^2 - 9x + 2 = 0 \\
&\Rightarrow (x-2)(4x-1) = 0 \Rightarrow x = \frac{1}{4} \text{ or } x = 2;
\end{aligned}$$

Check the solutions for admissibility: $x = \frac{1}{4}$ is not admissible; Only $x = 2$ is an admissible solution!

Equations Involving Rational Exponents

- Recall that $a^{m/n} = \sqrt[n]{a}^m = \sqrt[n]{a^m}$;
- Example: Solve the following equations:
  - $x^{2/3} = 4$
    $x^{2/3} = 4 \Rightarrow \sqrt[3]{x^2} = 4 \Rightarrow (\sqrt[3]{x^2})^3 = 4^3 \Rightarrow x^2 = 64 \Rightarrow x = \pm\sqrt{64} \Rightarrow x = -8$ or $x = 8$; Both are admissible solutions!
  - $(w-1)^{-2/5} = 4$
    $(w-1)^{-2/5} = 4 \Rightarrow ((w-1)^{-2/5})^{-5} = 4^{-5} \Rightarrow (w-1)^{(-\frac{2}{5})\cdot(-5)} = 4^{-5} \Rightarrow (w-1)^2 = \frac{1}{4^5} \Rightarrow (w-1)^2 = \frac{1}{1024} \Rightarrow w-1 = \pm\sqrt{\frac{1}{1024}} \Rightarrow w = 1 \pm \frac{1}{32} \Rightarrow w = \frac{31}{32}$ or $w = \frac{33}{32}$; Both are admissible solutions!

One More Example

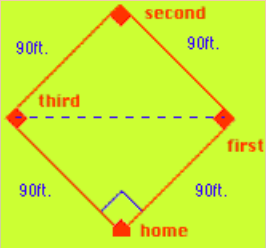
- Solve $(2x-3)^{-2/3} = -1$;

$$\begin{aligned}
&(2x-3)^{-2/3} = -1 \\
&\Rightarrow ((2x-3)^{-2/3})^{-3} = (-1)^{-3} \\
&\Rightarrow (2x-3)^{(-\frac{2}{3})\cdot(-3)} = \frac{1}{(-1)^3} \\
&\Rightarrow (2x-3)^2 = -1;
\end{aligned}$$

This equation does not have any real solutions!

Application: Baseball Diamond

- A baseball diamond is a square, 90 feet on each side; What is the distance from third to first base?



Let x denote the distance from third to first base; We use the Pythagorean Theorem to get

$$
\begin{aligned}
&90^2 + 90^2 = x^2 \quad \Rightarrow \quad 2 \cdot 90^2 = x^2 \\
&\Rightarrow \quad \pm\sqrt{2 \cdot 90^2} = x \quad \Rightarrow \quad x = +\sqrt{2}\sqrt{90^2} \\
&\Rightarrow \quad x = 90\sqrt{2};
\end{aligned}
$$

Therefore the distance from third to first is $90\sqrt{2}$ feet;