Introduction to Linear Algebra

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LSSU Math 305

1. Matrix Algebra
   - Matrix Operations
   - The Inverse of a Matrix
   - Characterization of Invertible Matrices
   - Partitioned Matrices
   - Matrix Factorizations

Subsection 1

Matrix Operations

Entries of a Matrix

- If A is an $m \times n$ matrix, then the scalar entry in the ith row and jth column of A is denoted by a_{ij} and is called the (i,j)-**entry** of A.
- For instance, the $(3,2)$-entry is the number a_{32} in the third row, second column.
- Each column of A is a list of m real numbers, which identifies a vector in $\mathbb{R}^m$.
- Often, these columns are denoted by $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$, and the matrix A is written as $A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{bmatrix}$.
- Observe that the number a_{ij} is the ith entry (from the top) of the jth column vector $\boldsymbol{a}_j$.

Diagonal and Zero Matrices

- The **diagonal entries** in an $m \times n$ matrix $A = [a_{ij}]$ are $a_{11}, a_{22}, a_{33}, \ldots$.
- They form the **main diagonal** of A.
- A **diagonal matrix** is a square $n \times n$ matrix whose nondiagonal entries are zero.
- An example is the $n \times n$ identity matrix, I_n.
- An $m \times n$ matrix whose entries are all zero is a **zero matrix** and is written as 0.
- The size of a zero matrix is usually clear from the context.

Addition of Matrices

- We say that two matrices are **equal** if they have the same size (i.e., the same number of rows and the same number of columns) and if their corresponding columns are equal, which amounts to saying that their corresponding entries are equal.
- If A and B are $m \times n$ matrices, then the **sum** $A + B$ is the $m \times n$ matrix whose columns are the sums of the corresponding columns in A and B.
- Since vector addition of the columns is done entrywise, each entry in $A + B$ is the sum of the corresponding entries in A and B.
- The sum $A + B$ is defined only when A and B are the same size.

Example

- Let
$$A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}, C = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix}.$$
- Then
$$A + B = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 6 \\ 2 & 8 & 9 \end{bmatrix}.$$
- $A + C$ is not defined because A and C have different sizes.

Scalar Multiplication

- If r is a scalar and A is a matrix, then the **scalar multiple** rA is the matrix whose columns are r times the corresponding columns in A.
- As with vectors, $-A$ stands for $(-1)A$, and $A-B$ is the same as $A+(-1)B$.

Example: If $A = \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix}$, then

$$\begin{aligned} 2B &= 2\begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix}, \\ A-2B &= \begin{bmatrix} 4 & 0 & 5 \\ -1 & 3 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 2 \\ 6 & 10 & 14 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -2 & 3 \\ -7 & -7 & -12 \end{bmatrix}. \end{aligned}$$

Properties of Addition and Scalar Multiplication

Theorem

Let A, B and C be matrices of the same size, and let r and s be scalars.

(a) $A + B = B + A$;

(b) $(A + B) + C = A + (B + C)$;

(c) $A + 0 = A$;

(d) $r(A + B) = rA + rB$;

(e) $(r + s)A = rA + sA$;

(f) $r(sA) = (rs)A$.

(b) If for a matrix A, we denote $A_{ij} = a_{ij}$, then

$$\begin{aligned} [(A+B)+C]_{ij} &= [A+B]_{ij} + c_{ij} = (a_{ij} + b_{ij}) + c_{ij} \\ &= a_{ij} + (b_{ij} + c_{ij}) = a_{ij} + [B+C]_{ij} \\ &= [A + (B+C)]_{ij}. \end{aligned}$$

(d) Similarly, we have

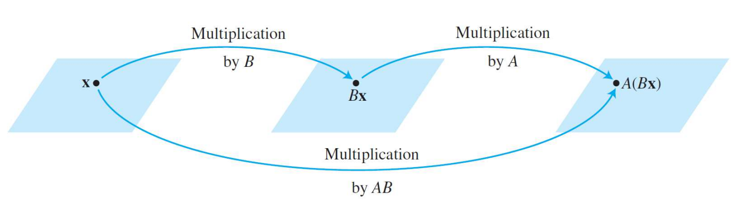
$$\begin{aligned} [r(A+B)]_{ij} &= r[A+B]_{ij} = r(a_{ij} + b_{ij}) \\ &= ra_{ij} + rb_{ij} = [rA]_{ij} + [rB]_{ij} = [rA + rB]_{ij}. \end{aligned}$$

Associativity

- Because of the associative property of addition, we can simply write
$$A + B + C$$
for the sum, which can be computed either as $(A + B) + C$ or as $A + (B + C)$.
- The same applies to sums of four or more matrices.

Composition of Transformations

- When a matrix B multiplies a vector $\boldsymbol{x}$, it transforms $\boldsymbol{x}$ into the vector $B\boldsymbol{x}$.
- If this vector is then multiplied in turn by a matrix A, the resulting vector is $A(B\boldsymbol{x})$.



- Thus $A(B\boldsymbol{x})$ is produced from $\boldsymbol{x}$ by a **composition** of mappings.
- Our goal is to represent this composite mapping as multiplication by a single matrix, denoted by AB, so that

$$A(B\boldsymbol{x}) = (AB)\boldsymbol{x}.$$

The Matrix Corresponding to Composition

- If A is $m \times n$, B is $n \times p$, and $\boldsymbol{x}$ is in $\mathbb{R}^p$, denote the columns of B by $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_p$ and the entries in $\boldsymbol{x}$ by $x_1, \ldots, x_p$.
- Then $B\boldsymbol{x} = x_1\boldsymbol{b}_1 + \cdots + x_p\boldsymbol{b}_p$.
- By the linearity of multiplication by A,

$$\begin{aligned} A(B\boldsymbol{x}) &= A(x_1\boldsymbol{b}_1) + \cdots + A(x_p\boldsymbol{b}_p) \\ &= x_1A\boldsymbol{b}_1 + \cdots + x_pA\boldsymbol{b}_p. \end{aligned}$$

- The vector $A(B\boldsymbol{x})$ is a linear combination of the vectors $A\boldsymbol{b}_1, \ldots, A\boldsymbol{b}_p$, using the entries in $\boldsymbol{x}$ as weights.
- In matrix notation, this linear combination is written as

$$A(B\boldsymbol{x}) = \begin{bmatrix} A\boldsymbol{b}_1 & A\boldsymbol{b}_2 & \cdots & A\boldsymbol{b}_p \end{bmatrix} \boldsymbol{x}.$$

Multiplication of Matrices

Definition

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_p$, then the **product** AB is the $m \times p$ matrix whose columns are $A\boldsymbol{b}_1, \ldots, A\boldsymbol{b}_p$. That is,

$$AB = A \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \cdots & \boldsymbol{b}_p \end{bmatrix} = \begin{bmatrix} A\boldsymbol{b}_1 & A\boldsymbol{b}_2 & \cdots & A\boldsymbol{b}_p \end{bmatrix}.$$

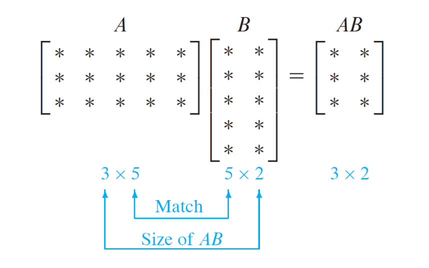
Example

- Compute AB, where $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$.
- Write $B = \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}_3 \end{bmatrix}$ and compute:

$$\begin{aligned} A\boldsymbol{b}_1 &= \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ -1 \end{bmatrix}; \\ A\boldsymbol{b}_2 &= \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 13 \end{bmatrix}; \\ A\boldsymbol{b}_3 &= \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ -9 \end{bmatrix}; \\ AB &= A \begin{bmatrix} \boldsymbol{b}_1 & \boldsymbol{b}_2 & \boldsymbol{b}_3 \end{bmatrix} = \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}. \end{aligned}$$

Example

- If A is a 3×5 matrix and B is a 5×2 matrix, what are the sizes of AB and BA, if they are defined?
- Since A has 5 columns and B has 5 rows, the product AB is defined and is a 3×2 matrix.



- The product BA is not defined because the 2 columns of B do not match the 3 rows of A.

The Row-Column Rule for Multiplication

Row-Column Rule for Computing AB

If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B.

If $(AB)_{ij}$ denotes the (i,j)-entry in AB, and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

- To verify this rule, let $B = \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_p \end{bmatrix}$.

  Column j of AB is $A\boldsymbol{b}_j$. We can compute $A\boldsymbol{b}_j$ by the row-vector rule for computing $A\boldsymbol{x}$. The ith entry in $A\boldsymbol{b}_j$ is the sum of the products of corresponding entries from row i of A and the vector $\boldsymbol{b}_j$. This is precisely the computation described in the rule for computing the (i,j)-entry of AB.

Example

- Calculate the $(1,3)$-entry of the product

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}.$$

- We compute

$$\begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\begin{aligned} (AB)_{13} &= a_{11}b_{13} + a_{12}b_{23} \\ &= 2 \cdot 6 + 3 \cdot 3 \\ &= 12 + 9 = 21. \end{aligned}$$

Example

- Find the entries in the second row of AB, where

$$A = \begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix}.$$

- We compute

$$\begin{bmatrix} 2 & -5 & 0 \\ -1 & 3 & -4 \\ 6 & -8 & -7 \\ -3 & 0 & 9 \end{bmatrix} \begin{bmatrix} 4 & -6 \\ 7 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} \square & \square \\ -4+21-12 & 6+3-8 \\ \square & \square \\ \square & \square \end{bmatrix} = \begin{bmatrix} \square & \square \\ 5 & 1 \\ \square & \square \\ \square & \square \end{bmatrix}.$$

Properties of Matrix Multiplication

Theorem

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

(a) $A(BC) = (AB)C$ (associative law of multiplication);

(b) $A(B + C) = AB + AC$ (**left distributive law**);

(c) $(B + C)A = BA + CA$ (**right distributive law**);

(d) $r(AB) = (rA)B = A(rB)$, for any scalar r;

(e) $I_m A = A = AI_n$ (**identity for matrix multiplication**).

(a) Suppose $C = \begin{bmatrix} \mathbf{c}_1 & \cdots & \mathbf{c}_p \end{bmatrix}$. By the definition of matrix multiplication, we get $BC = \begin{bmatrix} B\mathbf{c}_1 & \cdots & B\mathbf{c}_p \end{bmatrix}$ and
$A(BC) = \begin{bmatrix} A(B\mathbf{c}_1) & \cdots & A(B\mathbf{c}_p) \end{bmatrix}$.
But the definition of AB makes $A(B\mathbf{x}) = (AB)\mathbf{x}$, for all $\mathbf{x}$. So we get

$$A(BC) = \begin{bmatrix} (AB)\mathbf{c}_1 & \cdots & (AB)\mathbf{c}_p \end{bmatrix} = (AB)C.$$

Remarks: Associativity and Commutativity

- The associative and distributive laws say essentially that pairs of parentheses in matrix expressions can be inserted and deleted in the same way as in the algebra of real numbers.
- In particular, we can write ABC for the product, which can be computed either as $A(BC)$ or as $(AB)C$.
- Similarly, a product $ABCD$ of four matrices can be computed as $A(BCD)$ or $(ABC)D$ or $A(BC)D$, and so on.
- It does not matter how we group the matrices when computing the product, so long as the left-to-right order of the matrices is preserved.
- The left-to-right order in products is critical because AB and BA are usually not the same.
- If $AB = BA$, we say that A and B **commute** with one another.

Example

- Let $A = \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix}$.
  Show that these matrices do not commute, i.e., verify that $AB \neq BA$.
- We have

$$\begin{aligned} AB &= \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 3 \\ -2 & -6 \end{bmatrix}; \\ BA &= \begin{bmatrix} 2 & 0 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 29 & -2 \end{bmatrix}. \end{aligned}$$

  So $AB \neq BA$.

Warnings

1. In general, $AB \neq BA$.
2. The cancelation laws do not hold for matrix multiplication.

   That is, if $AB = AC$, then it is not true in general that $B = C$.

$$AB = AC \quad \not\Rightarrow \quad B = C.$$

3. If a product AB is the zero matrix, you cannot conclude in general that either $A = 0$ or $B = 0$.

$$AB = 0 \quad \not\Rightarrow \quad A = 0 \text{ or } B = 0.$$

Powers of a Matrix

- If A is an $n \times n$ matrix and if k is a positive integer, then A^k denotes the product of k copies of A:

$$A^k = \underbrace{A \cdots A}_{k}.$$

- If A is nonzero and if $\boldsymbol{x}$ is in $\mathbb{R}^n$, then $A^k\boldsymbol{x}$ is the result of left-multiplying $\boldsymbol{x}$ by A repeatedly k times.
- If $k = 0$, then $A^0\boldsymbol{x}$ should be $\boldsymbol{x}$ itself.

  Thus A^0 is interpreted as the identity matrix:

$$A^0 = I_n.$$

The Transpose of a Matrix

- Given an $m \times n$ matrix A, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

  Example: Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} -5 & 2 \\ 1 & -3 \\ 0 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \end{bmatrix}.$$

  Then

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \quad B^T = \begin{bmatrix} -5 & 1 & 0 \\ 2 & -3 & 4 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 & -3 \\ 1 & 5 \\ 1 & -2 \\ 1 & 7 \end{bmatrix}.$$

Properties of the Transpose

Theorem

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

(a) $(A^T)^T = A$;

(b) $(A+B)^T = A^T + B^T$;

(c) For any scalar r, $(rA)^T = rA^T$;

(d) $(AB)^T = B^T A^T$.

- The generalization of Part (d) to products of more than two factors can be stated in words as follows:

  The transpose of a product of matrices equals the product of their transposes in the reverse order.

Proof of Property (d)

- Given a matrix A, we denote its (i,j)-entry by A_{ij}.
- Let A be $m \times n$ and B be $n \times p$.

  Then, we have

$$\begin{aligned}(AB)^T_{ij} &= (AB)_{ji} \\ &= A_{j1}B_{1i} + A_{j2}B_{2i} + \cdots + A_{jn}B_{ni} \\ &= A^T_{1j}B^T_{i1} + A^T_{2j}B^T_{i2} + \cdots + A^T_{nj}B^T_{in} \\ &= B^T_{i1}A^T_{1j} + B^T_{i2}A^T_{2j} + \cdots + B^T_{in}A^T_{nj} \\ &= (B^TA^T)_{ij}.\end{aligned}$$

  We conclude that $(AB)^T = B^TA^T$.

Subsection 2

The Inverse of a Matrix

Invertible Matrices

- An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C such that
$$CA = I \quad \text{and} \quad AC = I,$$
where $I = I_n$, the $n \times n$ identity matrix.
- In this case, C is an **inverse** of A.
- In fact, C is uniquely determined by A, because if B were another inverse of A, then $B = BI = B(AC) = (BA)C = IC = C$.
- This unique inverse is denoted by A^{-1}, so that
$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I.$$
- A matrix that is not invertible is sometimes called a **singular matrix**, and an invertible matrix is called a **nonsingular matrix**.

Example

- Consider $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$.
- We have

$$\begin{aligned} AC &= \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \\ CA &= \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

- Thus, $C = A^{-1}$.

Formula for the Inverse of a 2×2 Matrix

Theorem

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

- The quantity $ad - bc$ is called the **determinant** of A, and we write

$$\det A = ad - bc.$$

- The theorem says that a 2×2 matrix A is invertible if and only if $\det A \neq 0$.

Example

- Find the inverse of $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$.
- We have

$$\det A = 3 \cdot 6 - 4 \cdot 5 = -2 \neq 0.$$

So A is invertible.

We have

$$A^{-1} = \frac{1}{-2} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ \frac{5}{2} & -\frac{3}{2} \end{bmatrix}.$$

Invertibility and Solutions of Linear Systems

Theorem

If A is an invertible $n \times n$ matrix, then for each $\boldsymbol{b}$ in $\mathbb{R}^n$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has the unique solution $\boldsymbol{x} = A^{-1}\boldsymbol{b}$.

- Take any $\boldsymbol{b}$ in $\mathbb{R}^n$.

  A solution exists because if $A^{-1}\boldsymbol{b}$ is substituted for $\boldsymbol{x}$, we get

  $$A\boldsymbol{x} = A(A^{-1}\boldsymbol{b}) = (AA^{-1})\boldsymbol{b} = I\boldsymbol{b} = \boldsymbol{b}.$$

  So $A^{-1}\boldsymbol{b}$ is a solution.

  To prove that the solution is unique, show that if $\boldsymbol{u}$ is any solution, then $\boldsymbol{u}$ must be $A^{-1}\boldsymbol{b}$.

  Indeed, if $A\boldsymbol{u} = \boldsymbol{b}$, we can multiply both sides by A^{-1} and obtain $A^{-1}A\boldsymbol{u} = A^{-1}\boldsymbol{b}$, implying $I\boldsymbol{u} = A^{-1}b$, which gives $\boldsymbol{u} = A^{-1}\boldsymbol{b}$.

Example

- Use the inverse matrix method to solve the system

$$\begin{cases} 3x_1 + 4x_2 & = & 3 \\ 5x_1 + 6x_2 & = & 7 \end{cases}$$

- Consider $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$.

  Compute its inverse $A^{-1} = \begin{bmatrix} -3 & 2 \\ \frac{5}{2} & -\frac{3}{2} \end{bmatrix}$.

  Then calculate $\boldsymbol{x}$:

$$\boldsymbol{x} = A^{-1}\boldsymbol{b} = \begin{bmatrix} -3 & 2 \\ \frac{5}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}.$$

Properties of Invertible Matrices

Theorem

(a) If A is an invertible matrix, then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

(b) If A and B are $n \times n$ invertible matrices, then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.

(c) If A is an invertible matrix, then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.

(a) Notice that the equation $A^{-1}C = I$ and $CA^{-1} = I$ are satisfied with A in place of C. Hence A^{-1} is invertible and A it its inverse.

(b) We compute

$$\begin{aligned} (AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I; \\ (B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I. \end{aligned}$$

(c) Similarly, we get

$$\begin{aligned} (A^{-1})^T A^T &= (AA^{-1})^T = I^T = I; \\ A^T(A^{-1})^T &= (A^{-1}A)^T = I^T = I. \end{aligned}$$

Elementary Matrices

- An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

Example: Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ and $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

Compute E_1A, E_2A and E_3A, and describe how these products can be obtained by elementary row operations on A.

Example (Cont'd)

- We obtain

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{bmatrix}.$$

Here we operated $R_3 \leftarrow R_3 - 4R_1$.
We obtain

$$E_2A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}.$$

The operation was $R_1 \leftrightarrow R_2$.
Finally,

$$E_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}.$$

So we get $R_3 \leftarrow 5R_3$.

Properties of Elementary Matrices

- If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.
- Each elementary matrix E is invertible.

  The inverse of E is the elementary matrix F of the same type that transforms E back into I.

  In fact, since row operations are reversible, elementary matrices are invertible, for if E is produced by a row operation on I, then there is another row operation of the same type that changes E back into I. Hence there is an elementary matrix F such that $FE = I$. Since E and F correspond to reverse operations, $EF = I$, too.

Example

- Find the inverse of $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$.
- To transform E_1 into I, we must perform $R_3 \leftarrow R_3 + 4R_1$.

  The elementary matrix that does this is

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +4 & 0 & 1 \end{bmatrix}.$$

Finding A^{-1}

Theorem

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n. In this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

- Suppose that A is invertible. Then, since the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for each $\boldsymbol{b}$, A has a pivot position in every row. Because A is square, the n pivot positions must be on the diagonal. This implies that the reduced echelon form of A is I_n. That is, $A \sim I_n$.

Finding A^{-1} (Converse)

- Suppose, conversely, that $A \sim I_n$. But each step of the row reduction of A corresponds to left-multiplication by an elementary matrix. So there exist elementary matrices $E_1, \ldots, E_p$ such that

$$A \sim E_1 A \sim E_2(E_1 A) \sim E_p(E_{p-1} \cdots E_1 A) = I_n.$$

That is, $E_p \cdots E_1 A = I_n$. Since the product $E_p \cdots E_1$ of invertible matrices is invertible, we get

$$\begin{array}{rcl} (E_p \cdots E_1)^{-1}(E_p \cdots E_1)A & = & (E_p \cdots E_1)^{-1} I_n \\ A & = & (E_p \cdots E_1)^{-1}. \end{array}$$

Thus A is invertible, as it is the inverse of an invertible matrix. Also, $A^{-1} = [(E_p \cdots E_1)^{-1}]^{-1} = E_p \cdots E_1$. Then $A^{-1} = E_p \cdots E_1 I_n$. So A^{-1} results from applying $E_1, \ldots, E_p$ successively to I_n. This is the same sequence that reduced A to I_n.

An Algorithm for Finding A^{-1}

- Row reduce the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$.
- If A is row equivalent to I, then $\begin{bmatrix} A & I \end{bmatrix}$ is row equivalent to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$.
- Otherwise, A does not have an inverse.

Example

- Find the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ if it exists.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 4R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 + 3R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right] \xrightarrow{R_3 \leftarrow \frac{1}{2} R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{9}{2} & 7 & -\frac{3}{2} \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & \frac{3}{2} & -2 & \frac{1}{2} \end{array}\right]$$

Since $A \sim I$, A is invertible, and $A^{-1} = \begin{bmatrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{bmatrix}$.

Another View of Matrix Inversion

- Denote the columns of I_n by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$.
- Then row reduction of $\begin{bmatrix} A & I \end{bmatrix}$ to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$ can be viewed as the simultaneous solution of the n systems

$$A\boldsymbol{x} = \boldsymbol{e}_1, \quad A\boldsymbol{x} = \boldsymbol{e}_2, \quad \ldots, \quad A\boldsymbol{x} = \boldsymbol{e}_n,$$

  where the "augmented columns" of these systems have all been placed next to A to form $\begin{bmatrix} A & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \cdots & \boldsymbol{e}_n \end{bmatrix}$.
- The equation $AA^{-1} = I$ and the definition of matrix multiplication show that the columns of A^{-1} are precisely the solutions of these systems.

Subsection 3

Characterization of Invertible Matrices

The Invertible Matrix Theorem

Theorem (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. Then the following statements are equivalent:

(a) A is an invertible matrix.

(b) A is row equivalent to the $n \times n$ identity matrix.

(c) A has n pivot positions.

(d) The equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.

(e) The columns of A form a linearly independent set.

(f) The linear transformation $\boldsymbol{x} \mapsto A\boldsymbol{x}$ is one-to-one.

(g) The equation $A\boldsymbol{x} = \boldsymbol{b}$ has at least one solution for each $\boldsymbol{b}$ in $\mathbb{R}^n$.

(h) The columns of A span $\mathbb{R}^n$.

(i) The linear transformation $\boldsymbol{x} \mapsto A\boldsymbol{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

(j) There is an $n \times n$ matrix C such that $CA = I$.

(k) There is an $n \times n$ matrix D such that $AD = I$.

(l) A^T is an invertible matrix.

Example

- Use the Invertible Matrix Theorem to decide if $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$ is invertible.
- We have

$$A \underset{R_3 \leftarrow R_3 + 5R_1}{\overset{R_2 \leftarrow R_2 - 3R_1}{\longrightarrow}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \end{bmatrix} \overset{R_3 \leftarrow R_3 + r_2}{\longrightarrow} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}.$$

So A has three pivot positions. Hence is invertible, by the Invertible Matrix Theorem.

Invertible Linear Transformations

- A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists a function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\begin{aligned} S(T(\boldsymbol{x})) &= \boldsymbol{x}, \quad \text{for all } \boldsymbol{x} \text{ in } \mathbb{R}^n; \\ T(S(\boldsymbol{x})) &= \boldsymbol{x}, \quad \text{for all } \boldsymbol{x} \text{ in } \mathbb{R}^n. \end{aligned}$$

- The next theorem shows that if such an S exists, it is unique and must be a linear transformation.
- We call S the **inverse** of T and write it as T^{-1}.

Invertible Transformations and Matrices

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\boldsymbol{x}) = A^{-1}\boldsymbol{x}$ is the unique function satisfying $S(T(\boldsymbol{x})) = \boldsymbol{x}$, for all $\boldsymbol{x}$ in $\mathbb{R}^n$, and $T(S(\boldsymbol{x})) = \boldsymbol{x}$, for all $\boldsymbol{x}$ in $\mathbb{R}^n$.

- Suppose that T is invertible. Then the second equation shows that T is onto $\mathbb{R}^n$: Let $\boldsymbol{b}$ be in $\mathbb{R}^n$. Set $\boldsymbol{x} = S(\boldsymbol{b})$. Then $T(\boldsymbol{x}) = T(S(\boldsymbol{b})) = \boldsymbol{b}$. So each $\boldsymbol{b}$ is in the range of T. Thus A is invertible, by the Invertible Matrix Theorem.

  Conversely, suppose that A is invertible, and let $S(\boldsymbol{x}) = A^{-1}\boldsymbol{x}$. Then, S is a linear transformation, and S obviously satisfies both equations. For instance, $S(T(\boldsymbol{x})) = S(A\boldsymbol{x}) = A^{-1}(A\boldsymbol{x}) = \boldsymbol{x}$. Thus T is invertible.

Invertible Transformations and Matrices (Cont'd)

- Now we show that S, with $S(\boldsymbol{x}) = A^{-1}\boldsymbol{x}$, is the unique transformation satisfying

$$S(T(\boldsymbol{x})) = \boldsymbol{x}, \quad \text{for all } \boldsymbol{x} \text{ in } \mathbb{R}^n,$$
$$T(S(\boldsymbol{x})) = \boldsymbol{x}, \quad \text{for all } \boldsymbol{x} \text{ in } \mathbb{R}^n.$$

Suppose that $S' : \mathbb{R}^n \to \mathbb{R}^n$ is another transformation satisfying

$$S'(T(\boldsymbol{x})) = \boldsymbol{x}, \quad \text{for all } \boldsymbol{x} \text{ in } \mathbb{R}^n,$$
$$T(S'(\boldsymbol{x})) = \boldsymbol{x}, \quad \text{for all } \boldsymbol{x} \text{ in } \mathbb{R}^n.$$

But then we have, for all $\boldsymbol{x}$ in $\mathbb{R}^n$,

$$S'(\boldsymbol{x}) = S'(T(S(\boldsymbol{x}))) = S(\boldsymbol{x}).$$

So $S' = S$ and S is unique.

Example

- What can you say about a one-to-one linear transformation T from $\mathbb{R}^n$ into $\mathbb{R}^n$?
- The columns of the standard matrix A of T are linearly independent.

  So A is invertible, by the Invertible Matrix Theorem.

  Thus, T maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.

  Also, T is invertible, by the preceding theorem.

Subsection 4

Partitioned Matrices

Partitioned (or Block) Matrices

- The matrix $A = \left[\begin{array}{ccc|cc|c} 3 & 0 & -1 & 5 & 9 & -2 \\ -5 & 2 & 4 & 0 & -3 & 1 \\ \hline -8 & -6 & 3 & 1 & 7 & -4 \end{array}\right]$ can be written as the 2×3 **partitioned** (or **block**) **matrix**

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

whose entries are the **blocks** (or **submatrices**)

$$A_{11} = \begin{bmatrix} 3 & 0 & -1 \\ -5 & 2 & 4 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 5 & 9 \\ 0 & -3 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$
$$A_{21} = \begin{bmatrix} -8 & -6 & 3 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 7 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} -4 \end{bmatrix}.$$

Addition and Scalar Multiplication

- If matrices A and B are the same size and are partitioned in exactly the same way, then it is natural to make the same partition of the ordinary matrix sum $A+B$.

  Example:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}\right] + \left[\begin{array}{cc|c} 3 & 5 & 7 \\ \hline 9 & 8 & 4 \\ 6 & 2 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 4 & 7 & 10 \\ \hline 13 & 13 & 10 \\ 13 & 10 & 10 \end{array}\right].$$

- In this case, each block of $A+B$ is the (matrix) sum of the corresponding blocks of A and B.
- Multiplication of a partitioned matrix by a scalar is also computed block by block.

  Example:

$$3\left[\begin{array}{ccc} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{array}\right] = \left[\begin{array}{ccc} 3A_{11} & 3A_{12} & 3A_{13} \\ 3A_{21} & 3A_{22} & 3A_{23} \end{array}\right].$$

Multiplication of Partitioned Matrices

- Partitioned matrices can be multiplied by the usual row-column rule as if the block entries were scalars, provided that for a product AB, the column partition of A matches the row partition of B.

  Example: Let

$$A = \left[\begin{array}{ccc|cc} 2 & -3 & 1 & 0 & -4 \\ 1 & 5 & -2 & 3 & -1 \\ \hline 0 & -4 & -2 & 7 & -1 \end{array}\right] = \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right],$$

$$B = \left[\begin{array}{cc} 6 & 4 \\ -2 & 1 \\ -3 & 7 \\ \hline -1 & 3 \\ 5 & 2 \end{array}\right] = \left[\begin{array}{c} B_1 \\ B_2 \end{array}\right].$$

  The 5 columns of A are partitioned into a set of 3 columns and then a set of 2 columns.

Example (Cont'd)

- The 5 rows of B are partitioned in the same way - into a set of 3 rows and then a set of 2 rows.
- We say that the partitions of A and B are **conformable for block multiplication**.
- We then have

$$
\begin{aligned}
AB &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ 1 & 5 & -2 \end{bmatrix} \begin{bmatrix} 6 & 4 \\ -2 & 1 \\ -3 & 7 \end{bmatrix} + \begin{bmatrix} 0 & -4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix} \\ \begin{bmatrix} 0 & -4 & -2 \end{bmatrix} \begin{bmatrix} 6 & -4 \\ -2 & 1 \\ -3 & 7 \end{bmatrix} + \begin{bmatrix} 7 & -1 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 5 & 2 \end{bmatrix} \end{bmatrix} \\
&= \begin{bmatrix} \begin{bmatrix} 15 & 12 \\ 2 & -5 \end{bmatrix} + \begin{bmatrix} -20 & -8 \\ -8 & 7 \end{bmatrix} \\ \begin{bmatrix} 14 & -18 \end{bmatrix} + \begin{bmatrix} -12 & 19 \end{bmatrix} \end{bmatrix} = \left[\begin{array}{cc} -5 & 4 \\ -6 & 2 \\ \hline 2 & 1 \end{array}\right].
\end{aligned}
$$

Example

- Let $A = \begin{bmatrix} -3 & 1 & 2 \\ 1 & -4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$.

  Verify that

$$AB = \text{col}_1(A)\text{row}_1(B) + \text{col}_2(A)\text{row}_2(B) + \text{col}_3(A)\text{row}_3(B).$$

- We have

$$\begin{aligned}
&\text{col}_1(A)\text{row}_1(B) + \text{col}_2(A)\text{row}_2(B) + \text{col}_3(A)\text{row}_3(B) \\
&= \begin{bmatrix} -3 \\ 1 \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} + \begin{bmatrix} 1 \\ -4 \end{bmatrix} \begin{bmatrix} c & d \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} \begin{bmatrix} e & f \end{bmatrix} \\
&= \begin{bmatrix} -3a & -3b \\ a & b \end{bmatrix} + \begin{bmatrix} c & d \\ -4c & -4d \end{bmatrix} + \begin{bmatrix} 2e & 2f \\ 5e & 5f \end{bmatrix} \\
&= \begin{bmatrix} -3a + c + 2e & -3b + d + 2f \\ a - 4c + 5e & b - 4d + 5f \end{bmatrix} = AB.
\end{aligned}$$

Column-Row Expansion of AB

Theorem

If A is $m \times n$ and B is $n \times p$, then

$$\begin{aligned} AB &= \begin{bmatrix} \text{col}_1(A) & \text{col}_2(A) & \cdots & \text{col}_n(A) \end{bmatrix} \begin{bmatrix} \text{row}_1(B) \\ \text{row}_2(B) \\ \vdots \\ \text{row}_n(B) \end{bmatrix} \\ &= \text{col}_1(A)\text{row}_1(B) + \cdots + \text{col}_n(A)\text{row}_n(B). \end{aligned}$$

- For each row index i and column index j, the (i,j)-entry in $\text{col}_k(A)\text{row}_k(B)$ is the product of a_{ik} from $\text{col}_k(A)$ and b_{kj} from $\text{row}_k(B)$. Hence the (i,j)-entry in the sum shown in the equation is $a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$. This sum is also the (i,j)-entry in AB, by the row-column rule.

Inverses of Partitioned Matrices

- A matrix of the form $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ is said to be **block upper triangular**.
  Assume that A_{11} is $p \times p$, A_{22} is $q \times q$, and A is invertible.
  Find a formula for A^{-1}.
- Denote A^{-1} by B and partition B so that

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}.$$

  This matrix equation provides four equations that will lead to the unknown blocks $B_{11}, \ldots, B_{22}$:

$$\begin{aligned} A_{11}B_{11} + A_{12}B_{21} &= I_p \\ A_{11}B_{12} + A_{12}B_{22} &= 0 \\ A_{22}B_{21} &= 0 \\ A_{22}B_{22} &= I_q \end{aligned}$$

Inverses of Partitioned Matrices (Cont'd)

- By itself, $A_{22}B_{22} = I_q$ does not show that A_{22} is invertible.

  However, since A_{22} is square, the Invertible Matrix Theorem and the last equation together show that A_{22} is invertible and $B_{22} = A_{22}^{-1}$.

  Next, left-multiply both sides of $A_{22}B_{21} = 0$ by A_{22}^{-1} and obtain $B_{21} = A_{22}^{-1}0 = 0$.

  So $A_{11}B_{11} + A_{12}B_{21} = I_p$ simplifies to $A_{11}B_{11} + 0 = I_p$.

  Since A_{11} is square, this shows that A_{11} is invertible and $B_{11} = A_{11}^{-1}$.

  Finally, use these results with $A_{11}B_{12} + A_{12}B_{22} = 0$ to find that $A_{11}B_{12} = -A_{12}B_{22} = -A_{12}A_{22}^{-1}$ and $B_{12} = -A_{11}^{-1}A_{12}A_{22}^{-1}$.

  Thus

$$A^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}.$$

Block Diagonal Matrices

- A **block diagonal matrix** is a partitioned matrix with zero blocks off the main diagonal (of blocks).
- Such a matrix is invertible if and only if each block on the diagonal is invertible.

Example

- Suppose $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$ is a square block diagonal matrix.

  Show that A is invertible if and only if B, C are invertible.
- Suppose that A is invertible. Assume A^{-1} is partitioned as $A^{-1} = \begin{bmatrix} D & E \\ F & G \end{bmatrix}$ so that the two partitions are conformal with block multiplication. Then we have:

$$AA^{-1} = I \Rightarrow \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} D & E \\ F & G \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}$$

$$\Rightarrow \begin{cases} BD & = & I_p \\ BE & = & 0 \\ CF & = & 0 \\ CG & = & I_q \end{cases} \Rightarrow \begin{cases} D & = & B^{-1} \\ E & = & 0 \\ F & = & 0 \\ G & = & C^{-1} \end{cases}$$

  So we get that B, C are invertible matrices, with inverses D, G, respectively. Moreover, we see that $A^{-1} = \begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix}$.

Example (Cont'd)

- Assume, conversely, that $A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}$ with B, C invertible. To show that A is invertible, form the matrix $\begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix}$. Note that its partition and that of A are conformable with block multiplication and compute:

$$\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix} = \begin{bmatrix} BB^{-1} & 0 \\ 0 & CC^{-1} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} = I;$$
$$\begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} = \begin{bmatrix} B^{-1}B & 0 \\ 0 & C^{-1}C \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix} = I;$$

Thus A is invertible and $A^{-1} = \begin{bmatrix} B^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix}$.

Subsection 5

Matrix Factorizations

LU Factorization

- Assume that A is an $m \times n$ matrix that can be row reduced to echelon form, without row interchanges.
- Then A can be written in the form $A = LU$, where L is an $m \times m$ lower triangular matrix with 1's on the diagonal and U is an $m \times n$ echelon form of A.

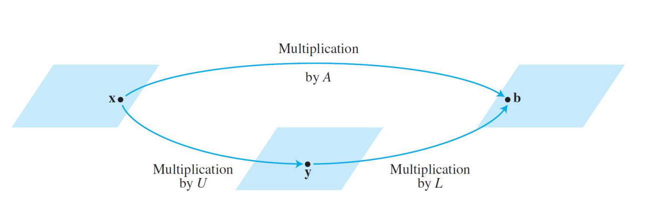
$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix} \begin{bmatrix} \square & * & * & * & * \\ 0 & \square & * & * & * \\ 0 & 0 & 0 & \square & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- Such a factorization is called an **LU factorization** of A.
- The matrix L is invertible and is called a **unit lower triangular matrix**.

Solving Equations Using the LU Factorization

- When $A = LU$, the equation $A\boldsymbol{x} = \boldsymbol{b}$ can be written as $L(U\boldsymbol{x}) = \boldsymbol{b}$.
- Writing $\boldsymbol{y}$ for $U\boldsymbol{x}$, we can find $\boldsymbol{x}$ by solving the pair of equations

$$\begin{aligned} L\boldsymbol{y} &= \boldsymbol{b} \\ U\boldsymbol{x} &= \boldsymbol{y} \end{aligned}$$



- First solve $L\boldsymbol{y} = \boldsymbol{b}$ for $\boldsymbol{y}$;
- Then solve $U\boldsymbol{x} = \boldsymbol{y}$ for $\boldsymbol{x}$.
- Each equation is easy to solve because L and U are triangular.

Example

- Using the LU factorization

$$A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$= LU$, solve $A\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{b} = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}$.

Example (Cont'd)

- First solve $L\boldsymbol{y} = \boldsymbol{b}$:

$$\begin{bmatrix} L & \boldsymbol{b} \end{bmatrix}$$

$$= \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{array}\right] \xrightarrow[\substack{R_3 \leftarrow R_3 - 2R_1 \\ R_4 \leftarrow R_4 + 3R_1}]{R_2 \leftarrow R_2 + R_1} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & -5 & 1 & 0 & 25 \\ 0 & 8 & 3 & 1 & -16 \end{array}\right]$$

$$\xrightarrow[R_4 \leftarrow R_4 - 8R_2]{R_3 \leftarrow R_3 + 5R_2} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 3 & 1 & 16 \end{array}\right] \xrightarrow{R_4 \leftarrow R_4 - 3R_3} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

$$= \begin{bmatrix} I & \boldsymbol{y} \end{bmatrix}.$$

Example (Cont'd)

- Now solve $U\boldsymbol{x} = \boldsymbol{y}$:

$$\begin{bmatrix} U & \boldsymbol{y} \end{bmatrix} = \left[\begin{array}{cccc|c} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right] \xrightarrow{R_4 \leftarrow (-1)R_4} \left[\begin{array}{cccc|c} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[\substack{R_2 \leftarrow R_2 - 2R_4 \\ R_3 \leftarrow R_3 - R_4}]{R_1 \leftarrow R_1 - 2R_4} \left[\begin{array}{cccc|c} 3 & -7 & -2 & 0 & -7 \\ 0 & -2 & -1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_3 \leftarrow (-1)R_3} \left[\begin{array}{cccc|c} 3 & -7 & -2 & 0 & -7 \\ 0 & -2 & -1 & 0 & -2 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow[R_2 \leftarrow R_2 + R_3]{R_1 \leftarrow R_1 + 2R_3} \left[\begin{array}{cccc|c} 3 & -7 & 0 & 0 & -19 \\ 0 & -2 & 0 & 0 & -8 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_2 \leftarrow (-\frac{1}{2})R_2} \left[\begin{array}{cccc|c} 3 & -7 & 0 & 0 & -19 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftarrow R_1 + 7R_2} \left[\begin{array}{cccc|c} 3 & 0 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{3}R_1} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] = [I_4 \ \ \boldsymbol{x}].$$

Obtaining an LU Factorization

- Suppose A can be reduced to an echelon form U using only row replacements that add a multiple of one row to another row below it.
- In this case, there exist unit lower triangular elementary matrices $E_1, \ldots, E_p$ such that $E_p \cdots E_1 A = U$.
- Then $A = (E_p \cdots E_1)^{-1} U = LU$, where $L = (E_p \cdots E_1)^{-1}$.
- It can be shown that products and inverses of unit lower triangular matrices are also unit lower triangular.
- Thus L is unit lower triangular.
- Note that the row operations which reduce A to U, also reduce the matrix L to I, because $E_p \cdots E_1 L = (E_p \cdots E_1)(E_p \cdots E_1)^{-1} = I$.

The LU Factorization Algorithm

Algorithm for an LU Factorization

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the same sequence of row operations reduces L to I.

- Step 1 is not always possible, but when it is, the argument in the preceding slide shows that an LU factorization exists.
- By construction, L will satisfy $(E_p \cdots E_1)L = I$ using the same $E_1, \ldots, E_p$ as the one reducing A to U.
- Thus L will be invertible, by the Invertible Matrix Theorem, with $(E_p \cdots E_1) = L^{-1}$.
- Since $E_p \cdots E_1 A = U$, $L^{-1}A = U$, and $A = LU$.

Example

- Find an LU factorization of $A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$.
-

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 \leftarrow R_3 - R_1 \\ R_4 \leftarrow R_4 + 3R_1}]{R_2 \leftarrow R_2 + 2R_1} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix}$$

$$\xrightarrow[R_4 \leftarrow R_4 - 4R_2]{R_3 \leftarrow R_3 + 3R_2} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4 - 2R_3} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$$= U.$$

Example (Cont'd): Pivot Columns

- To transform A to U, we looked at the following pivot columns:

$$A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix} \xrightarrow[\substack{R_3 \leftarrow R_3 - R_1 \\ R_4 \leftarrow R_4 + 3R_1}]{R_2 \leftarrow R_2 + 2R_1} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & -9 & -3 & -4 & 10 \\ 0 & 12 & 4 & 12 & -5 \end{bmatrix}$$

$$\xrightarrow[R_4 \leftarrow R_4 - 4R_2]{R_3 \leftarrow R_3 + 3R_2} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 & 7 \end{bmatrix} \xrightarrow{R_4 \leftarrow R_4 - 2R_3} \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ 0 & 3 & 1 & 2 & -3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

$$= U.$$

Example (Cont'd)

- Since A has four rows, L should be 4×4.
- We look at the colored entries of the preceding slide that were used to determine the sequence of row operations that transformed A to U.

$$\begin{bmatrix} 2 \\ -4 \\ 2 \\ -6 \end{bmatrix} \begin{bmatrix} 3 \\ -9 \\ 12 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \end{bmatrix}.$$

- Divide each by the pivot and use the result to form L:

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{bmatrix}.$$

- This is because the same row operations we used to transform A to U transform L to I_4.