Mathematical Logic

(Based on lecture slides by Stan Burris)

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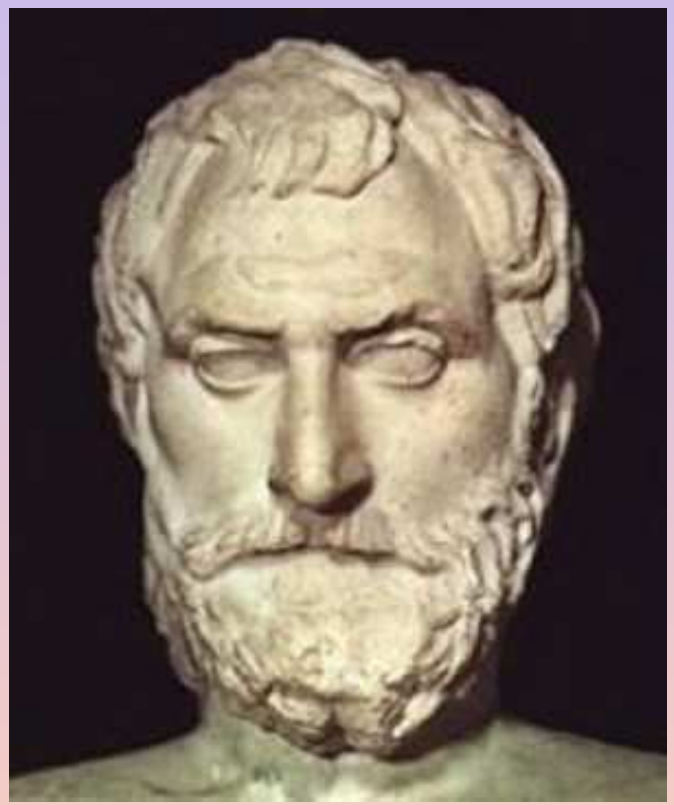
Subsection 1

Natural Philosophy and Mathematics

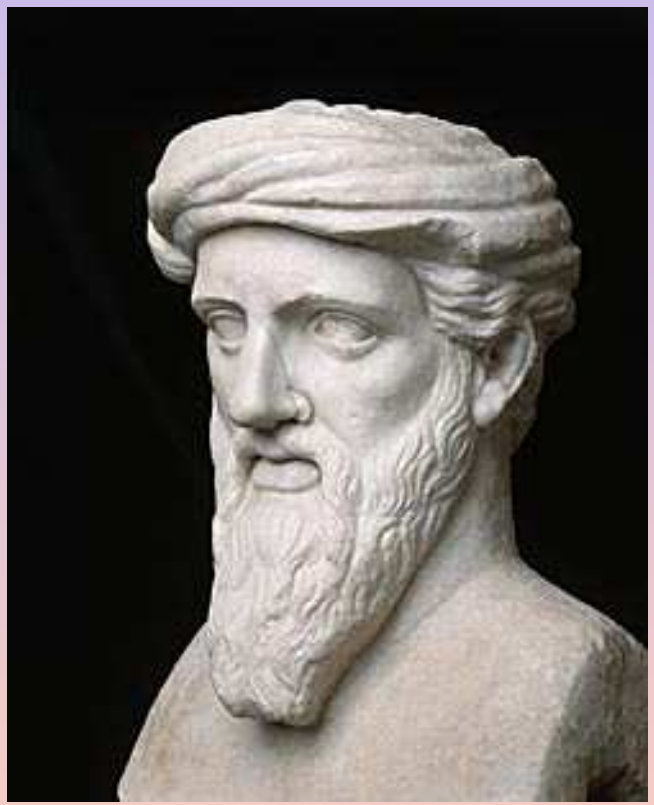
At the Beginning Greeks Created Deduction...

- Thalis (Thales) of Miletus (624-545 B.C.)



Natural Numbers as the Basis of the Universe

- Pythagoras of Samos (566-497 B.C.)

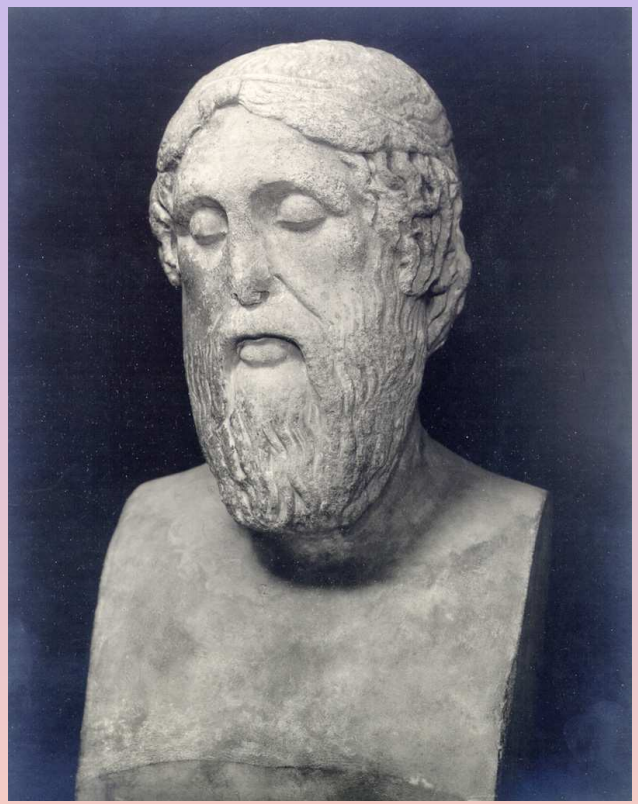


Subsection 2

Sophistry

Epimenides of Knossos

- Epimenides of Knossos ($\sim$ 600 B.C.)



Another Version of The Liar Paradox

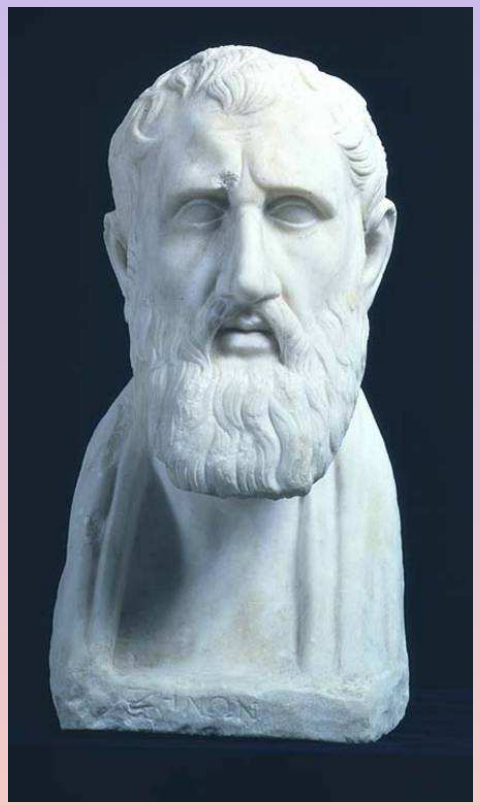
- Epistle of Saint Paul to Titus:

> One of themselves, a prophet of their own, said, "*Cretans are always liars, wily beasts, lazy gluttons*". This testimony is true.

- If that prophet was telling the truth, then, as a Cretan, he was lying! Thus, contrary to what Paul claims, Cretans cannot be attributed with all three qualities that the Cretan claimed characterize them!

Zeno of Elea

- Zeno of Elea (490-430 B.C.)



The Achilles and the Tortoise Paradox

- Zeno was probing:

> If a tortoise is given a head start in a race with Achilles, then Achilles can never catch the tortoise because Achilles must always get to where the tortoise has already been.

- This "paradox" is related to the fact that a sum of infinitely many numbers may be finite, as you well know from the theory of infinite series!
  E.g.,

$$\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2.$$

Protagoras of Abdera

- Protagoras of Abdera (490-420 B.C.)

The Protagoras and Evathlos Paradox

- Evathlos was studying rhetoric and argumentation with the master Protagoras to become a lawyer.
- He had paid half his tuition fees in advance and agreed to pay the remaining half when he won his first case in court.
- Evathlos kept procrastinating in taking up the legal practice.
- Protagoras decided to sue him to recover the remaining fees.
- Protagoras' argument in court:

> If I win this case, then Evathlos should pay me. If I lose the case, then, because of our agreement he should pay me.

- Evathlos' argument in court:

> If I win this case, by the court's decision, I should not pay. If I lose the case, then, because of our agreement, I should not pay.

Miguel de Cervantes Saavedra

- Miguel de Cervantes Saavedra, born in Alcalá de Henares, Castile (1547-1616)

The Judges and Gallows Paradox

- In **Don Quixote** to cross a certain bridge of the island of Baratavia, which was governed by Sancho Panza, one had to answer the questions of four judges.
- The law stipulated:

> Anyone who crosses this river shall first take oath as to whither he is bound and why. If he swears to the truth, he shall be permitted to pass, but if he tells a falsehood, he shall die without hope of pardon on the gallows that has been set up there.

- A traveler came one day and said to the judges:

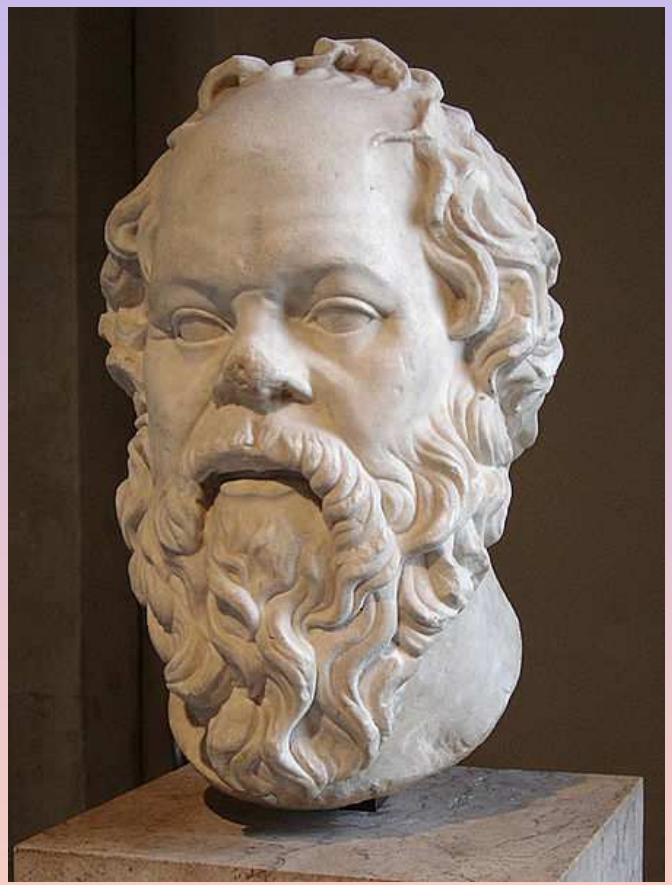
> My destination is to die upon the gallows....

Subsection 3

Aristotle

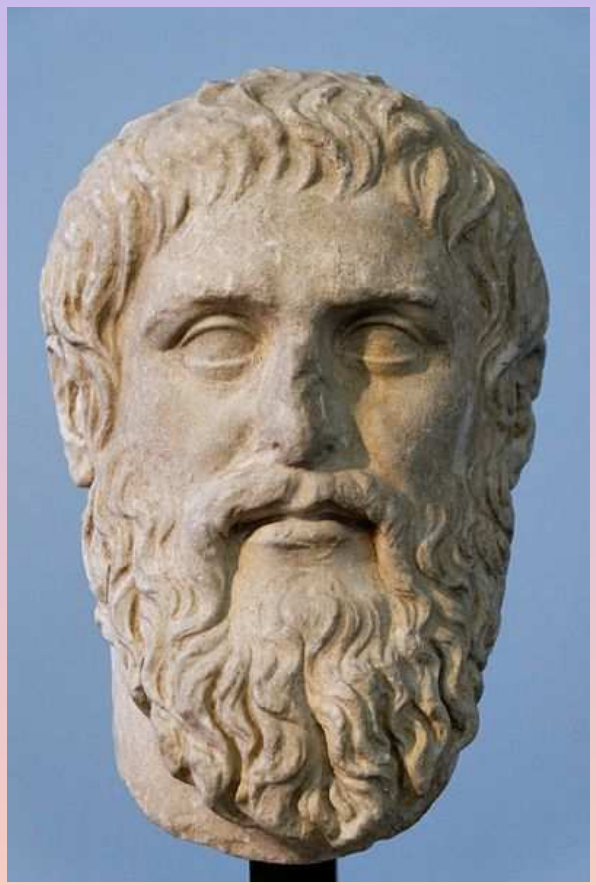
Socrates of Athens

- Socrates of Athens (469-399 B.C.)



Plato of Athens

- Plato of Athens (424-348 B.C.)



Aristotelis (Aristotle) of Stageira

- Aristotelis of Stageira (384-322 B.C.)

....and then Aristotelis created LOGIC

- One of Aristotelis' syllogisms:

> All men are mortal.
> Socrates is a man.
> $\therefore$ Socrates is mortal.

- Another of Aristotelis' syllogisms:

> Some students are clever.
> Some clever people are lazy.
> $\therefore$ Some students are lazy.

- The first is a valid syllogism; the second is not valid. (Why?)

Aristotelian Types of Statements

- Let S be a **subject** and P be a **predicate**;
- In his categorical syllogisms, Aristotelis permitted four kinds of **statements**:
  - **Universal Affirmative** A: All S is P
  - **Universal Negative** E: No S is P
  - **Particular Affirmative** I: Some S is P
  - **Particular Negative** O: Some S is not P

Aristotelian Syllogisms

- A **syllogism** is a 3-line argument of the form

  Major Premiss: ____□____□ (Squares can be P and M)
  Minor Premiss: ____□____□ (Squares can be S and M)
  Conclusion: ____S____P

- The major premiss is the one with the predicate of the conclusion.
- The minor premiss is the one with the subject of the conclusion.

The Number of Aristotelian syllogisms

- There are $2 \times 2 \times 2 \times 1 = 8$ possibilities for the major premiss $\_\_\_\_\square\_\_\_\_\square$;
- There are $2 \times 2 \times 2 \times 1 = 8$ possibilities for the minor premiss $\_\_\_\_\square\_\_\_\_\square$;
- There are only $2 \times 1 \times 2 \times 1 = 4$ possibilities for the conclusion $\_\_\_\_S\_\_\_\_P$;
- Therefore, we have $8 \times 8 \times 4 = 256$ different Aristotelian categorical syllogisms;
- A main goal of Aristotelian logic was to determine the valid categorical syllogisms.

Classification of Aristotelian Syllogisms: Moods

- The **mood** XYZ of a syllogism is the AEIO classification of the three statements in a syllogism, where the first letter X refers to the major premiss, Y to the minor premiss and Z to the conclusion;
- Example: The syllogism

> All students are clever.
> No clever people are lazy.
> $\therefore$ No students are lazy.

  has the mood EAE.
- There are $4 \times 4 \times 4 = 64$ distinct moods.

Classification of Aristotelian Syllogisms: Figures

- The **figure** of a syllogism refers to whether or not the middle term M comes first or second in each of the premisses.
- The four figures for syllogisms:

Figure 1

$$\begin{array}{l} \_\_\_\_M\_\_\_\_P \\ \_\_\_\_S\_\_\_\_M \\ \hline \_\_\_\_S\_\_\_\_P \end{array}$$

Figure 2

$$\begin{array}{l} \_\_\_\_P\_\_\_\_M \\ \_\_\_\_S\_\_\_\_M \\ \hline \_\_\_\_S\_\_\_\_P \end{array}$$

Figure 3

$$\begin{array}{l} \_\_\_\_M\_\_\_\_P \\ \_\_\_\_M\_\_\_\_S \\ \hline \_\_\_\_S\_\_\_\_P \end{array}$$

Figure 4

$$\begin{array}{l} \_\_\_\_P\_\_\_\_M \\ \_\_\_\_M\_\_\_\_S \\ \hline \_\_\_\_S\_\_\_\_P \end{array}$$

Venn Diagrams for A, E, I, O Statements

- Recall the statements:

  A: All S is P E: No S is P I: Some S is P O: Some S is not P

- SHADED regions have NO ELEMENTS in them!

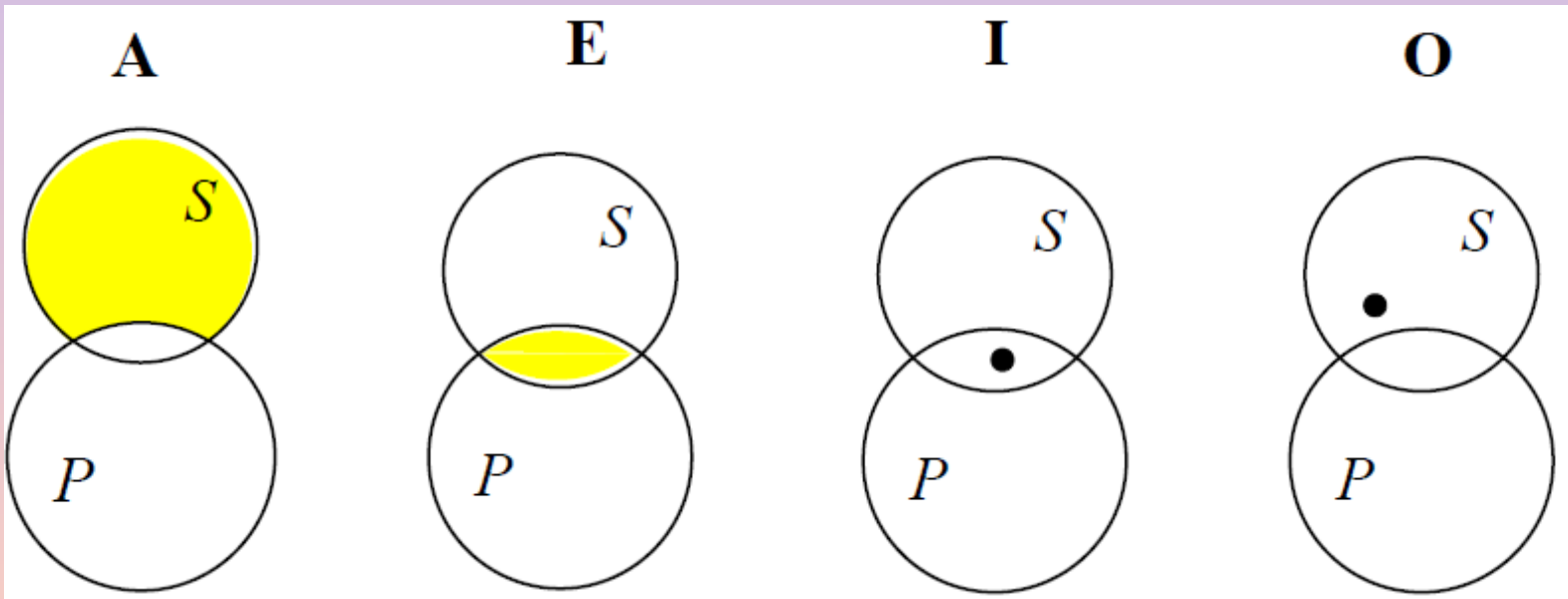
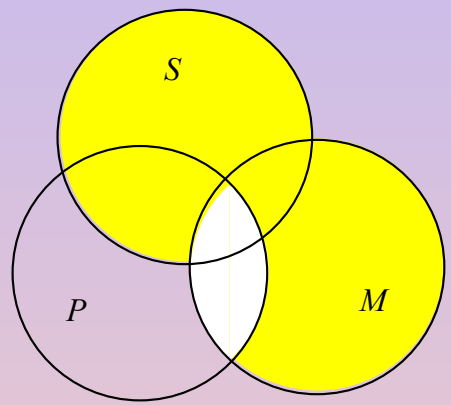


Figure 1 AAI Syllogism - The Modern Approach

- As an example we consider the validity of **Figure 1 AAI** syllogism.
- Note that it is determined by its figure and mood classification.

All M is P
All S is M
$\therefore$ Some S is P.



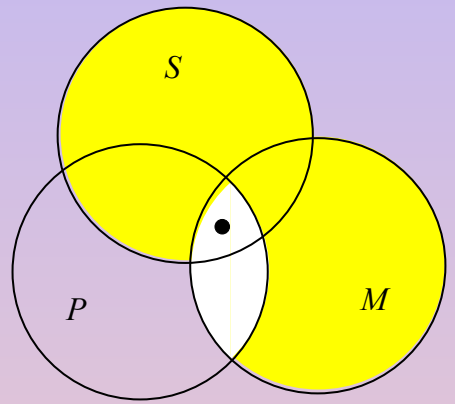
- This is not a valid syllogism!
- Think, for example, about the following: $M = \{0\}, P = \{0\}$ and $S = \emptyset$.

All M is P. ✓
All S is M. ✓
$\therefore$ Some S is P. ✠

Figure 1 AAI Syllogism - The Aristotelian Approach

- By Aristotelis' standards, the Figure 1 AAI syllogism is valid.

All M is P
All S is M
$\therefore$ Some S is P.

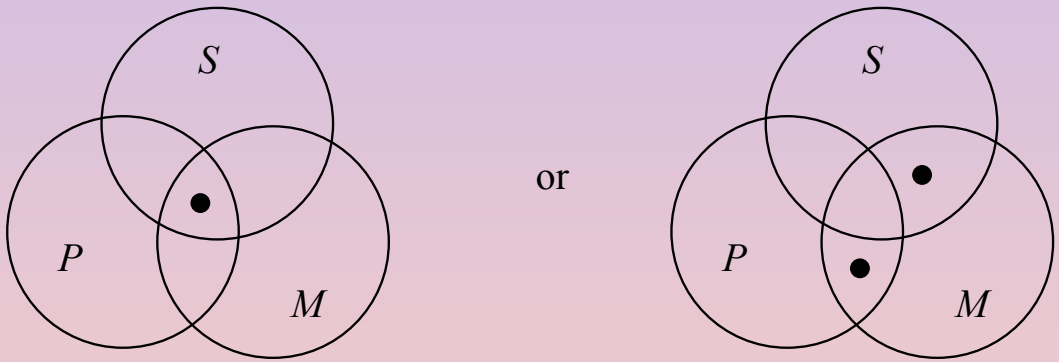


- The previous counterexample with $M = \{0\}, P = \{0\}$ and $S = \emptyset$ would not be considered by Aristotelis.
- Aristotelis would not admit empty sets or properties with empty extensions in his reasoning.
- This is a substantial difference (breakthrough?) when comparing with the modern approach.

Figure 3 III Syllogism - The Modern Approach

- As another example we consider the validity of **Figure 3 III** syllogism.

Some M is P
Some M is S
$\therefore$ Some S is P.



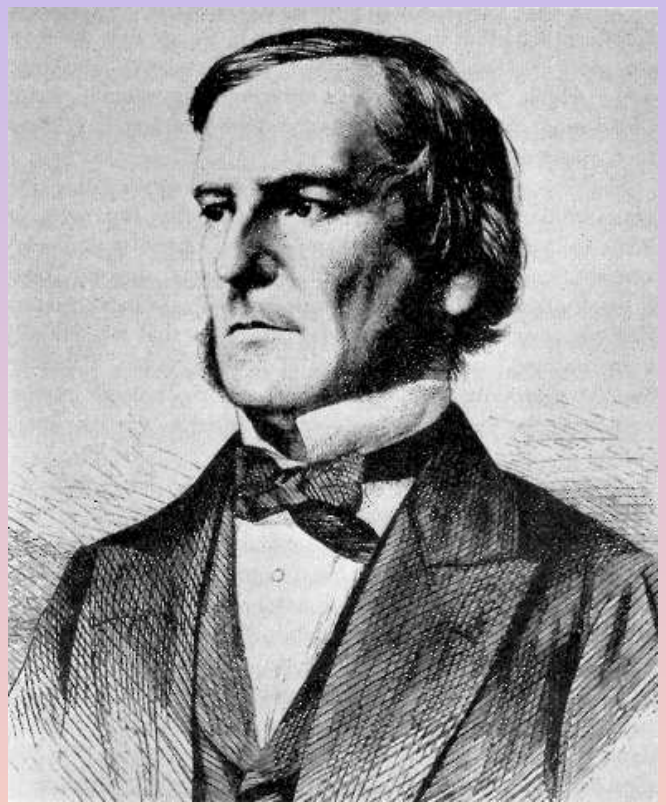
- This is not a valid syllogism, since the second case in the figure above presents a counterexample.
- We should be able to present this counterexample more formally!!

Subsection 4

George Boole

George Boole

- George Boole, born in Ballintemple, County Cork, Ireland (1815-1864)



....and then Boole created ALGEBRAIC LOGIC

- Boole's Key Idea: Use Equations to express Logical Statements!

Universal Statement...	...becomes the Equation
All S is P	$S \cap P' = 0 \quad (SP' = 0)$
No S is P	$S \cap P = 0 \quad (SP = 0)$

- Boole also had equations for the particular statements.
- But, by the end of the 1800s, they were considered a bad idea.

Example: A Syllogism in Equational Form

Figure 1 AAA Syllogism:

All M is P
All S is M
$\therefore$ All S is P

In Equational Form:

$$\begin{aligned} MP' &= 0 \\ SM' &= 0 \\ \therefore\ SP' &= 0 \end{aligned}$$

- To see that this is a valid argument, one may now use an equational argument!!

$$\begin{aligned} SP' &= S1P' \quad \text{(Intersection with universe)} \\ &= S(M \cup M')P' \quad (M \cup M' = 1) \\ &= SMP' \cup SM'P' \quad \text{(Distributivity)} \\ &= S0 \cup 0P' \quad \text{(Hypotheses)} \\ &= 0 \cup 0 \quad \text{(Intersection with } \emptyset) \\ &= 0. \quad (\emptyset \cup \emptyset = \emptyset) \end{aligned}$$

An Equational Syllogism

1: PQ' = 0
2: QR' = 0
3: RS' = 0
$\therefore$ PS' = 0.

Proceed as follows:

$$\begin{aligned} PS' &= P(Q \cup Q')S' \\ &= PQS' \cup PQ'S' \\ &= PQS' \\ &= PQ(R \cup R')S' \\ &= PQRS' \cup PQR'S' \\ &= 0 \cup 0 \\ &= 0. \end{aligned}$$

Another Big Step Forward

- Boole applied the algebra of equations to arguments with many premisses and many variables.
- These give rise to systems of many equations in many variables:

$$
\begin{aligned}
&F_1(A_1, \ldots, A_m, B_1, \ldots, B_n) = 0 \\
&F_2(A_1, \ldots, A_m, B_1, \ldots, B_n) = 0 \\
&\vdots \\
&F_k(A_1, \ldots, A_m, B_1, \ldots, B_n) = 0 \\
&\therefore \ F(B_1, \ldots, B_n) = 0
\end{aligned}
$$

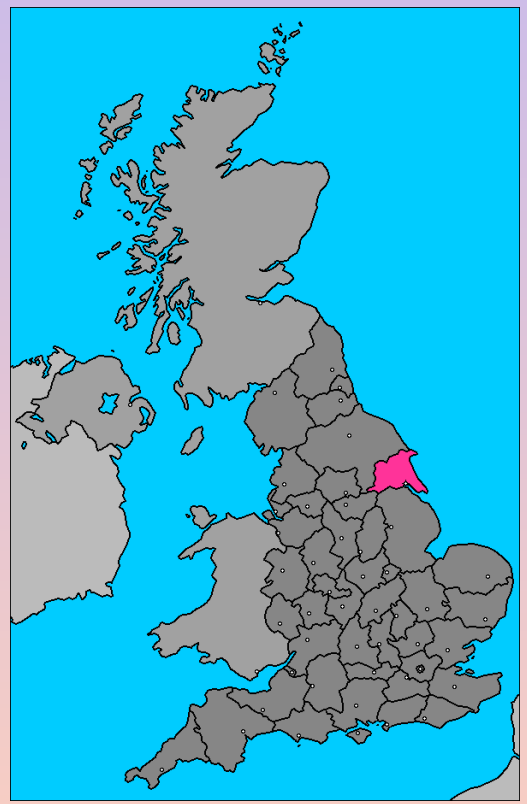
- Boole's work marks the end of the focus on Aristotelis' syllogisms and the beginning of Mathematical Logic.

Subsection 5

Venn Diagrams

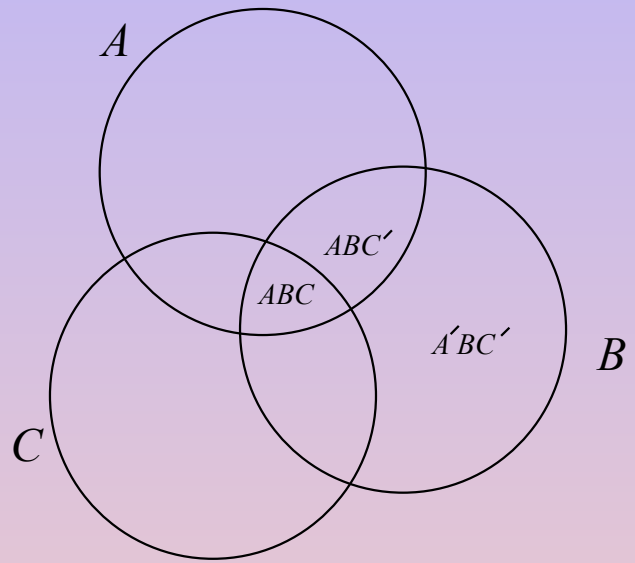
John Venn

- John Venn, born in Kingston upon Hull, Yorkshire, England (1834-1923)

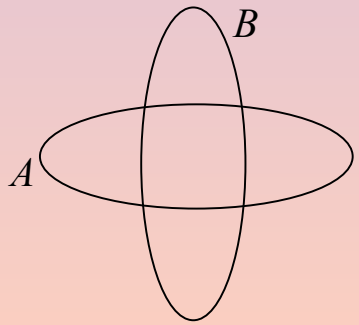


Venn Diagrams

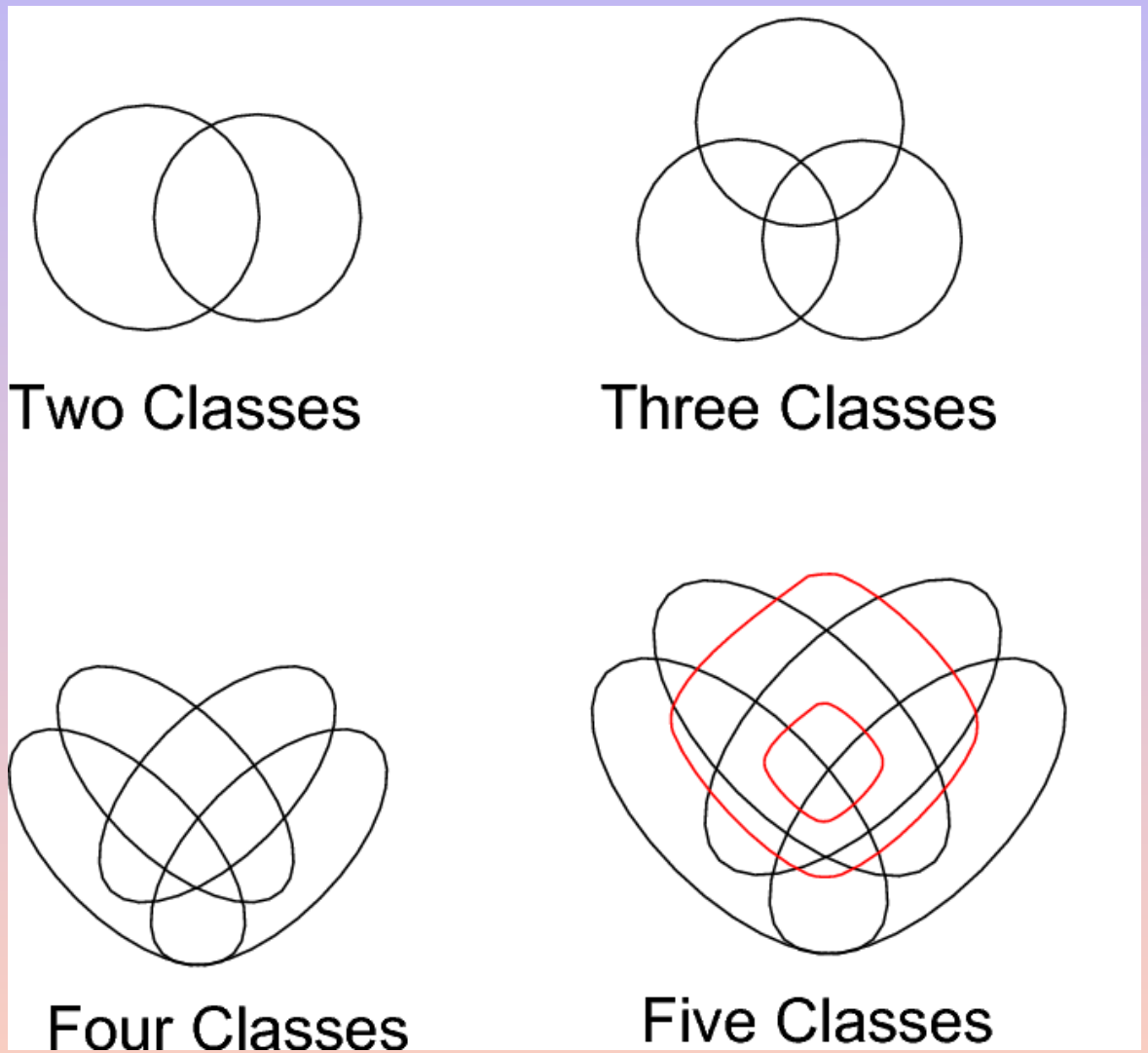
- **Venn Diagrams** subdivide the plane into *connected constituents*.



- The following is not legal. (Why?)

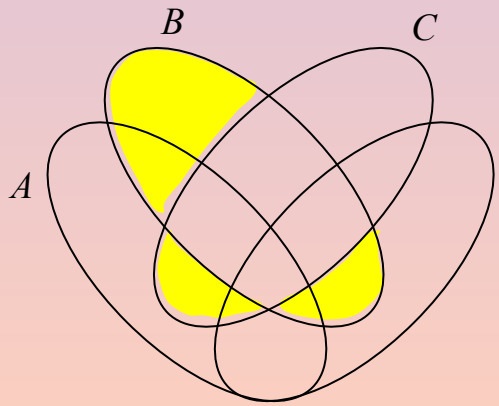


Venn's Diagrams for Two, Three, Four and Five Sets



Simplifying Premisses Before Shading Regions

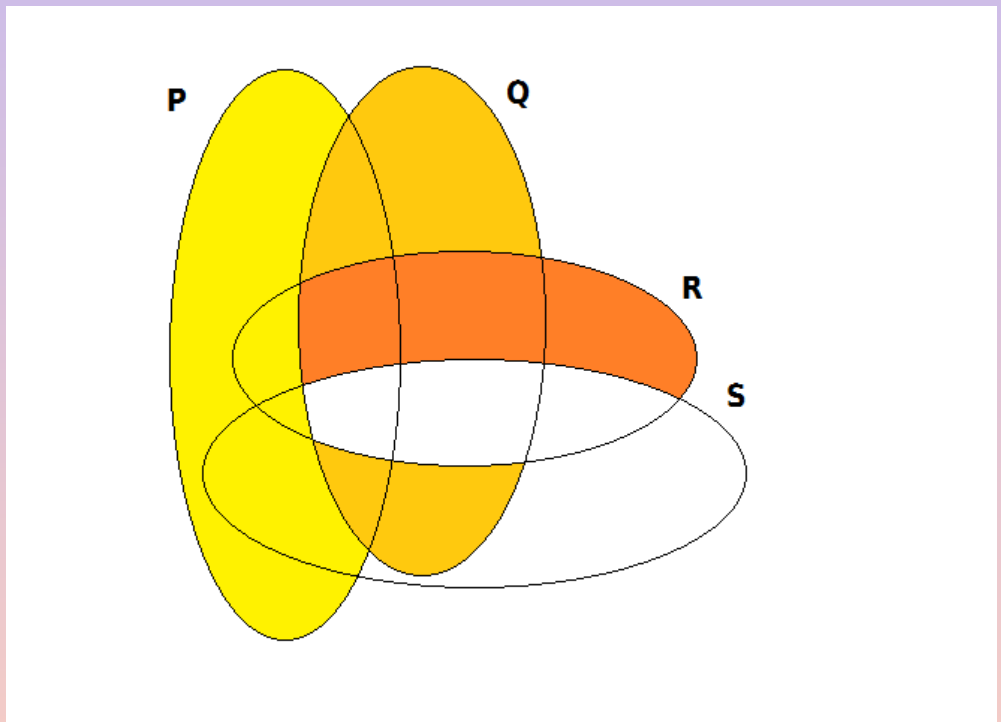
- Write each premiss as a union of intersections of classes or their complements. Then put each of the intersections equal to 0.
- An Example: Suppose that the premiss $A(B'C)' = 0$ is given. Rewrite, using De Morgan's identity, $A(B \cup C') = 0$. Use the Distributive Law to rewrite $AB \cup AC' = 0$. Thus, we must have $AB = 0$ and $AC' = 0$.
- Another Example: Suppose $(AC \cup B)(AB' \cup C') = 0$ is given. Rewrite using Distributivity: $ACAB' \cup BAB' \cup ACC' \cup BC' = 0$. Note that $BAB' = 0$ and $ACC' = 0$, whence $AB'C \cup BC' = 0$. Thus, we get $AB'C = 0$ and $BC' = 0$.
- Now shade



An Equational Syllogism

1: PQ' = 0
2: QR' = 0
3: RS' = 0
$\therefore$ PS' = 0.

Fill-in Venn diagram regions:

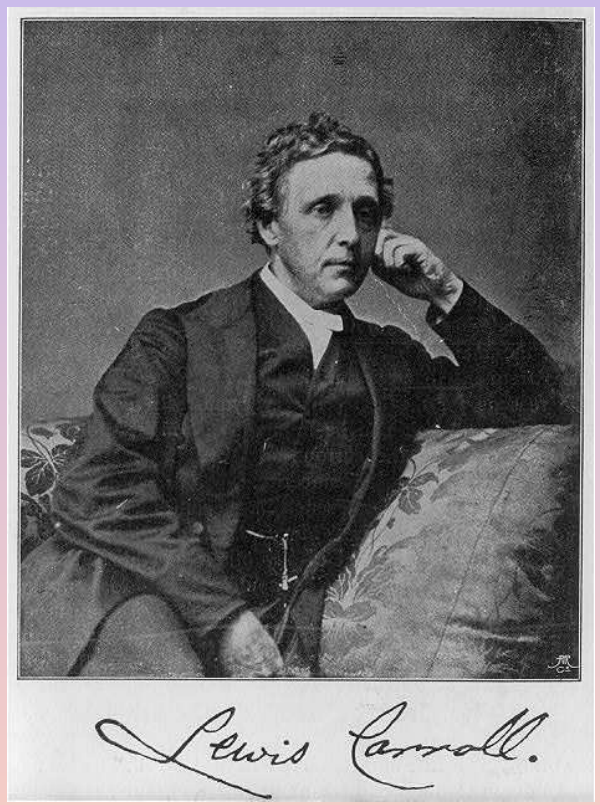


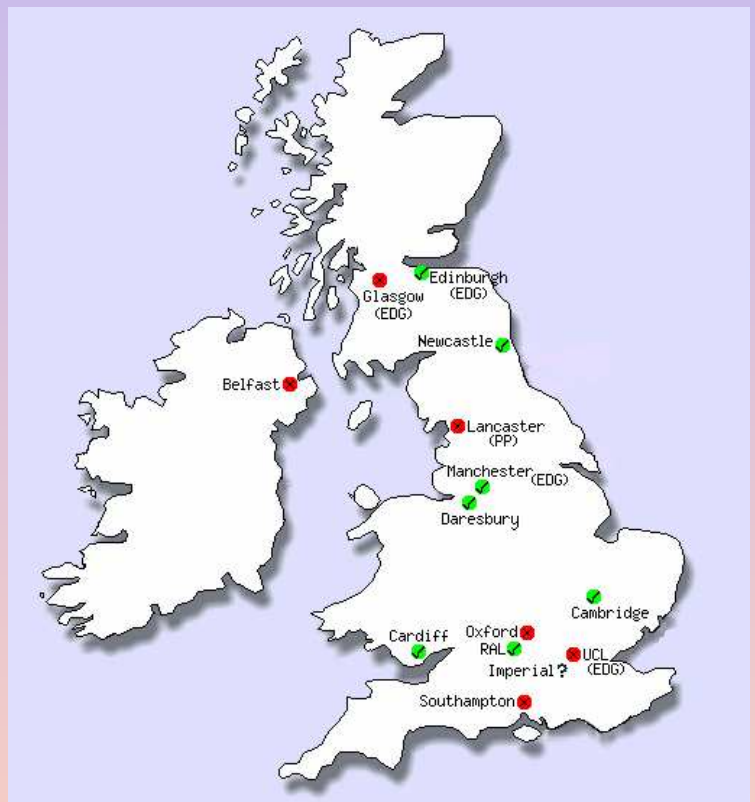
Subsection 6

Carroll's Tree Method

Lewis Carroll

- Lewis Carroll, born in Daresbury, Cheshire, England (1832-1898)





Lewis Carroll's Tree Method

- Suppose we want to show that $F = 0$.
- Since $F = FX \cup FX'$, this is equivalent to $FX = 0$ and $FX' = 0$.
- Therefore, to show a conclusion $F = 0$ is valid simply build a tree:
  - We start at the top with the conclusion;
  - Each branch should multiply out to 0.
- Example: Suppose we want to show the validity of the argument:

1: PQ' = 0
2: QR' = 0
3: RS' = 0
$\therefore$ PS' = 0.