Introduction to Linear Algebra

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 305

Subsection 1

Introduction to Determinants

2×2 Determinants

- For a 1×1 matrix $A = [a_{11}]$, we define
$$\det A = a_{11}.$$
- Recall that the determinant of a 2×2 matrix, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, is the number
$$\det A = a_{11}a_{22} - a_{12}a_{21}.$$
- Recall also that a 2×2 matrix is invertible if and only if its determinant is nonzero.

3×3 Determinants

- Consider an invertible $A = [a_{ij}]$ with $a_{11} \neq 0$.
- We have:

$$A \xrightarrow[R_3 \leftarrow a_{11} R_3]{R_2 \leftarrow a_{11} R_2} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} a_{21} & a_{11} a_{22} & a_{11} a_{23} \\ a_{11} a_{31} & a_{11} a_{32} & a_{11} a_{33} \end{bmatrix}$$

$$\xrightarrow[R_3 \leftarrow R_3 - a_{31} R_1]{R_2 \leftarrow R_2 - a_{21} R_1} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11} a_{22} - a_{12} a_{21} & a_{11} a_{23} - a_{13} a_{21} \\ 0 & a_{11} a_{32} - a_{12} a_{31} & a_{11} a_{33} - a_{13} a_{31} \end{bmatrix}.$$

- Since A is invertible, either the $(2,2)$-entry or the $(3,2)$-entry on the right is nonzero.
- Let us suppose that the $(2,2)$-entry is nonzero. (Otherwise, we can make a row interchange before proceeding.)

3×3 Determinants (Cont'd)

- Continuing, we get

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22}-a_{12}a_{21} & a_{11}a_{23}-a_{13}a_{21} \\ 0 & a_{11}a_{32}-a_{12}a_{31} & a_{11}a_{33}-a_{13}a_{31} \end{bmatrix} \xrightarrow{R_3 \leftarrow (a_{11}a_{22}-a_{12}a_{21})R_3}$$
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22}-a_{12}a_{21} & a_{11}a_{23}-a_{13}a_{21} \\ 0 & (a_{11}a_{22}-a_{12}a_{21})(a_{11}a_{32}-a_{12}a_{31}) & (a_{11}a_{22}-a_{12}a_{21})(a_{11}a_{33}-a_{13}a_{31}) \end{bmatrix}$$
$$\xrightarrow{R_3 \leftarrow R_3-(a_{11}a_{32}-a_{12}a_{31})R_2} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22}-a_{12}a_{21} & a_{11}a_{23}-a_{13}a_{21} \\ 0 & 0 & a_{11}\Delta \end{bmatrix},$$

where

$$\begin{aligned} \Delta = \; & a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32} \\ & - a_{11}a_{23}a_{32}-a_{12}a_{21}a_{33}-a_{13}a_{22}a_{31}. \end{aligned}$$

- Since A is invertible, Δ must be nonzero.
- The converse is true, too, as we will see in the following section.
- We call Δ the **determinant** of the 3×3 matrix A.

Rewriting the 3×3 Determinant

- To generalize the definition of the determinant to larger matrices, we use 2×2 determinants to rewrite the 3×3 Δ described above.
- The terms in Δ can be grouped as

$$(a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32}) - (a_{12}a_{21}a_{33} - a_{12}a_{23}a_{31}) + (a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}).$$

- Thus, we get

$$\Delta = a_{11} \cdot \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}.$$

- For brevity, write

$$\Delta = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13},$$

where A_{11}, A_{12} and A_{13} are obtained from A by deleting the first row and one of the three columns.

Submatrices of A

- For any square matrix A, let A_{ij} denote the submatrix formed by deleting the ith row and jth column of A.

Example: Let $A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix}$.

Then A_{32} is obtained by crossing out row 3 and column 2,

$$A = \begin{bmatrix} 1 & -2 & 5 & 0 \\ 2 & 0 & 4 & -1 \\ 3 & 1 & 0 & 7 \\ 0 & 4 & -2 & 0 \end{bmatrix} \quad \Rightarrow \quad A_{32} = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}.$$

$n \times n$ Determinant

- We can now give a recursive definition of a determinant.
- When $n = 3$, $\det A$ is defined using determinants of the 2×2 submatrices A_{1j}.
- When $n = 4$, $\det A$ uses determinants of the 3×3 submatrices A_{1j}.
- In general, an $n \times n$ determinant is defined by determinants of $(n-1) \times (n-1)$ submatrices.

Definition

For $n \geq 2$, the **determinant** of an $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} \det_{A1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j}. \end{aligned}$$

Example and Notation

- Compute the determinant of $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.
- We compute:

$$\begin{aligned} \det A &= a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13} \\ &= 1 \cdot \det \begin{bmatrix} 4 & -1 \\ -2 & 0 \end{bmatrix} - 5 \cdot \det \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \det \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix} \\ &= 1 \cdot (0-2) - 5 \cdot (0-0) + 0 \cdot (-4-0) = -2. \end{aligned}$$

- Another common notation for the determinant of a matrix uses a pair of vertical lines in place of brackets.
- Thus the calculation can be written as

$$\det A = 1 \begin{vmatrix} 4 & -1 \\ -2 & 0 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} + 0 \begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix}.$$

Cofactors and Cofactor Expansion

- Given $A = [a_{ij}]$, the (i,j)-**cofactor** of A is the number C_{ij} given by
$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$
- Then $\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$.
- This formula is the **cofactor expansion across the first row** of A.

Theorem

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors above is
$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}.$$
The cofactor expansion down the jth column is
$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}.$$

Example

- Use a cofactor expansion across the third row to compute $\det A$, where $A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix}$.
- Compute

$$\begin{aligned} \det A &= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \\ &= (-1)^{3+1}a_{31}\det A_{31} + (-1)^{3+2}a_{32}\det A_{32} \\ &\qquad + (-1)^{3+3}a_{33}\det A_{33} \\ &= 0\begin{vmatrix} 5 & 0 \\ 4 & -1 \end{vmatrix} - (-2)\begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} + 0\begin{vmatrix} 1 & 5 \\ 2 & 4 \end{vmatrix} \\ &= 0 + 2(-1) + 0 = -2. \end{aligned}$$

Example

- Compute $\det A$, where $A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}$.
- The cofactor expansion down the first column of A has all terms equal to zero except the first:

$$\det A = 3 \begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix}.$$

Example (Cont'd)

- Next, expand this 4×4 determinant down the first column, in order to take advantage of the zeros there:

$$\det A = 3 \begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} = 3 \cdot 2 \cdot \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix}.$$

This 3×3 determinant was computed in a previous example and found to equal -2:

$$\det A = 3 \cdot 2 \cdot (-2) = -12.$$

Matrices with Many Zeroes

- The matrix in the last example was nearly triangular.
- The method in that example is easily adapted to prove the following theorem.

Theorem

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

- The strategy the last example of looking for zeros works extremely well when an entire row or column consists of zeros.

  In such a case, the cofactor expansion along such a row or column is a sum of zeros and the determinant is zero.

Subsection 2

Properties of Determinants

Row Operations

Theorem (Row Operations)

Let A be a square matrix.

(a) If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.

(b) If two rows of A are interchanged to produce B, then $\det B = -\det A$.

(c) If one row of A is multiplied by k to produce B, then $\det B = k \cdot \det A$.

Row Operations (Part (b))

(b) We show only the special case in which the first and second rows are interchanged. The general case is similar.

$$\begin{vmatrix} a_{21} & \cdots & a_{2n} \\ a_{11} & \cdots & a_{1n} \\ a_{31} & \cdots & a_{3n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \sum_{i=1}^{n} a_{1i}(-1)^{2+i}\det A_{1i}$$

$$\begin{aligned} &= \sum_{i=1}^{n}(-1)a_{1i}(-1)^{1+i}\det A_{1i} \\ &= -\sum_{i=1}^{n} a_{1i}(-1)^{1+i}\det A_{1i} \\ &= -\det A. \end{aligned}$$

- Note that this implies that, if a matrix A has two identical rows then its determinant is zero, because $\det A = -\det A$.

Row Operations (Part (a))

(a) We show only the special case in which the first row is first plus c times the second row. The general case is similar.

$$\begin{vmatrix} a_{11}+ca_{21} & \cdots & a_{1n}+ca_{2n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \sum_{i=1}^{n}(a_{1i}+ca_{2i})(-1)^{1+i}\det A_{1i}$$

$$= \sum_{i=1}^{n} a_{1i}(-1)^{1+i}\det A_{1i} + c\sum_{i=1}^{n} a_{2i}(-1)^{1+i}\det A_{1i}$$

$$= \det A + \begin{vmatrix} a_{21} & \cdots & a_{2n} \\ a_{21} & \cdots & a_{2n} \\ a_{31} & \cdots & a_{3n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \det A + 0 = \det A.$$

Row Operations (Part (c))

(c) We show only the special case in which the first row has been multiplied by c. The general case is similar.

$$\begin{aligned}
\begin{vmatrix} ca_{11} & \cdots & ca_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} &= \sum_{i=1}^n (ca_{1i})(-1)^{1+i}\det A_{1i} \\
&= c\sum_{i=1}^n a_{1i}(-1)^{1+i}\det A_{1i} \\
&= c\det A.
\end{aligned}$$

Example

- Compute $\det A$, where $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$.
- The strategy is to reduce A to echelon form and then to use the fact that the determinant of a triangular matrix is the product of the diagonal entries.

  The first two row replacements in column 1 do not change the value:

$$\det A = \begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}.$$

  An interchange of rows 2 and 3 reverses the sign of the determinant:

$$\det A = -\begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -1 \cdot 3 \cdot (-5) = 15.$$

Example

- Compute $\det A$, where $A = \begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$.
- To simplify the arithmetic, we want a 1 in the upper left corner.

  We could interchange rows 1 and 4.

  Instead, we factor out 2 from the top row, and then proceed with row replacements in the first column:

$$\det A = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix}.$$

Example (Cont'd)

- Next, we could factor out another 2 from row 3 or use the 3 in the second column as a pivot.
  We choose the latter operation, adding 4 times row 2 to row 3:

$$\det A = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix}.$$

  Finally, adding $-\frac{1}{2}$ times row 3 to row 4, and computing the "triangular determinant, we find that

$$\det A = 2\begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2 \cdot 1 \cdot 3 \cdot (-6) \cdot 1 = -36.$$

Determinant and Echelon Form

- Suppose a square matrix A has been reduced to an echelon form U by row replacements and row interchanges.
- If there are r interchanges, then the preceding theorem shows that
$$\det A = (-1)^r \det U.$$
- Since U is in echelon form, it is triangular, and so $\det U$ is the product of the diagonal entries $u_{11}, \ldots, u_{nn}$.
- If A is invertible, the entries u_{ii} are all pivots (because $A \sim I_n$ and the u_{ii} have not been scaled to 1's).
- Otherwise, at least u_{nn} is zero, and the product $u_{11} \cdots u_{nn}$ is zero.
- Thus
$$\det A = \begin{cases} (-1)^r \cdot (\text{product of pivots in } U), & \text{if } A \text{ is invertible;} \\ 0, & \text{if } A \text{ is not invertible.} \end{cases}$$

Invertibility and Determinants

Theorem

A square matrix A is invertible if and only if $\det A \neq 0$.

- This theorem adds the statement "$\det A \neq 0$" to the Invertible Matrix Theorem.
- A useful corollary is that $\det A = 0$ when the columns of A are linearly dependent.
- Also, $\det A = 0$ when the rows of A are linearly dependent.
- In practice, linear dependence is obvious when two columns or two rows are the same or a column or a row is zero.

Example

- Compute $\det A$, where $A = \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ -6 & 7 & -7 & 4 \\ -5 & -8 & 0 & 9 \end{bmatrix}$.
- Add 2 times row 1 to row 3 to obtain

$$\det A = \det \begin{bmatrix} 3 & -1 & 2 & -5 \\ 0 & 5 & -3 & -6 \\ 0 & 5 & -3 & -6 \\ -5 & -8 & 0 & 9 \end{bmatrix} = 0,$$

because the second and third rows of the second matrix are equal.

Example

- Compute $\det A$, where $A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ -2 & -5 & 4 & -2 \end{bmatrix}$.
- A good way to begin is to use the 2 in column 1 as a pivot, eliminating the -2 below it.

  Then use a cofactor expansion to reduce the size of the determinant, followed by another row replacement operation.
- We have

$$\det A = \begin{vmatrix} 0 & 1 & 2 & -1 \\ 2 & 5 & -7 & 3 \\ 0 & 3 & 6 & 2 \\ 0 & 0 & -3 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 2 \\ 0 & -3 & 1 \end{vmatrix} = -2 \begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}.$$

Example (Cont'd)

- An interchange of rows 2 and 3 in

$$-2\begin{vmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & -3 & 1 \end{vmatrix}$$

would produce a "triangular determinant".

Another approach is to make a cofactor expansion down the first column:

$$\det A = (-2)\cdot 1 \cdot \begin{vmatrix} 0 & 5 \\ -3 & 1 \end{vmatrix} = -2\cdot 15 = -30.$$

Determinant of the Transpose

Theorem

If A is an $n \times n$ matrix, then $\det A^T = \det A$.

- The theorem is obvious for $n = 1$.

  Suppose the theorem is true for $k \times k$ determinants and let $n = k + 1$. Then we have: $\det A^T$

$$= \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix} = \sum_{i=1}^n a_{i1}(-1)^{1+i}\det(A^T)_{1i}$$

$$= \sum_{i=1}^n a_{i1}(-1)^{i+1}\det(A_{i1})^T \overset{\text{I.H.}}{=} \sum_{i=1}^n a_{i1}(-1)^{i+1}\det A_{i1}$$

$$= \det A.$$

  By the principle of induction, the theorem is true for all $n \geq 1$.

Multiplicative Property

Theorem (Multiplicative Property)

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$.

Example: Verify the theorem for $A = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$.

- We have
$$AB = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 25 & 20 \\ 14 & 13 \end{bmatrix}.$$

  Now we compute
$$\begin{aligned} \det AB &= 25 \cdot 13 - 20 \cdot 14 = 325 - 280 = 45; \\ (\det A)(\det B) &= 9 \cdot 5 = 45 = \det AB. \end{aligned}$$

Linearity Property

- For an $n \times n$ matrix A, we can consider $\det A$ as a function of the n column vectors in A.
- Suppose that the jth column of A is allowed to vary, and write

$$A = \begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_{j-1} & \boldsymbol{x} & \boldsymbol{a}_{j+1} & \cdots & \boldsymbol{a}_n \end{bmatrix}.$$

- Define a transformation T from $\mathbb{R}^n$ to $\mathbb{R}$ by

$$T(\boldsymbol{x}) = \det \begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_{j-1} & \boldsymbol{x} & \boldsymbol{a}_{j+1} & \cdots & \boldsymbol{a}_n \end{bmatrix}.$$

- Then,

$$\begin{array}{rcll} T(c\boldsymbol{x}) & = & cT(\boldsymbol{x}), & \text{for all scalars } c \text{ and all } \boldsymbol{x} \text{ in } \mathbb{R}^n, \\ T(\boldsymbol{u}+\boldsymbol{v}) & = & T(\boldsymbol{u}) + T(\boldsymbol{v}), & \text{for all } \boldsymbol{u}, \boldsymbol{v} \text{ in } \mathbb{R}^n. \end{array}$$

Subsection 3

Cramer's Rule, Volume & Linear Transformations

Cramer's Rule

- For any $n \times n$ matrix A and any $\boldsymbol{b}$ in $\mathbb{R}^n$, let $A_i(\boldsymbol{b})$ be the matrix obtained from A by replacing column i by the vector $\boldsymbol{b}$:

$$A_i(\boldsymbol{b}) = [\ \boldsymbol{a}_1 \quad \cdots \quad \underbrace{\boldsymbol{b}}_{\text{col. } i} \quad \cdots \quad \boldsymbol{a}_n\ ]$$

Theorem (Cramer's Rule)

Let A be an invertible $n \times n$ matrix. For any $\boldsymbol{b}$ in $\mathbb{R}^n$, the unique solution x of $A\boldsymbol{x} = \boldsymbol{b}$ has entries given by

$$x_i = \frac{\det A_i(\boldsymbol{b})}{\det A}, \quad i = 1, \ldots, n.$$

- Denote the columns of A by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ and the columns of the $n \times n$ identity matrix I by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$.

Cramer's Rule (Cont'd)

- If $A\boldsymbol{x} = \boldsymbol{b}$, the definition of matrix multiplication shows that

$$\begin{array}{rcl} A \cdot I_i(\boldsymbol{x}) & = & A \left[\begin{array}{ccccc} \boldsymbol{e}_1 & \cdots & \boldsymbol{x} & \cdots & \boldsymbol{e}_n \end{array} \right] \\ & = & \left[\begin{array}{ccccc} A\boldsymbol{e}_1 & \cdots & A\boldsymbol{x} & \cdots & A\boldsymbol{e}_n \end{array} \right] \\ & = & \left[\begin{array}{ccccc} \boldsymbol{a}_1 & \cdots & \boldsymbol{b} & \cdots & \boldsymbol{a}_n \end{array} \right] \\ & = & A_i(\boldsymbol{b}). \end{array}$$

By the multiplicative property of determinants,

$$(\det A)(\det I_i(\boldsymbol{x})) = \det A_i(\boldsymbol{b}).$$

The second determinant on the left is simply x_i (make a cofactor expansion along the ith row). Hence $(\det A) \cdot x_i = \det A_i(\boldsymbol{b})$. This proves the required equation because A is invertible and $\det A \neq 0$.

Example

- Use Cramer's rule to solve the system $\begin{cases} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{cases}$.
- View the system as $A\boldsymbol{x} = \boldsymbol{b}$. Using the notation introduced above,

$$A = \begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}, \quad A_1(\boldsymbol{b}) = \begin{bmatrix} 6 & -2 \\ 8 & 4 \end{bmatrix}, \quad A_2(\boldsymbol{b}) = \begin{bmatrix} 3 & 6 \\ -5 & 8 \end{bmatrix}.$$

Since $\det A = 2$, the system has a unique solution. By Cramer's rule,

$$\begin{aligned} x_1 &= \frac{\det A_1(\boldsymbol{b})}{\det A} = \frac{24+16}{2} = 20; \\ x_2 &= \frac{\det A_2(\boldsymbol{b})}{\det A} = \frac{24+30}{2} = 27. \end{aligned}$$

Example

- Consider the system $\left\{ \begin{array}{rcl} 3sx_1 - 2x_2 & = & 4 \\ -6x_1 + sx_2 & = & 1 \end{array} \right.$ in which s is an unspecified parameter.
  Determine the values of s for which the system has a unique solution, and use Cramer's rule to describe the solution.
- View the system as $A\boldsymbol{x} = \boldsymbol{b}$. Then

$$A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}, \quad A_1(\boldsymbol{b}) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix}, \quad A_2(\boldsymbol{b}) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix}.$$

  Since $\det A = 3s^2 - 12 = 3(s+2)(s-2)$, the system has a unique solution precisely when $s \neq \pm 2$. For such an s, we have

$$\begin{array}{rcl} x_1 & = & \dfrac{\det A_1(\boldsymbol{b})}{\det A} = \dfrac{4s+2}{3(s+2)(s-2)}, \\ x_2 & = & \dfrac{\det A_2(\boldsymbol{b})}{\det A} = \dfrac{3s+24}{3(s+2)(s-2)} = \dfrac{s+8}{(s+2)(s-2)}. \end{array}$$

A Formula for A^{-1}

- The jth column of A^{-1} is a vector $\boldsymbol{x}$ that satisfies $A\boldsymbol{x} = \boldsymbol{e}_j$ where $\boldsymbol{e}_j$ is the jth column of the identity matrix, and the ith entry of $\boldsymbol{x}$ is the (i,j)-entry of A^{-1}.
- By Cramer's rule,

$$(i,j)\text{-entry of } A^{-1} = x_i = \frac{\det A_i(\boldsymbol{e}_i)}{\det A}.$$

- Recall that A_{ji} denotes the submatrix of A formed by deleting row j and column i.
- A cofactor expansion down column i of $A_i(\boldsymbol{e}_j)$ shows that

$$\det A_i(\boldsymbol{e}_j) = (-1)^{i+j} \det A_{ji} = C_{ji},$$

  where C_{ji} is a cofactor of A.
- Thus, we get that the (i,j)-entry of A^{-1} is the cofactor C_{ji} divided by $\det A$. [Note that the subscripts on C_{ji} are the reverse of (i,j).]

The Adjoint of a Matrix

- We conclude that

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

- The matrix of cofactors on the right side is called the **adjugate** (or **classical adjoint**) of A, denoted by $\mathrm{adj} A$.

Theorem (An Inverse Formula)

Let A be an invertible $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A}\mathrm{adj} A.$$

Example

- Find the inverse of the matrix $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$.
- The nine cofactors are

$$\begin{array}{lll} C_{11} = +\begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} = -2, & C_{12} = -\begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} = 3, & C_{13} = +\begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5, \\ C_{21} = -\begin{vmatrix} 1 & 3 \\ 4 & -2 \end{vmatrix} = 14, & C_{22} = +\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = 7, & C_{23} = -\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 7, \\ C_{31} = +\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} = 4, & C_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 1, & C_{33} = +\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} = -3. \end{array}$$

The adjugate matrix is the transpose of the matrix of cofactors:

$$\mathrm{adj}A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}.$$

Example (Cont'd)

- We could compute $\det A$ directly, but the following computation provides a check on the calculations above and produces $\det A$:

$$(\text{adj}A) \cdot A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} = 14I.$$

  Since $(\text{adj}A)A = 14I$, the theorem shows that $\det A = 14$.

  Hence

$$A^{-1} = \frac{1}{14} \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} & 1 & \frac{2}{7} \\ \frac{3}{14} & -\frac{1}{2} & \frac{1}{14} \\ \frac{5}{14} & -\frac{1}{2} & -\frac{3}{14} \end{bmatrix}.$$

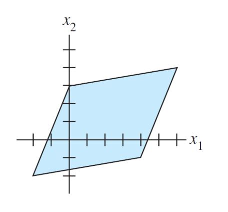
Determinants as Area or Volume

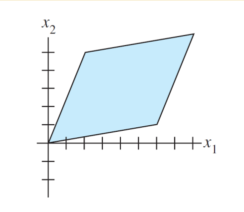
Theorem

If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the volume of the parallelepiped determined by the columns of A is $|\det A|$.

Example: Calculate the area of the parallelogram determined by the points $(-2, 2)$, $(0, 3)$, $(4, -1)$, $(6, 4)$.

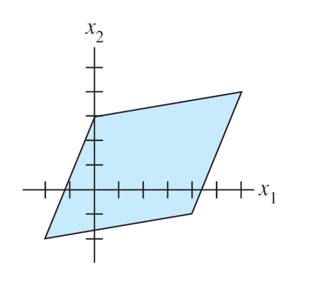
- First translate the parallelogram to one having the origin as a vertex.

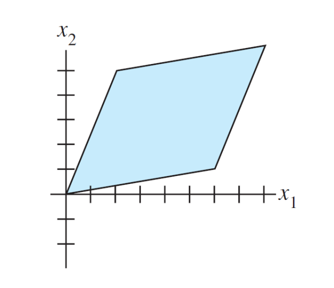




For example, subtract the vertex $(-2, -2)$ from each vertex.

Determinants as Area or Volume





The new parallelogram has the same area, and its vertices are $(0,0)$, $(2,5)$, $(6,1)$ and $(8,6)$. This parallelogram is determined by the columns of $A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}$. We have $|\det A_j| = |-28|$. So the area of the parallelogram is 28.

Linear Transformations

- If T is a linear transformation and S is a set in the domain of T, let $T(S)$ denote the set of images of points in S.
- When S is a region bounded by a parallelogram, we also refer to S as a parallelogram.

Theorem

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det A| \cdot \{\text{area of } S\}.$$

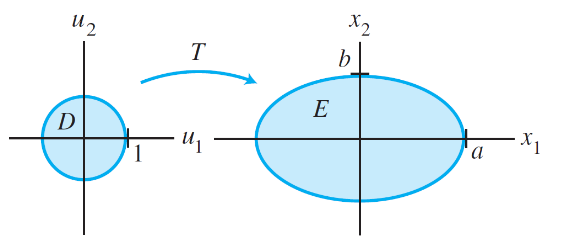
If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\text{volume of } T(S)\} = |\det A| \cdot \{\text{volume of } S\}.$$

- The conclusions also hold whenever S is a region in $\mathbb{R}^2$ with finite area or a region in $\mathbb{R}^3$ with finite volume.

Example

- Let a and b be positive numbers. Find the area of the region E bounded by the ellipse whose equation is
$$\frac{x_1^2}{a^2}+\frac{x_2^2}{b^2}=1.$$
- We claim that E is the image of the unit disk D under the linear transformation T determined by the matrix $A=\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.



Example (Cont'd)

- If $\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $\boldsymbol{x} = A\boldsymbol{u}$, then

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

  i.e., $u_1 = \frac{x_1}{a}$ and $u_2 = \frac{x_2}{b}$.
  It follows that $\boldsymbol{u}$ is in the unit disk, with $u_1^2 + u_2^2 \leq 1$, if and only if $\boldsymbol{x}$ is in E, with $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} \leq 1$.
  Thus, we get

$$\begin{aligned} \{\text{area of ellipse}\} &= \{\text{area of } T(D)\} \\ &= |\det A| \cdot \{\text{area of } D\} \\ &= ab \cdot \pi \cdot 1^2 = \pi ab. \end{aligned}$$