Mathematical Logic

(Based on lecture slides by Stan Burris)

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LSSU Math 300



Central Themes in Mathematical Logic

- What is truth? Which things/statements/ideas etc. are true?
- Closely related, what is falsity? Which statements are false?
 - It turns out to be mind-boggling to answer if we are too ambitious!
- To make reasonable advances, we restrict our vocabulary (language) to a small "artificial" one and allow sentences (formulas) only using that small vocabulary.

In other words, we deviate from our natural language; we do not want to (or cannot) be too inclusive and too ambitious!

• We also restrict what things we talk about (models). We talk mostly about "mathematical" structures; not arbitrary situations in real life; Again, we do not want to (or cannot) be too inclusive and too ambitious, because the difficulty in reasoning about very general "real-life" situations is a Herculean task...

Sets (or Classes or Collections)

- We take the notion of "A is a set" or "A is a class" as understood; we provide no formal definition.
- Intuitively a set (or a class or a collection) is any collection of things; the things in the set are called its elements or its members.



• **Remark:** We use the words **set** and **class** interchangeably, just as was the custom before 1900. There are some more contemporary theoretical reasons for which

these terms are sometimes distinguished nowadays.

Membership, Equality and Empty Set

We say "x is a member of the set A" or "x belongs to the set A" or, simply, "x is in A" and we write x ∈ A to mean



• If x is not in A, we write $x \notin A$:



- Two sets A and B are equal, written A = B, if they have exactly the same elements; Equivalently, A and B are equal, if every element of A is a member of B and every element of B is a member of A.
- The empty set Ø is the set with no elements in it.

Exercises

- Recall A = B if every element of A is in B and vice-versa.
- Let A be the set of all male human beings. Let B be the set of all sons of a human being. Show that A = B.
 Show first that every element of A is also an element of B.
 Next show that every element of B is also an element of A.
 Finally, conclude that A and B have exactly the same elements.
- Show that there can be only one empty set.
 Assume there are two such Ø and Ø'.
 Next show that every element of Ø is a member of Ø';
 Now show, also, that every element of Ø' is a member of Ø;
 But this means that Ø = Ø'; that is there can only exist one emptyset!

The Intention and the Extension of a Concept

- From a philosophical viewpoint, a notion has both intension and extension.
- Intension has to do with properties:

x is a rose. (The property itself)

• Extension has to do with sets:

The set of all roses. (Set of objects having the property)

- From the mathematical point of view
 - a property (intention) is either true or false for a specific object;
 - the set (extension) consists of all objects for which the corresponding property is true.
- Can we provide a simple example from elementary mathematics?

Using Properties to Create Sets

- We can express the construction of a **set** (an extension) using a **property** (an intention) with **set-builder notation**.
- We write

$$\{x: P(x)\}$$

- This is read "the set of all x, such that property P holds for x".
- For example the set Roses of all roses can be written

$$Roses = \{x : x \text{ is a rose}\}.$$

This "naive" way of constructing sets has led to paradoxes when one considers classes that are "too big".
 Even though such problems do not arise in our work, we present a famous one in the following slides.

Eubulides of Miletus

• Eubulides (Ef-vou-li-this) of Miletus (4th Century B.C.)



Eubulides' Liar Paradox

• Eubulides asked the following question:

A man says that he is lying. Is what he says true or false?

- Trying to assign a truth value to the man's statement leads to a paradox!
- If what he says is true, then he is lying. But if he is lying, then what he says is false. It follows that if what he says is true, then what he says is false.
- If what he says is false, then he is not lying. But if he is not lying, then what he says is true. It follows that if what he says is false, then what he says is true.
- The man's sentence employs self-reference; This is a common way of obtaining paradoxes and contradictions.

Introduction and Primer on Sets

Bertrand Arthur William Russell

 Bertrand Arthur William Russell, born in Trellech, Monmouthshire, United Kingdom (1872-1970)





The Russell Paradox (Bertrand Russell 1901)

- Let P(x) be the property " $x \notin x$ ".
- An object has this property if it does not belong to itself.
- Consider a set $A = \{x : P(x)\}$. Then

$$A \in A$$
 iff $P(A)$
iff $A \notin A$.

- This is clearly a contradiction.
- We have to be more careful about how we build sets (using an intension (property) to build an extension (set)) to avoid such problems!

Philip Edward Bertrand Jourdain

• Philip Edward Bertrand Jourdain, born in Ashbourne, Derbyshire, United Kingdom (1879-1919)



Jourdain's Card Paradox

• Consider the two faces of a card:

Front: The sentence on the other side of this card is TRUE.

Back: The sentence on the other side of this card is FALSE.

- Trying to assign a truth value to either of them leads to a paradox!
- If the first statement is true, then so is the second. But if the second statement is true, then the first statement is false. It follows that if the first statement is true, then the first statement is false.
- If the first statement is false, then the second is false, too. But if the second statement is false, then the first statement is true. It follows that if the first statement is false, then the first statement is true.
- The same mechanism applies to the second statement.
- Neither of the sentences employs self-reference; Instead this is a case of circular reference.

Using Properties Carefully to Create Sets

Axiom of Specification or Comprehension

Given any property P(x) and set A there is a set whose elements are the elements x of A for which P(x) is true.

• The usual way of describing this set is by set-builder notation, namely

 $\{x \in A : P(x)\}.$

- This is read: "the set of all x in A, for which P holds".
- Recall the notation for the various number systems:

 In natural (it includes 0),
 Z integer,
 Q rational,
 R real and
 C complex;
- Example: Assume all previous number systems are given. An application of the Axiom of Comprehension gives the set of all even natural numbers:

$$\{x \in \mathbb{N} : x \text{ is even}\}.$$

Leonhard Euler

• Leonhard Euler, born in Basel, Switzerland (1707-1783)





Subsets of a Set

• Given two sets A and B, we say that A is a **subset** of B, and write $A \subseteq B$, if every element of A is an element of B.



(An Euler Diagram)

- The notation $A \not\subseteq B$ means that "A is not a subset of B".
- The notation $A \subseteq B$ (also $A \subset B$) means that " $A \subseteq B$ and $A \neq B$ ".
- In case A ⊊ B, we say "A is a proper subset of B"; Note that this means that "A is a subset of B and B has at least one element that is not an element of A".
- Let us create a couple of examples!

Union of Sets

• Given two sets A and B we define the **union** $A \cup B$ of A and B by

 $x \in A \cup B$ iff $x \in A$ or $x \in B$.



Figure : The elements of $A \cup B$ are in the shaded region.

- Caution! The textbook uses a different convention... It shades those regions that are known to be empty.
- Example: $\{0, 2, 4, 6, 8, 10\} \cup \{0, 3, 6, 9\} = \{0, 2, 3, 4, 6, 8, 9, 10\}.$

Intersection of Sets

Given two sets A and B we define the intersection A ∩ B of A and B by

 $x \in A \cap B$ iff $x \in A$ and $x \in B$.



Figure : The elements of $A \cap B$ are in the shaded region.

• Example: $\{0, 2, 4, 6, 8, 10\} \cap \{0, 3, 6, 9\} = \{0, 6\}.$

Difference of Sets

• Given two sets A and B we define the **difference** $A \setminus B$ (also written A - B) of A minus B by

$$x \in A \setminus B$$
 iff $x \in A$ and $x \notin B$.



Figure : The elements of $A \setminus B$ are in the shaded region.

• Example: $\{0, 2, 4, 6, 8, 10\} \setminus \{0, 3, 6, 9\} = \{2, 4, 8, 10\}.$

Universe U and Complements

- In a given context it is understood that the sets being considered are all subsets of a given set *U* called the **universe** (of discourse).
- Given a set A (a subset of the universe U) we define the **complement** A' of A by

$$x \in A'$$
 iff $x \notin A$.



Figure : The elements of A' are in the shaded region.

• Example: Suppose $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Then $\{0, 2, 4, 6, 8, 10\}' = \{1, 3, 5, 7, 9\}$.

Some Exercises with Sets

• Show that, for a set A in a universe U, we have

$$(A')' = A.$$

• Show that, for any sets A, B in a universe U, we have

$$A\cup (B\backslash A)=A\cup B.$$

• Show that, for any sets A, B, C in a universe U, we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cup C).$$

• Show that, for any sets A, B in a universe U, we have

$$(A\cup B)'=A'\cap B'.$$

• Show that, for any sets A, B, C in a universe U, we have

$$(A \backslash B) \backslash C \subseteq A \backslash C.$$

Proof of $A \cup (B \setminus A) = A \cup B$

- We show first that $A \cup (B \setminus A) \subseteq A \cup B$. If $x \in A \cup (B \setminus A)$, then
 - $x \in A$ or In this case $x \in A \cup B$.
 - $x \in B \setminus A$. In this case $x \in B$. Therefore, $x \in A \cup B$.
- We show next that $A \cup B \subseteq A \cup (B \setminus A)$. If $x \in A \cup B$, then
 - $x \in A$ or
 - In this case $x \in A \cup (B \setminus A)$.
 - *x* ∈ *B*.
 - If $x \in A$, then $x \in A \cup (B \setminus A)$. If $x \notin A$, then $x \in B \setminus A$, whence, again $x \in A \cup (B \setminus A)$.

Proof of $(A \setminus B) \setminus C \subseteq A \setminus C$

• We show that
$$(A \setminus B) \setminus C \subseteq A \setminus C$$
.
If $x \in (A \setminus B) \setminus C$, then
 $x \in (A \setminus B)$ and $x \notin C$
whence $x \in A$ and $x \notin B$ and $x \notin C$
and, therefore, $x \in A \setminus C$.