Intermediate Algebra

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Subsection 1

Factoring and Completing the Square

Review of Factoring

Zero-Factor Property

$ab = 0$ is equivalent to $a = 0$ or $b = 0$.

- To solve a quadratic equation $ax^2 + bx + c = 0$ for x,
  1. Write the equation with 0 on one side;
  2. Factor the other side;
  3. Use the Zero-Factor Property;
  4. Solve the simpler equations;
- Example: Solve $3x^2 - 4x = 15$ by factoring;

$$\begin{aligned} & 3x^2 - 4x = 15 \Rightarrow 3x^2 - 4x - 15 = 0 \\ & \Rightarrow (3x+5)(x-3) = 0 \\ & \Rightarrow 3x + 5 = 0 \text{ or } x - 3 = 0 \\ & \Rightarrow x = -\frac{5}{3} \text{ or } x = 3; \end{aligned}$$

Review of the Even-Root Property

Even-Root Property

If n is even, then

- If $k > 0$, then $x^n = k$ is equivalent to $x = \pm\sqrt[n]{k}$;
- If $k = 0$, then $x^n = k (= 0)$ is equivalent to $x = 0$;
- If $k < 0$, then $x^n = k$ has no real solution;

- Example: Solve $(x-1)^2 = 9$;

$$\begin{aligned} &(x-1)^2 = 9 \\ &\Rightarrow x - 1 = \pm\sqrt{9} \\ &\Rightarrow x = 1 \pm 3 \\ &\Rightarrow x = -2 \text{ or } x = 4; \end{aligned}$$

Completing the Square I

- If we start with the binomial $x^2 + bx$ and add $(\frac{b}{2})^2$, then we get
$$x^2 + bx + (\frac{b}{2})^2 = x^2 + 2 \cdot x \cdot \frac{b}{2} + (\frac{b}{2})^2 = (x + \frac{b}{2})^2;$$
- This process is called **completing the square**;
- Note we are adding the square of one-half of b;
- Example: Find the perfect square trinomial whose first two terms are given:
  - $x^2 + 8x$
    We must add the square of one-half of 8, i.e. $(\frac{8}{2})^2 = 16$. Thus, the perfect square trinomial is $x^2 + 8x + 16 = (x + 4)^2$;
  - $x^2 - 5x$
    We must add the square of one-half of 5, i.e. $(\frac{5}{2})^2 = \frac{25}{4}$. Thus, the perfect square trinomial is $x^2 - 5x + \frac{25}{4} = (x - \frac{5}{2})^2$;
  - $x^2 + \frac{4}{7}x$
    We must add the square of one-half of $\frac{4}{7}$, i.e. $(\frac{2}{7})^2 = \frac{4}{49}$. Thus, the perfect square trinomial is $x^2 + \frac{4}{7}x + \frac{4}{49} = (x + \frac{2}{7})^2$;

Factoring Perfect Square Trinomials

- To factor a trinomial, we must keep in mind the forms:

$$(a+b)^2 = a^2 + 2 \cdot a \cdot b + b^2 \quad \text{and} \quad (a-b)^2 = a^2 - 2 \cdot a \cdot b + b^2;$$

- Example: Factor each trinomial:
  - $x^2 + 12x + 36$
    We attempt to match one of the right-hand sides above:
    $$x^2 + 12x + 36 = x^2 + 2 \cdot x \cdot 6 + 6^2 = (x+6)^2;$$
  - $y^2 - 7y + \frac{49}{4}$
    We attempt to match one of the right-hand sides above:
    $$y^2 - 7y + \frac{49}{4} = y^2 - 2 \cdot y \cdot \frac{7}{2} + \left(\frac{7}{2}\right)^2 = \left(y - \frac{7}{2}\right)^2;$$
  - $z^2 - \frac{4}{3}z + \frac{4}{9}$
    We attempt to match one of the right-hand sides above:
    $$z^2 - \frac{4}{3}z + \frac{4}{9} = z^2 - 2 \cdot z \cdot \frac{2}{3} + \left(\frac{2}{3}\right)^2 = \left(z - \frac{2}{3}\right)^2;$$

Solving an Equation by Completing the Square

- Solve $x^2 + 6x + 5 = 0$ by completing the square;

  We consider $x^2 + 6x$; To complete this square, we must add the square of one-half of 6, i.e., 9; We must do this to both sides so as to obtain an equivalent equation!

$$\begin{aligned} & x^2 + 6x + 5 = 0 \\ & \Rightarrow (x^2 + 6x + 9) + 5 = 9 \\ & \Rightarrow x^2 + 2 \cdot x \cdot 3 + 3^2 = 9 - 5 \\ & \Rightarrow (x+3)^2 = 4 \\ & \Rightarrow x + 3 = \pm\sqrt{4} \\ & \Rightarrow x + 3 = -2 \text{ or } x + 3 = 2 \\ & \Rightarrow x = -5 \text{ or } x = -1; \end{aligned}$$

Strategy for Solving by Completing the Square

- To solve a quadratic equation $ax^2 + bx + c = 0$ by completing the square, we apply the following steps:
  1. If $a \neq 1$, divide each side of the equation by a;
  2. Leave only the x^2 and x terms on the left;
  3. Add to both sides the square of one-half the coefficient of x;
  4. Factor the left-hand side as the square of a binomial;
  5. Apply the even-root property;
  6. Solve for x;
- Example: Solve $2x^2 + 3x - 2 = 0$ by completing the square;

$$\begin{array}{ll} 2x^2 + 3x - 2 = 0 & \Rightarrow \quad x^2 + \frac{3}{2}x - 1 = 0 \\ \quad \Rightarrow \quad x^2 + \frac{3}{2}x = 1 & \Rightarrow \quad x^2 + \frac{3}{2}x + \left(\frac{3}{4}\right)^2 = 1 + \frac{9}{16} \\ \quad \Rightarrow \quad \left(x + \frac{3}{4}\right)^2 = \frac{25}{16} & \Rightarrow \quad x + \frac{3}{4} = \pm\sqrt{\frac{25}{16}} \\ \quad \Rightarrow \quad x + \frac{3}{4} = -\frac{5}{4} \text{ or } x + \frac{3}{4} = \frac{5}{4} & \\ \quad \Rightarrow \quad x = -2 \text{ or } x = \frac{1}{2}; & \end{array}$$

Completing the Square II

- Example: Solve $x^2 - 3x - 6 = 0$ by completing the square;

$$\begin{aligned} & x^2 - 3x - 6 = 0 \\ & \quad \Rightarrow \quad x^2 - 3x = 6 \\ & \quad \Rightarrow \quad x^2 - 3x + \left(\tfrac{3}{2}\right)^2 = 6 + \tfrac{9}{4} \\ & \quad \Rightarrow \quad \left(x - \tfrac{3}{2}\right)^2 = \tfrac{33}{4} \\ & \quad \Rightarrow \quad x - \tfrac{3}{2} = \pm\sqrt{\tfrac{33}{4}} \\ & \quad \Rightarrow \quad x - \tfrac{3}{2} = -\tfrac{\sqrt{33}}{2} \text{ or } x - \tfrac{3}{2} = \tfrac{\sqrt{33}}{2} \\ & \quad \Rightarrow \quad x = \frac{3 - \sqrt{33}}{2} \text{ or } x = \frac{3 + \sqrt{33}}{2}; \end{aligned}$$

Equations Containing Radicals

- Solve $x+3=\sqrt{153-x}$;

$$\begin{aligned} & x+3=\sqrt{153-x} \\ & \Rightarrow (x+3)^2=(\sqrt{153-x})^2 \\ & \Rightarrow x^2+6x+9=153-x \\ & \Rightarrow x^2+7x-144=0 \\ & \Rightarrow (x+16)(x-9)=0 \\ & \Rightarrow x+16=0 \text{ or } x-9=0 \\ & \Rightarrow x=-16 \text{ or } x=9; \end{aligned}$$

Check the solutions! $x=-16$ is not admissible!
Only $x=9$ is an admissible solution!

Equations Containing Rational Expressions

- Solve the equation $\frac{1}{x} + \frac{3}{x-2} = \frac{5}{8}$;

$$\begin{aligned}
&\frac{1}{x} + \frac{3}{x-2} = \frac{5}{8} \\
&\Rightarrow 8x(x-2)\left(\frac{1}{x} + \frac{3}{x-2}\right) = 8x(x-2)\cdot\frac{5}{8} \\
&\Rightarrow 8(x-2) + 24x = 5x(x-2) \\
&\Rightarrow 8x - 16 + 24x = 5x^2 - 10x \\
&\Rightarrow 5x^2 - 42x + 16 = 0 \\
&\Rightarrow (5x-2)(x-8) = 0 \\
&\Rightarrow 5x - 2 = 0 \text{ or } x - 8 = 0 \\
&\Rightarrow x = \tfrac{2}{5} \text{ or } x = 8;
\end{aligned}$$

Check whether each solution works!
Both $x = \frac{2}{5}$ and $x = 8$ are admissible!

Subsection 2

The Quadratic Formula

The Quadratic Formula

- The solution to $ax^2 + bx + c = 0$, with $a \neq 0$, is given by the formula
$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.}$$
- Usually, we first compute the **discriminant** $D = b^2 - 4ac$;
If $D \geq 0$, then we apply $x = \dfrac{-b \pm \sqrt{D}}{2a}$; If $D < 0$, there are no real solutions;
- Example: Solve the equation $x^2 + 2x - 15 = 0$ using the quadratic formula;

We have $a = 1, b = 2$ and $c = -15$; Therefore,
$D = b^2 - 4ac = 2^2 - 4 \cdot 1 \cdot (-15) = 4 + 60 = 64$; Hence
$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-2 \pm \sqrt{64}}{2 \cdot 1} = \frac{-2 \pm 8}{2} = \begin{cases} -5 \\ 3 \end{cases}$$
Thus, the two solutions are $x = -5$ and $x = 3$;

Using the Formula II

- Example: Solve the equation $4x^2 = 12x - 9$ using the quadratic formula;

  We first make the right-hand side zero: $4x^2 - 12x + 9 = 0$; We have $a = 4, b = -12$ and $c = 9$; Therefore,
  $D = b^2 - 4ac = (-12)^2 - 4 \cdot 4 \cdot 9 = 144 - 144 = 0$; Hence

  $$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-12) \pm \sqrt{0}}{2 \cdot 4} = \frac{12}{8} = \frac{3}{2}$$

  Thus, the only solution is $x = \frac{3}{2}$;

Using the Formula III

- Example: Solve the equation $\frac{1}{3}x^2 + x + \frac{1}{2} = 0$ using the quadratic formula;

  We first multiply both sides by 6 to clear denominators and make life easier: $6(\frac{1}{3}x^2 + x + \frac{1}{2}) = 0$, whence we get $2x^2 + 6x + 3 = 0$; We have $a = 2, b = 6$ and $c = 3$; Therefore, $D = b^2 - 4ac = 6^2 - 4 \cdot 2 \cdot 3 = 36 - 24 = 12$; Hence

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-6 \pm \sqrt{12}}{2 \cdot 2} = \frac{-6 \pm 2\sqrt{3}}{4} = \frac{-3 \pm \sqrt{3}}{2};$$

  Thus, the solutions are $x = \dfrac{-3 - \sqrt{3}}{2}$ and $x = \dfrac{-3 + \sqrt{3}}{2}$;

Summary of Methods for Solving $ax^2 + bx + c = 0$

- We have developed several methods for solving $ax^2 + bx + c = 0$:
  1. **Even-Root Property**: This, we use when $b = 0$, i.e., there is no x-term; E.g., $(x-2)^2 = 8 \Rightarrow x - 2 = \pm\sqrt{8} \Rightarrow x = 2 \pm 2\sqrt{2}$;
  2. **Factoring**: This we use whenever we are able to factor; E.g., $x^2 + 5x + 6 = 0 \Rightarrow (x+3)(x+2) = 0 \Rightarrow x + 3 = 0$ or $x + 2 = 0 \Rightarrow$ $x = -3$ or $x = -2$;
  3. **Quadratic Formula**: This solves any quadratic equation (the most powerful weapon); E.g.,
     $$x^2 + 5x + 3 = 0 \Rightarrow x = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1} \Rightarrow x = \frac{-5 \pm \sqrt{13}}{2};$$
  4. **Completing Square**: Also solves any quadratic, but is slower than the quadratic formula; E.g., $x^2 - 6x + 7 = 0 \Rightarrow x^2 - 6x = -7 \Rightarrow$ $x^2 - 6x + 9 = -7 + 9 \Rightarrow (x-3)^2 = 2 \Rightarrow x - 3 = \pm\sqrt{2} \Rightarrow x = 3 \pm \sqrt{2}$;

Number of Solutions

- A byproduct of computing $D = b^2 - 4ac$ in the application of the quadratic formula is that we can tell right away how many solutions $ax^2 + bx + c = 0$ has:
  - If $D > 0$, it has two real solutions;
  - If $D = 0$, it has one real solution;
  - If $D < 0$, it does not have any real solutions;
- Example: Determine the number of real solutions of the given quadratic; You do not need to find the solutions (if there are any);
  - $x^2 - 3x - 5 = 0$
    $D = b^2 - 4ac = (-3)^2 - 4 \cdot 1 \cdot (-5) = 9 + 20 = 29 > 0$; Therefore, $x^2 - 3x - 5 = 0$ has two real solutions;
  - $x^2 = 3x - 9$ Rewrite $x^2 - 3x + 9 = 0$;
    $D = b^2 - 4ac = (-3)^2 - 4 \cdot 1 \cdot 9 = 9 - 36 = -27 < 0$; Therefore, $x^2 = 3x - 9 = 0$ has no real solutions;
  - $4x^2 - 12x + 9 = 0$
    $D = b^2 - 4ac = (-12)^2 - 4 \cdot 4 \cdot 9 = 144 - 144 = 0$; Therefore, $4x^2 - 12x + 9 = 0$ has one real solution;

Application: Area of Tabletop

- The area of a rectangular tabletop is 6 square feet. If the width is 2 feet shorter than the length, what are its dimensions?

Let x be the width. Then, the length is $x+2$; Therefore, since the area is 6 square feet, we get

$$x(x+2)=6 \Rightarrow x^2+2x=6 \Rightarrow x^2+2x+1=6+1$$
$$\Rightarrow (x+1)^2=7 \Rightarrow x+1=\pm\sqrt{7} \Rightarrow x=-1\pm\sqrt{7};$$

Since only the $+$ will work, we have length $1+\sqrt{7}$ feet and width $-1+\sqrt{7}$ feet;

Subsection 3

More on Quadratic Equations

Writing an Equation with Given Solutions

- Write a quadratic equation having the given pair of solutions
  - $4, -6$
    Reverse the factoring method:
$$\begin{gathered} x = 4 \text{ or } x = -6 \\ x - 4 = 0 \text{ or } x + 6 = 0 \\ (x-4)(x+6) = 0 \\ x^2 + 2x - 24 = 0. \end{gathered}$$
  - $-\sqrt{2}, \sqrt{2}$
    Reverse the factoring method:
$$\begin{gathered} x = -\sqrt{2} \text{ or } x = \sqrt{2} \\ x + \sqrt{2} = 0 \text{ or } x - \sqrt{2} = 0 \\ (x+\sqrt{2})(x-\sqrt{2}) = 0 \\ x^2 - 2 = 0. \end{gathered}$$

Correspondence Between Solutions and Factors

If a and b are solutions to a quadratic equation, then the equation is equivalent to $(x-a)(x-b) = 0$.

Using the Discriminant to Factor

- Recall that a polynomial is **prime** if it cannot be factored into linear factors with integer coefficients;

Testing for Primality Using the Discriminant

If $ax^2 + bx + c$ is a quadratic polynomial with integer coefficients having greatest common divisor 1, then it is prime if and only if its discriminant $D = b^2 - 4ac$ is not a perfect square.

- Example: Determine whether each of the following polynomials can be factored:
  - $6x^2 + x - 15$
    $D = b^2 - 4ac = 1^2 - 4 \cdot 6 \cdot (-15) = 1 + 360 = 361$; Since $361 = 19^2$ is a perfect square, the quadratic polynomial $6x^2 + x - 15$ is not prime, i.e., it can be factored; In fact, $6x^2 + x - 15 = (2x - 3)(3x + 5)$;
  - $5x^2 - 3x + 2$
    $D = b^2 - 4ac = (-3)^2 - 4 \cdot 5 \cdot 2 = 9 - 40 = -31$; this is not a perfect square; so the quadratic polynomial $5x^2 - 3x + 2$ is prime, i.e., it cannot be factored;

Equations Quadratic in Form I

- Solve the equation $(x+15)^2 - 3(x+15) - 18 = 0$;

$$\begin{aligned}
& (x+15)^2 - 3(x+15) - 18 = 0 \\
\overset{y=x+15}{\Rightarrow} \quad & y^2 - 3y - 18 = 0 \\
\Rightarrow \quad & (y+3)(y-6) = 0 \\
\Rightarrow \quad & y+3 = 0 \text{ or } y-6 = 0 \\
\Rightarrow \quad & y = -3 \text{ or } y = 6 \\
\overset{y=x+15}{\Rightarrow} \quad & x+15 = -3 \text{ or } x+15 = 6 \\
\Rightarrow \quad & x = -18 \text{ or } x = -9;
\end{aligned}$$

Equations Quadratic in Form II

- Solve the equation $x^4 - 6x^2 + 8 = 0$;

$$\begin{aligned}
&x^4 - 6x^2 + 8 = 0 \\
&\Rightarrow \quad (x^2)^2 - 6x^2 + 8 = 0 \\
&\overset{y=x^2}{\Rightarrow} \quad y^2 - 6y + 8 = 0 \\
&\Rightarrow \quad (y-2)(y-4) = 0 \\
&\Rightarrow \quad y - 2 = 0 \text{ or } y - 4 = 0 \\
&\Rightarrow \quad y = 2 \text{ or } y = 4 \\
&\overset{y=x^2}{\Rightarrow} \quad x^2 = 2 \text{ or } x^2 = 4 \\
&\Rightarrow \quad x = \pm\sqrt{2} \text{ or } x = \pm\sqrt{4} \\
&\Rightarrow \quad x = -\sqrt{2} \text{ or } x = \sqrt{2} \text{ or } x = -2 \text{ or } x = 2;
\end{aligned}$$

Equations Quadratic in Form III

- Solve the equation $(x^2+2x)^2-11(x^2+2x)+24=0$;

$$\begin{aligned}
& (x^2+2x)^2-11(x^2+2x)+24=0 \\
\stackrel{y=x^2+2x}{\Rightarrow} \quad & y^2-11y+24=0 \\
\Rightarrow \quad & (y-3)(y-8)=0 \\
\Rightarrow \quad & y-3=0 \text{ or } y-8=0 \\
\Rightarrow \quad & y=3 \text{ or } y=8 \\
\stackrel{y=x^2+2x}{\Rightarrow} \quad & x^2+2x=3 \text{ or } x^2+2x=8 \\
\Rightarrow \quad & x^2+2x-3=0 \text{ or } x^2+2x-8=0 \\
\Rightarrow \quad & (x+3)(x-1)=0 \text{ or } (x+4)(x-2)=0 \\
\Rightarrow \quad & x+3=0 \text{ or } x-1=0 \text{ or } x+4=0 \text{ or } x-2=0 \\
\Rightarrow \quad & x=-3 \text{ or } x=1 \text{ or } x=-4 \text{ or } x=2;
\end{aligned}$$

Equations Quadratic in Form IV

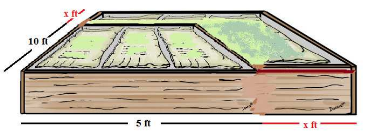
- Solve the equation $x - 9x^{1/2} + 14 = 0$;

$$\begin{aligned} & x - 9x^{1/2} + 14 = 0 \\ \Rightarrow \quad & (x^{1/2})^2 - 9x^{1/2} + 14 = 0 \\ \stackrel{y=x^{1/2}}{\Rightarrow} \quad & y^2 - 9y + 14 = 0 \\ \Rightarrow \quad & (y-2)(y-7) = 0 \\ \Rightarrow \quad & y - 2 = 0 \text{ or } y - 7 = 0 \\ \Rightarrow \quad & y = 2 \text{ or } y = 7 \\ \stackrel{y=x^{1/2}}{\Rightarrow} \quad & x^{1/2} = 2 \text{ or } x^{1/2} = 7 \\ \Rightarrow \quad & (x^{1/2})^2 = 2^2 \text{ or } (x^{1/2})^2 = 7^2 \\ \Rightarrow \quad & x = 4 \text{ or } x = 49; \end{aligned}$$

Application: Increasing Dimensions

- Lorraine's flower bed is rectangular in shape with a length of 10 feet and a width of 5 feet; She wants to increase the length and width by the same amount to obtain a flower bed with an area of 75 square feet; What should the amount of increase be?

  Suppose that she needs to increase both the length and the width by x feet;

  Therefore the new length is $x+10$ feet and the new width is $x+5$ feet; Therefore, since the new area is to be 75 square feet, we get

  $$(x+5)(x+10)=75 \Rightarrow x^2+15x+50=75 \Rightarrow x^2+15x-25=0$$
  $$\Rightarrow x=\frac{-15\pm\sqrt{225-4\cdot 1\cdot(-25)}}{2\cdot 1} \Rightarrow x=\frac{-15\pm\sqrt{325}}{2};$$

  Since only the $+$ will work, we have $x=\dfrac{-15+\sqrt{325}}{2}\approx 1.51$ feet;

Subsection 4

Quadratic Functions and Graphs

Quadratic Functions

Definition of Quadratic Functions

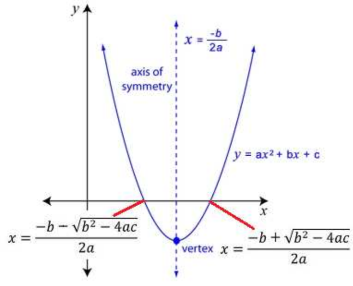
A **quadratic function** is one of the form $f(x) = ax^2 + bx + c$, where a, b, c are real constants, with $a \neq 0$;

- Example: Given the function $f(x)$ and the values for x or y, find the value for y or x, respectively, so that the pair (x, y) be on the graph of the function;
  - $f(x) = x^2 - x - 6$ and $x = 2$;
    $y = f(2) = 2^2 - 2 - 6 = -4$;
  - $f(x) = x^2 - x - 6$ and $y = 0$;
    $f(x) = 0 \Rightarrow x^2 - x - 6 = 0 \Rightarrow (x+2)(x-3) = 0 \Rightarrow x + 2 = 0$ or $x - 3 = 0 \Rightarrow x = -2$ or $x = 3$;
  - $f(x) = -16x^2 + 48x + 84$ and $x = 0$;
    $y = f(0) = 16 \cdot 0^2 + 48 \cdot 0 + 84 = 84$;
  - $f(x) = -16x^2 + 48x + 84$ and $y = 20$;
    $f(x) = 20 \Rightarrow -16x^2 + 48x + 84 = 20 \Rightarrow -16x^2 + 48x + 64 = 0 \Rightarrow x^2 - 3x - 4 = 0 \Rightarrow (x+1)(x-4) = 0 \Rightarrow x + 1 = 0$ or $x - 4 = 0 \Rightarrow x = -1$ or $x = 4$;

Graphing Quadratic Functions

- The graph of the quadratic function is a **parabola** opening either up or down;

1. The **vertex** is the lowest or highest point; Its x-coordinate is $x = -\frac{b}{2a}$;
2. The parabola opens up if $a > 0$ and down if $a < 0$;
3. Its y-intercept is the point $(0, c)$;

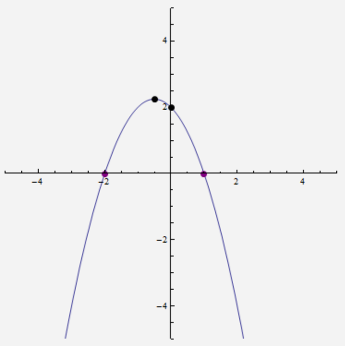


4. Finally, its x-intercepts are the points with $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$;

Examples of Quadratic Function Graphs I

- Find the vertex, the opening direction, the intercepts and sketch the graph of $f(x) = -x^2 - x + 2$;

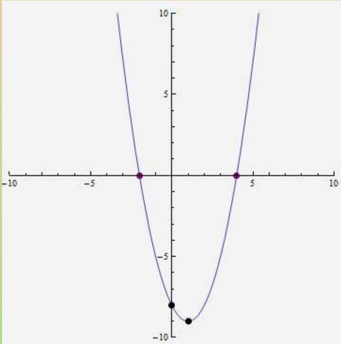
  1. The vertex has x-coordinate $x = -\frac{b}{2a} = -\frac{-1}{2\cdot(-1)} = -\frac{1}{2}$; Its y-coordinate, therefore, is $y = f(-\frac{1}{2}) = -(-\frac{1}{2})^2 - (-\frac{1}{2}) + 2 = -\frac{1}{4} + \frac{1}{2} + 2 = \frac{9}{4}$;
  2. The parabola opens down since $a = -1 < 0$;
  3. Its y-intercept is $(0, 2)$;



  4. Finally, its x-intercepts are the solutions of $-x^2 - x + 2 = 0 \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x+2)(x-1) = 0 \Rightarrow x + 2 = 0$ or $x - 1 = 0 \Rightarrow x = -2$ or $x = 1$;

Examples of Quadratic Function Graphs II

- Find the vertex, the opening direction, the intercepts and sketch the graph of $f(x) = x^2 - 2x - 8$;

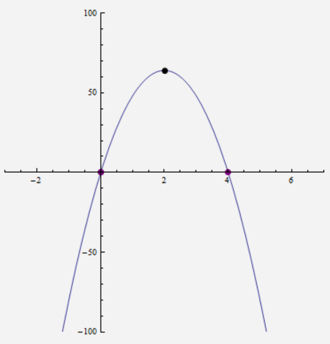


1. The vertex has x-coordinate $x = -\frac{b}{2a} = -\frac{-2}{2 \cdot 1} = 1$; Its y-coordinate, therefore, is $y = f(1) = 1^2 - 2 \cdot 1 - 8 = 1 - 2 - 8 = -9$;
2. The parabola opens up since $a = 1 > 0$;
3. Its y-intercept is $(0, -8)$;
4. Finally, its x-intercepts are the solutions of $x^2 - 2x - 8 = 0 \Rightarrow (x+2)(x-4) = 0 \Rightarrow x + 2 = 0$ or $x - 4 = 0 \Rightarrow x = -2$ or $x = 4$;

Examples of Quadratic Function Graphs III

- Find the vertex, the opening direction, the intercepts and sketch the graph of $f(x) = -16x^2 + 64x$;

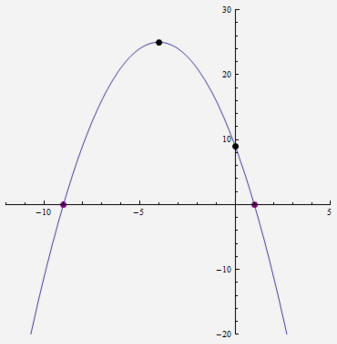
  1. The vertex has x-coordinate $x = -\frac{b}{2a} = -\frac{64}{2\cdot(-16)} = 2$; Its y-coordinate, therefore, is $y = f(2) = -16\cdot 2^2 + 64\cdot 2 = -64 + 128 = 64$;
  2. The parabola opens down since $a = -16 < 0$;
  3. Its y-intercept is $(0, 0)$;



  4. Finally, its x-intercepts are the solutions of $-16x^2 + 64x = 0 \Rightarrow x^2 - 4x = 0 \Rightarrow x(x-4) = 0 \Rightarrow x = 0$ or $x - 4 = 0 \Rightarrow x = 0$ or $x = 4$;

Examples of Quadratic Function Graphs IV

- Find the vertex, the opening direction, the intercepts and sketch the graph of $f(x) = -x^2 - 8x + 9$;



1. The vertex has x-coordinate $x = -\frac{b}{2a} = -\frac{-8}{2\cdot(-1)} = -4$; Its y-coordinate, therefore, is $y = f(-4) = -(-4)^2 - 8\cdot(-4) + 9 = -16 + 32 + 9 = 25$;
2. The parabola opens down since $a = -1 < 0$;
3. Its y-intercept is $(0, 9)$;
4. Finally, its x-intercepts are the solutions of $-x^2 - 8x + 9 = 0 \Rightarrow x^2 + 8x - 9 = 0 \Rightarrow (x+9)(x-1) = 0 \Rightarrow x + 9 = 0$ or $x - 1 = 0 \Rightarrow x = -9$ or $x = 1$;

Subsection 5

Quadratic Inequalities

Solving Quadratic Inequalities Graphically

- A quadratic inequality has one of the following forms

$$ax^2 + bx + c > 0, \qquad ax^2 + bx + c \geq 0,$$
$$ax^2 + bx + c < 0, \qquad ax^2 + bx + c \leq 0,$$

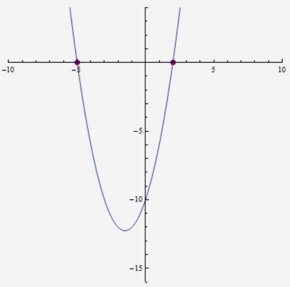
where a, b, c are real constants and $a \neq 0$;

Strategy for Solving Graphically

1. Rewrite the inequality in one of these four forms (one side zero);
2. Find the roots of the quadratic polynomial;
3. Graph the parabola passing through the x-intercepts found in the previous step;
4. Read the solution set to the inequality from the graph;

Applying the Method I

- Solve the quadratic inequality $x^2 + 3x > 10$;
Rewrite $x^2 + 3x - 10 > 0$; Find the roots of $x^2 + 3x - 10 = 0 \Rightarrow$ $(x+5)(x-2) = 0 \Rightarrow x + 5 = 0$ or $x - 2 = 0 \Rightarrow x = -5$ or $x = 2$;
Roughly sketch the graph (we know intercepts and that parabola opens up):

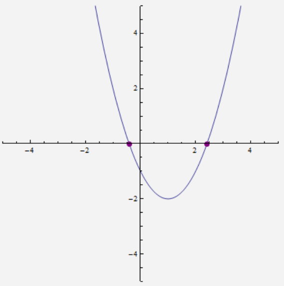


Since we want $y = x^2 + 3x - 10 > 0$, we have to pick

$$x < -5 \text{ or } x > 2; \text{ in interval notation } x \text{ in } (-\infty, -5) \cup (2, \infty);$$

Applying the Method II

- Solve the quadratic inequality $x^2 - 2x - 1 \leq 0$;
  The form $x^2 - 2x - 1 \leq 0$ is the one we want; Find the roots of $x^2 - 2x - 1 = 0 \Rightarrow x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow x = \dfrac{2 \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-1)}}{2 \cdot 1} \Rightarrow x = \dfrac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$; Roughly sketch the graph (we know intercepts and that parabola opens up):



Since we want $y = x^2 - 2x - 1 \leq 0$, we have to pick

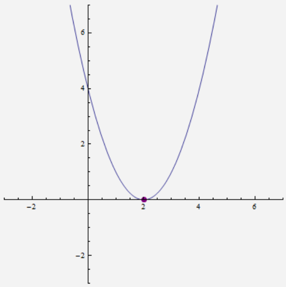
$$1 - \sqrt{2} \leq x \leq 1 + \sqrt{2};$$

or, in interval notation

$$x \text{ in } [1 - \sqrt{2}, 1 + \sqrt{2}];$$

Applying the Method III

- Solve the quadratic inequality $x^2 - 4x + 4 \leq 0$;
  The form $x^2 - 4x + 4 \leq 0$ is the one we want; Find the roots of $x^2 - 4x + 4 = 0 \Rightarrow (x-2)^2 = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2$; Roughly sketch the graph (we know intercept and that parabola opens up):



Since we want $y = x^2 - 4x + 4 \leq 0$, we have to pick

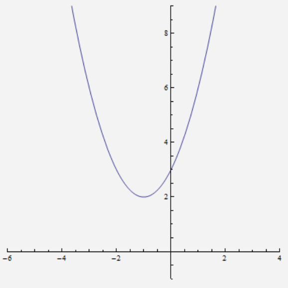
$$x = 2;$$

or, in interval notation

$$x \text{ in } [2,2] = \{2\};$$

Applying the Method IV

- Solve the quadratic inequality $x^2+2x+3>0$;
  The form $x^2+2x+3>0$ is the one we want; Find the roots of $x^2+2x+3=0 \Rightarrow x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a} \Rightarrow x=\dfrac{-2\pm\sqrt{2^2-4\cdot 1\cdot 3}}{2\cdot 1} \Rightarrow x=\dfrac{-2\pm\sqrt{-8}}{2}$; Thus, we have no real roots and the parabola opens up:



Since we want $y=x^2+2x+3>0$, we have to pick

$$x \text{ any real number;}$$

or, in interval notation

$$x \text{ in } (-\infty,\infty);$$

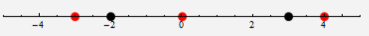
Solving Quadratic Inequalities Using Test Points

Strategy for Solving Using Test Point Method

1. Rewrite the inequality with zero on the right;
2. Solve the equation resulting from replacing inequality by equality;
3. Locate the solutions on the number line;
4. Select a test point on each interval determined by the solutions;
5. Check whether inequality is satisfied by each test point;
6. Write the final solution interval using interval notation;

Example of Solving Inequalities Using Test Points I

- Example: Solve $x^2 - x \leq 6$;
  Rewrite $x^2 - x - 6 \leq 0$;
  Solve $x^2 - x - 6 = 0 \Rightarrow (x+2)(x-3) = 0 \Rightarrow x = -2$ or $x = 3$;
  The black points are the solutions on the real line



The red points are test points in the three intervals formed by the two solutions;
We perform the following tests, using the test points:

Test	Yes/No
$(-3)^2 - (-3) - 6 \stackrel{?}{\leq} 0$	✠
$0^2 - 0 - 6 \stackrel{?}{\leq} 0$	✓
$4^2 - 4 - 6 \stackrel{?}{\leq} 0$	✠

Thus, the solution interval is $[-2, 3]$;

Example of Solving Inequalities Using Test Points II

- Example: Solve $x^2 - 4x > 6$; Rewrite $x^2 - 4x - 6 > 0$; Solve $x^2 - 4x - 6 = 0 \Rightarrow x = -\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot (-6)}}{2 \cdot 1} = \frac{4 \pm \sqrt{40}}{2} = 2 \pm \sqrt{10}$; The black points are the solutions on the real line

The red points are test points in the three intervals formed by the two solutions; We perform the following tests, using the test points:

Test	Yes/No
$(-2)^2 - 4(-2) - 6 \stackrel{?}{>} 0$	✓
$2^2 - 4 \cdot 2 - 6 \stackrel{?}{>} 0$	✠
$6^2 - 4 \cdot 6 - 6 \stackrel{?}{>} 0$	✓

Thus, the solution interval is $(-\infty, 2 - \sqrt{10}) \cup (2 + \sqrt{10}, \infty)$;

Example of Solving Inequalities Using Test Points III

- Example: Solve $x^2 + 6x + 10 \geq 0$;
  Solve $x^2 + 6x + 10 = 0$;
  Note $D = b^2 - 4ac = 6^2 - 4 \cdot 1 \cdot 10 = -4 < 0$; Thus, $x^2 + 6x + 10 = 0$ has no real solutions;
  The black points are the solutions on the real line

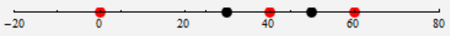
  The red point is a test point in the only interval formed;
  We perform the following test, using the test point:

Test	Yes/No
$0^2 + 6 \cdot 0 + 10 \stackrel{?}{\geq} 0$	✓

  Thus, the solution interval is $(-\infty, \infty)$;

Application: Making a Profit

- Anna's daily profit P (in dollars) for selling x magazine subscriptions is determined by $P(x) = -x^2 + 80x - 1500$; For which values of x is her profit positive?
  We would like to solve $-x^2 + 80x - 1500 > 0$;
  Solve $-x^2 + 80x - 1500 = 0 \Rightarrow x^2 - 80x + 1500 = 0 \Rightarrow$ $(x - 30)(x - 50) = 0 \Rightarrow x = 30$ or $x = 50$;
  The black points are the solutions and the red test points:



We perform the following tests, using the test points:

Test	Yes/No
$-0^2 + 80 \cdot 0 - 1500 \stackrel{?}{>} 0$	✠
$-40^2 + 80 \cdot 40 - 1500 \stackrel{?}{>} 0$	✓
$-60^2 + 80 \cdot 60 - 1500 \stackrel{?}{>} 0$	✠

Thus, profit is positive, when x is in $(30, 50)$, i.e., when Anna sells between 31 and 49 magazine subscriptions;