Introduction to Real Analysis

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1. Sequences of Real Numbers, Convergence
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Subsection 1

Bounded Sequences

Bounded Sequences

Definition (Bounded Sequence)

A sequence (x_n) of real numbers is said to be **bounded** if the set $\{x_n : n \in \mathbb{P}\}$ is bounded.
A sequence that is not bounded is said to be **unbounded**.

- A sequence (x_n) in $\mathbb{R}$ is bounded if and only if there exists a positive real number K such that $|x_n| \leq K$, for all n.

  If $a \leq x_n \leq b$, for all n, and if $K = |a| + |b|$, then $|a| \leq K$ and $|b| \leq K$, whence $-K \leq -|a| \leq a \leq x_n \leq b \leq |b| \leq K$. Therefore, $|x_n| \leq K$.

Boundedness of Sum and Product

- Example: Every constant sequence ($x_n = x$, for all n) is bounded.
- Example: The sequence $x_n = (-1)^n$ is bounded.
- Example: The sequence $x_n = n$ is unbounded: For every real number K, there exists, by the Archimedean property, a positive integer n, such that $n = n \cdot 1 > K$, whence the set of all x_n is not bounded above.

Theorem

If (x_n) and (y_n) are bounded sequences in $\mathbb{R}$, then the sequences $(x_n + y_n)$ and $(x_n y_n)$ are also bounded.

- If $|x_n| \leq K$ and $|y_n| \leq K'$, then $|x_n + y_n| \leq |x_n| + |y_n| \leq K + K'$ and $|x_n y_n| = |x_n||y_n| \leq KK'$.

Subsection 2

Ultimately, Frequently

Ultimately, Frequently

Definition (Ultimately, Frequently)

Let (x_n) be a sequence in a set X and let A be a subset of X.

(i) We say that $x_n \in A$ **ultimately** if x_n belongs to A from some index onward, i.e., there is an index N, such that $x_n \in A$, for all $n \geq N$. Symbolically,

$$\exists N(n \geq N \Rightarrow x_n \in A).$$

(Equivalently, $\exists N(n > N \Rightarrow x_n \in A)$, because $n > N$ means the same thing as $n \geq N + 1$.)

(ii) We say that $x_n \in A$ **frequently** if, for every index N, there is an index $n \geq N$, for which $x_n \in A$. Symbolically,

$$(\forall N)(\exists n \geq N)(x_n \in A).$$

(Equivalently, $(\forall N)(\exists n > N)(x_n \in A)$.)

Examples

- Example: Let $x_n = \frac{1}{n}$, let $\epsilon > 0$ and let $A = (0, \epsilon)$. Then $x_n \in A$ ultimately.

  Choose an index N such that $\frac{1}{N} < \epsilon$. Then $n \geq N$ implies $\frac{1}{n} \leq \frac{1}{N} < \epsilon$.
- Example: For each positive integer n, let S_n be a statement (which may be either true or false). Let

$$A = \{n \in \mathbb{P} : S_n \text{ is true}\}.$$

We say that:

- S_n is **true frequently** if $n \in A$ frequently;
- S_n is **true ultimately** if $n \in A$ ultimately.

The following illustrate the usage:

- $n^2 - 5n + 6 > 0$ ultimately (in fact, for $n \geq 4$).
- n is frequently divisible by 5 (in fact, for $n = 5, n = 10, n = 15$, etc.).

Relation Between Ultimately and Frequently

Theorem

Let (x_n) be a sequence in a set X and let A be a subset of X. One and only one of the following conditions holds:

(1) $x_n \in A$ ultimately;

(2) $x_n \notin A$ frequently.

- To say that (1) is false means that, for every index N, the implication

$$n \geq N \Rightarrow x_n \in A$$

  is false. So there must exist an index $n \geq N$ for which $x_n \notin A$. This is precisely the meaning of (2).
- Example: If (x_n) is a sequence in $\mathbb{R}$, then either $x_n < 5$ ultimately, or $x_n \geq 5$ frequently, but not both.

Subsection 3

Null Sequences

Null Sequences

Definition (Null Sequence)

A sequence (x_n) in $\mathbb{R}$ is said to be **null** if, for every positive real number ϵ, $|x_n| < \epsilon$ ultimately.

- Example: The sequence $(\frac{1}{n})$ is null.
- The concept of null sequence can be expressed as follows:

  Given any $\epsilon > 0$ (no matter how small), the distance from $|x_n|$ to the origin is ultimately smaller than ϵ (in this sense, x_n "approaches" 0).
- A more informal way to express the same concept:

  x_n is arbitrarily small provided n is sufficiently large.
  - "arbitrarily small" is understood to suggest that the degree of smallness is specified in advance, before any indices are selected;
  - "sufficiently large" is understood in the sense of "ultimately" (not merely "frequently").

Properties of Null Sequences

Theorem

Let (x_n) and (y_n) be null sequences and let $c \in \mathbb{R}$. Then:

(1) (x_n) is bounded.

(2) (cx_n) is null.

(3) $(x_n + y_n)$ is null.

(4) If (b_n) is a bounded sequence then $(b_n x_n)$ is null.

(5) If (z_n) is such that $|z_n| \leq |x_n|$ ultimately, then (z_n) is also null.

(1) Let $\epsilon = 1$. There exists an index N, such that $|x_n| < 1$, for all $n > N$. If K is the largest of the numbers $1, |x_1|, |x_2|, \ldots, |x_N|$, then $|x_n| \leq K$, for every positive integer n. Thus (x_n) is bounded.

(2) For $\frac{\epsilon}{|c|} > 0$, there exists N, such that $|x_n| < \frac{\epsilon}{|c|}$, for all $n > N$. Thus, $|cx_n| < \epsilon$, for all $n > N$, showing that (cx_n) is null.

Proof of the Theorem on Null Sequences

(3) Let $\epsilon > 0$. Since (x_n) is null, there is an index N_1 such that $n \geq N_1$ implies $|x_n| < \frac{\epsilon}{2}$. Similarly, there is an index N_2 such that $n \geq N_2$ implies that $|y_n| < \frac{\epsilon}{2}$. If N is the larger of N_1 and N_2, then $n \geq N$ implies that $|x_n + y_n| \leq |x_n| + |y_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. This proves that $(x_n + y_n)$ is null.

(4) Let K be a positive real number such that $|b_n| \leq K$, for all n. Given any $\epsilon > 0$, choose an index N such that $n \geq N$ implies $|x_n| < \frac{\epsilon}{K}$. Then $n \geq N$ implies $|b_n x_n| = |b_n||x_n| \leq K|x_n| < \epsilon$. Thus, $(b_n x_n)$ is null.

Note that (2) is also a special case of (4).

(5) By assumption, there is an index N_1, such that $n \geq N_1$ implies $|z_n| \leq |x_n|$. Given any $\epsilon > 0$, choose an index N_2, such that $n \geq N_2$ implies $|x_n| < \epsilon$. If $N = \max\{N_1, N_2\}$, then $n \geq N$ implies $|z_n| \leq |x_n| < \epsilon$. Thus, (z_n) is null.

An Additional Theorem

Theorem

If $a_n \uparrow a$ or $a_n \downarrow a$, then the sequence $(a_n - a)$ is null.

- If $a_n \downarrow a$, then $a_n - a = a_n + (-a) \downarrow a + (-a) = 0$. In particular, $a_n - a \geq 0$ and $\inf(a_n - a) = 0$. Given any $\epsilon > 0$, choose an index N such that $a_N - a < \epsilon$. Then $n \geq N$ implies $|a_n - a| = a_n - a$ $\leq a_N - a < \epsilon$. Thus, $(a_n - a)$ is null.

  For $a_n \uparrow a$, we apply the preceding case to $-a_n \downarrow -a$.
- Example: If $|x| < 1$, then the sequence (x^n) is null.

  Writing $c = |x|$, we have $c^n \downarrow 0$. Thus, the sequence $(|x^n|) = (c^n)$ is null. Therefore, (x^n) is null.
- Example: Fix $x \in \mathbb{R}$ and let $x_n = x$, for all n. The constant sequence (x_n) is null if and only if $x = 0$.

  The condition "$|x_n| < \epsilon$ ultimately" means $|x| < \epsilon$. If this happens for every $\epsilon > 0$, then $x = 0$.

Subsection 4

Convergent Sequences

Convergent Sequences

Definition (Convergent Sequence)

A sequence (a_n) in $\mathbb{R}$ is said to be **convergent** in $\mathbb{R}$ if there exists a real number a, such that the sequence $(a_n - a)$ is null, and **divergent** if no such number exists.

- Such a number a (if it exists) is unique:
  Suppose that both $(a_n - a)$ and $(a_n - b)$ are null. Let $x_n = (a_n - b) - (a_n - a) = a - b$. Being the difference of null sequences, (x_n) is null. But the constant $(a - b)$ is null if and only if $a - b = 0$, i.e., $a = b$.

Definition (Limit)

Let (a_n) be a convergent sequence, such that $(a_n - a)$ is null. Then the number a is called the **limit** of the convergent sequence (a_n), and the sequence is said to **converge** to a. This is expressed by writing $\lim_{n\to\infty} a_n = a$, $a_n \to a$ as $n \to \infty$, or, more concisely, $\lim a_n = a$ or $a_n \to a$.

Examples

- If $a_n \uparrow a$ or $a_n \downarrow a$, then $a_n \to a$. Thus, every bounded monotone sequence is convergent.
- $a_n \to 0$ iff (a_n) is null.

  Since $a_n - 0 = a_n$, this is immediate from the definition.
- $a_n \to a$ iff $a_n - a \to 0$.

  This follows from the previous remark.
- $x^n \to 0$ iff $|x| < 1$.

  If $|x| < 1$, then $x_n \to 0$ since $(|x^n|)$ is decreasing and bounded below.
  If $|x| > 1$, then $|x^n| = |x|^n > 1$, for all n, whence (x_n) is not null.
- If $|x| < 1$ and $a_n = 1 + x + x^2 + \cdots + x^{n-1}$, then $a_n \to \frac{1}{1-x}$.

$$a_n - \frac{1}{1-x} = \frac{1-x^n}{1-x} - \frac{1}{1-x} = \frac{-1}{1-x} \cdot x^n.$$

  This is a constant multiple of a null sequence. Hence it is null.

Properties of Convergent Sequences

Theorem (Convergent Sequences)

Let (a_n), (b_n) be convergent sequences in $\mathbb{R}$, say $a_n \to a$ and $b_n \to b$, and let $c \in \mathbb{R}$. Then:

(1) (a_n) is bounded.

(2) $ca_n \to ca$.

(3) $a_n + b_n \to a + b$.

(4) $a_n b_n \to ab$.

(5) $|a_n| \to |a|$.

(6) If $b \neq 0$, then $|b_n|$ is ultimately bounded away from 0, in the sense that there exists an $r > 0$ (for example, $r = \frac{1}{2}|b|$) such that $|b_n| \geq r$ ultimately.

(7) If b and the b_n are all nonzero, then $\frac{a_n}{b_n} \to \frac{a}{b}$.

(8) If $a_n \leq b_n$, for all n, then $a \leq b$.

Proof of the Theorem

(1) $(a_n - a)$ is null. So it is bounded. Since $a_n = (a_n - a) + a$, (a_n) is the sum of two bounded sequences.

(2) $ca_n - ca = c(a_n - a)$ is a scalar multiple of a null sequence. So it is null.

(3) $(a_n + b_n) - (a + b) = (a_n - a) + (b_n - b)$ is the sum of two null sequences. Therefore it is null.

(4) $a_n b_n - ab = a_n(b_n - b) + (a_n - a)b$. Since (a_n) is bounded and $(b_n - b)$, $(a_n - a)$ are null, it follows that $(a_n b_n - ab)$ is null.

(5) $||a_n| - |a|| < |a_n - a|$, where $(a_n - a)$ is null. So $(|a_n| - |a|)$ is null.

Proof of the Theorem (Cont'd)

(6) Let $r = \frac{1}{2}|b|$ and choose an integer N such that $n \geq N$ implies $|b_n - b| \leq r$. Then, for all $n \geq N$, $2r = |b| = |(b - b_n) + b_n| \leq |b - b_n| + |b_n| \leq r + |b_n|$ whence $|b_n| \geq r$.

(7) From (6), it follows that the sequence $\left(\dfrac{1}{b_n}\right)$ is bounded. Thus $\dfrac{1}{b_n} - \dfrac{1}{b} = \dfrac{1}{b_n b}(b - b_n)$ is the product of a bounded sequence and a null sequence. Therefore it is null. This implies that $\dfrac{1}{b_n} \to \dfrac{1}{b}$. Hence $\dfrac{a_n}{b_n} = a_n \dfrac{1}{b_n} \to a\dfrac{1}{b} = \dfrac{a}{b}$.

(8) Let $c_n = b_n - a_n$ and $c = b - a$. Then $c_n \geq 0$ and $c_n \to c$. Our problem is to show that $c \geq 0$. By (5), $|c_n| \to |c|$, i.e., $c_n \to |c|$. Since $c_n \to c$, $c = |c| \geq 0$ by the uniqueness of limits.

Suprema and Infima as Limits of Sequences

Theorem

If $A \subseteq \mathbb{R}$ is nonempty and bounded above, and if $M = \sup A$, then there exists a sequence (x_n) in A such that $x_n \to M$.

- For each positive integer n, choose $x_n \in A$ so that $M - \frac{1}{n} < x_n \leq M$. Then $\lim (M - \frac{1}{n}) \leq \lim x_n \leq \lim M$, whence $\lim x_n = M$.
- Similarly, if $A \subseteq \mathbb{R}$ is nonempty and bounded below, then $\inf A$ is the limit of a sequence in A.

Subsection 5

Subsequences, Bolzano-Weierstraß Theorem

Subsequences

- Given a sequence (x_n), there are various ways of forming "subsequences":
  - Take every other term $x_1, x_3, x_5, \ldots$;
  - Take all of the terms from some index onward $x_6, x_7, x_8, \ldots$;
  - Take all terms for which the index is a prime number $x_2, x_3, x_5, x_7, \ldots$.

  One is free to discard any terms, as long as infinitely many remain.

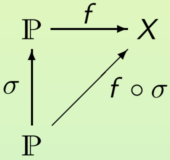
Definition (Subsequence)

Let (x_n) be any sequence. Choose a strictly increasing sequence of positive integers $n_1 < n_2 < n_3 < \cdots$ and define $y_k = x_{n_k}$, $k = 1, 2, 3, \ldots$. One calls (y_k) a **subsequence** of (x_n). This is also expressed by saying that $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ is a subsequence of $x_1, x_2, x_3, \ldots$, or that (x_{n_k}) is a subsequence of (x_n).

- Forming a subsequence amounts to choosing a sequence of indices, the essential thing being that the chosen indices must form a strictly increasing sequence.

Some Remarks

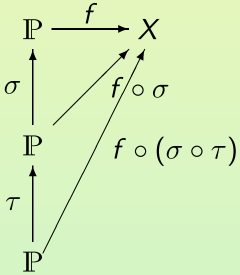
- Let (n_k) be a strictly increasing sequence of positive integers $n_1 < n_2 < n_3 < \cdots$. For every positive integer N, there exists a positive integer k, such that $n_k > N$ (whence $n_j > N$, for all $j \geq k$). It suffices to show that $n_k \geq k$, for all positive integers k. This is obvious for $k = 1$. Assuming inductively that $n_k \geq k$, we have $n_{k+1} \geq n_k + 1 \geq k + 1$.
- A sequence (x_n) in a set X can be thought of as a function $f : \mathbb{P} \to X$, where $f(n) = x_n$. A subsequence of (x_n) is obtained by specifying a strictly increasing function $\sigma : \mathbb{P} \to \mathbb{P}$ and taking the composite function $f \circ \sigma$:



Writing $n_k = \sigma(k)$, we have $(f \circ \sigma)(k) = f(\sigma(k)) = f(n_k) = x_{n_k}$.

One More Remark

- An application of the preceding remark: If (y_k) is a subsequence of (x_n), then every subsequence of (y_k) is also a subsequence of (x_n).
  The essence lies in the fact that, if $\sigma : \mathbb{P} \to \mathbb{P}$ and $\tau : \mathbb{P} \to \mathbb{P}$ are strictly increasing, then so is $\sigma \circ \tau : \mathbb{P} \to \mathbb{P}$.
  In detail, suppose that $f : \mathbb{P} \to X$ defines the sequence (x_n), i.e., $f(n) = x_n$, and that $\sigma : \mathbb{P} \to \mathbb{P}$ defines the subsequence (y_k), i.e., $y_k = f(\sigma(k))$. Write $g = f \circ \sigma$. Then $g : \mathbb{P} \to X$, with $g(k) = y_k$.
  Suppose (z_i) is a subsequence of (y_k), say defined by $\tau : \mathbb{P} \to \mathbb{P}$, so that $z_i = g(\tau(i))$. Then $g \circ \tau = (f \circ \sigma) \circ \tau = f \circ (\sigma \circ \tau)$, so (z_i) is defined by the strictly increasing function $\sigma \circ \tau : \mathbb{P} \to \mathbb{P}$.



Properties of Subsequences

Theorem

Let (a_n) be a sequence in $\mathbb{R}$ and let (a_{n_k}) be a subsequence of (a_n).

(1) If (a_n) is bounded, then so is (a_{n_k}).

(2) If (a_n) is null, then so is (a_{n_k}).

(3) If (a_n) is convergent, then so is (a_{n_k}); more precisely, if $a_n \to a$ as $n \to \infty$, then also $a_{n_k} \to a$ as $k \to \infty$.

(4) If $a_n \uparrow a$, then also $a_{n_k} \uparrow a$, and similarly for decreasing sequences.

(1) If $|a_n| \leq K$, for all n, then, in particular, $|a_{n_k}| \leq K$, for all k.

(2) Write $b_k = a_{n_k}$, $k = 1, 2, 3, \ldots$. Let $\epsilon > 0$. By assumption, there is an index N, such that $|a_n| < \epsilon$, for all $n \geq N$. Choose k so that $n_k \geq N$. Then $j \geq k$ implies $n_j \geq n_k \geq N$, which implies $|a_{n_j}| < \epsilon$. Thus, (b_k) is null.

Properties of Subsequences (Cont'd)

(3) By assumption, $(a_n - a)$ is null, whence its subsequence $(a_{n_k} - a)$ is also null and, consequently, $a_{n_k} \to a$.

(4) If $a_n \uparrow a$, the subsequence (a_{n_k}) is certainly increasing and bounded above (by a). Writing $b = \sup\{a_{n_k} : k \in \mathbb{P}\}$, we know that $b \leq a$ and we have to show that $b = a$. Given any positive integer n, there is a k such that $n_k > n$. Therefore, $a_n \leq a_{n_k} \leq b$. Thus, $a_n \leq b$, for every positive integer n, whence $a \leq b$.

If $a_n \downarrow a$, then $-a_n \uparrow -a$, therefore $-a_{n_k} \uparrow -a$ and, consequently, $a_{n_k} \downarrow a$.

Frequently and Subsequences

Theorem

Let (x_n) be a sequence and let (P) be a property that a term x_n may or may not have. Then the following conditions are equivalent:

(a) x_n has property (P) frequently;

(b) There exists a subsequence (x_{n_k}) of (x_n) such that every x_{n_k} has property (P).

- Let $A = \{n \in \mathbb{P} : x_n \text{ has property } (P)\}$.

(a)⇒(b) By assumption, $n \in A$ frequently. Choose $n_1 \in A$. Choose $n_2 \in A$, so that $n_2 > n_1$. Choose $n_3 \in A$, so that $n_3 > n_2$ and so on. The subsequence $x_{n_1}, x_{n_2}, x_{n_3}, \ldots$ has the desired property.

(b)⇒(a) By assumption, $n_k \in A$, for all k. Given any index N, the claim is that A contains an integer $n \geq N$. Indeed, $n_k > N$, for some k.

- Example: Suppose we are trying to show that $|a_n - a| < \epsilon$ ultimately. The alternative is that $|a_n - a| \geq \epsilon$ frequently, i.e., $|a_{n_k} - a| \geq \epsilon$, for some subsequence (a_{n_k}).

Monotone Subsequences

Theorem

Every sequence in $\mathbb{R}$ has a monotone subsequence.

- Assuming (a_n) is any sequence of real numbers, we seek a subsequence (a_{n_k}) that is either increasing or decreasing. Call a positive integer n a peak point for the sequence if $a_n \geq a_k$ for all $k \geq n$. Think of the sequence as a function $f : \mathbb{P} \to \mathbb{R}$, $f(n) = a_n$. For n to be a peak point means that no point of the graph of f from n onward is higher than (n, a_n). There are two possibilities:
  (1) n is frequently a peak point: If $n_1 < n_2 < n_3 < \cdots$ are peak points, then the subsequence (a_{n_k}) is decreasing: For, $a_{n_1} \geq a_{n_2}$ (because n_1 is a peak point), $a_{n_2} \geq a_{n_3}$ (because n_2 is a peak point), etc.
  (2) From some index N onward, n is not a peak point: Let $n_1 = N$. Since n_1 is not a peak point, there is an index $n_2 > n_1$, such that $a_{n_2} > a_{n_1}$. But n_2 is not a peak point either. So there is an $n_3 > n_2$, such that $a_{n_3} > a_{n_2}$. Continuing in this way, we obtain an increasing subsequence of (a_n).

Bolzano-Weierstraß Theorem

Bolzano-Weierstraß Theorem

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

- Let (a_n) be a bounded sequence of real numbers. By the preceding theorem, (a_n) has a monotone subsequence (a_{n_k}). Suppose, for example, that (a_{n_k}) is increasing. It is also bounded, so $a_{n_k} \uparrow a$ for a suitable real number a and $a_{n_k} \to a$.

Corollary (Closed Interval Version)

In a closed interval $[a, b]$, every sequence has a subsequence that converges to a point of the interval.

- Suppose $x_n \in [a, b], n = 1, 2, 3, \ldots$. By the theorem, some subsequence is convergent to a point of $\mathbb{R}$, say $x_{n_k} \to x$. Since $a \leq x_{n_k} \leq b$, for all k, it follows that $a \leq x \leq b$. Thus, $x \in [a, b]$.

Subsection 6

Cauchy's Criterion for Convergence

Cauchy's Criterion

- The criterion for a monotone sequence to converge is that it be bounded.
- Cauchy's criterion for convergence applies to sequences that are not necessarily monotone:

Theorem (Cauchy's Criterion)

For a sequence (a_n) in $\mathbb{R}$, the following conditions are equivalent:

(a) (a_n) is convergent;

(b) For every $\epsilon > 0$, there is an index N, such that $|a_m - a_n| < \epsilon$, whenever $m, n \geq N$, in symbols,

$$(\forall \epsilon > 0)(\exists N)(m, n \geq N \Rightarrow |a_m - a_n| < \epsilon).$$

- (a)⇒(b): Say $a_n \to a$. If $\epsilon > 0$, then $|a_n - a| < \frac{\epsilon}{2}$ ultimately, say for $n \geq N$. If both $m, n \geq N$, then, by the triangle inequality, $|a_m - a_n| = |(a_m - a) + (a - a_n)| \leq |a_m - a| + |a - a_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Cauchy's Criterion (Cont'd)

- (b)⇒(a): Assuming (b), we show, first, that the sequence (a_n) is bounded. Choose an index M, such that $|a_m - a_n| < 1$, for all $m, n \geq M$. Then, for all $n \geq M$,

$$|a_n| = |(a_n - a_M) + a_M| \leq |a_n - a_M| + |a_M| < 1 + |a_M|,$$

whence the sequence (a_n) is bounded. Explicitly, if $r = \max\{|a_1|, |a_2|, \ldots, |a_{M-1}|, 1 + |a_M|\}$, then $|a_n| \leq r$, for all n.
By the Bolzano-Weierstraß Theorem, (a_n) has a convergent subsequence, say $a_{n_k} \to a$. We will show that $a_n \to a$.
Let $\epsilon > 0$. By hypothesis, there is an index N, such that $m, n \geq N$ imply $|a_m - a_n| < \frac{\epsilon}{2}$. Since $a_{n_k} \to a$, there is an index K, such that $k \geq K$ implies $|a_{n_k} - a| < \frac{\epsilon}{2}$. Choose an index $k \geq K$, such that $n_k \geq N$. Then, for all $n \geq N$,

$$|a_n - a| = |(a_n - a_{n_k}) + (a_{n_k} - a)| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

Thus, the sequence $(a_n - a)$ is null, so $a_n \to a$.

Subsection 7

lim sup and lim inf of a Bounded Sequence

Bounded Sequences

- For a monotone sequence to be convergent it has to be bounded.
- For a bounded sequence, which condition ensures convergence?
- Let (a_n) be a bounded sequence in $\mathbb{R}$, say $|a_n| \leq K$, for all n.
- For each n, let A_n be the set of all terms from n onward,

$$A_n = \{a_n, a_{n+1}, a_{n+2}, \ldots\} = \{a_k : k \geq n\}$$

- A_n is bounded, indeed $A_n \subseteq [-K, K]$, and we may define

$$b_n = \sup A_n = \sup_{k \geq n} a_k, \quad c_n = \inf A_n = \inf_{k \geq n} a_k.$$

- This produces two sequences (b_n) and (c_n), with $c_n \leq b_n$, for all n.
- The sequences are bounded: $-K \leq c_n \leq b_n \leq K$, for all n.
- Moreover, (c_n) is increasing and (b_n) is decreasing: Since $A_n \supseteq A_{n+1}$,

$$c_n = \inf A_n \leq \inf A_{n+1} = c_{n+1} \text{ and } b_n = \sup A_n \geq \sup A_{n+1} = b_{n+1}.$$

- Thus, with the notation $c = \sup c_n = \sup \{c_n : n \in \mathbb{P}\}$, $b = \inf b_n = \inf \{b_n : n \in \mathbb{P}\}$, we have $c_n \uparrow c$ and $b_n \downarrow b$.

Limit Superior and Limit Inferior

Definition (Limit Superior and Limit Inferior)

With the above notations, b is called the **limit superior** of the bounded sequence (a_n), written

$$\limsup a_n = b = \inf_{n\geq 1} b_n = \inf_{n\geq 1} (\sup_{k\geq n} a_k)$$

and c is called the **limit inferior** of the sequence (a_n), written

$$\liminf a_n = c = \sup_{n\geq 1} c_n = \sup_{n\geq 1} (\inf_{k\geq n} a_k).$$

- Example:
  - (i) For the sequence $1, -1, 1, -1, \ldots$, $A_n = \{-1, 1\}$, for all n, so $b_n = 1$ and $c_n = -1$, for all n, therefore $b = 1$ and $c = -1$.
  - (ii) For the sequence $1, -1, 1, 1, 1, \ldots$, $A_n = \{1\}$, for $n \geq 3$, so $b_n = c_n = 1$, for $n \geq 3$, therefore $b = c = 1$.
  - (iii) For the sequence $\frac{1}{2}, \frac{2}{3}, \frac{1}{3}, \frac{3}{4}, \frac{1}{4}, \frac{4}{5}, \ldots, \frac{1}{k}, \frac{k}{k+1}, \ldots$, $b_n = 1$ and $c_n = 0$, for all n, therefore, $b = 1$ and $c = 0$.

Relation Between the Limits

Theorem

For every bounded sequence (a_n) in $\mathbb{R}$, $\liminf a_n \leq \limsup a_n$.

- In the preceding notations, the problem is to show that $c \leq b$. Since $c_n \to c$ and $b_n \to b$ and $c_n \leq b_n$, for all n, it follows that $c \leq b$.
- Another way to see this: If m and n are any two positive integers and $p = \max\{m, n\}$, then $m \leq p$ and $n \leq p$, whence $c_m \leq c_p \leq b_p \leq b_n$. This shows that each c_m is a lower bound for all the b_n, so $c_m \leq \text{GLB} b_n = b$. Then b is an upper bound for all the c_m, so $c = \text{LUB} c_m \leq b$.

Criterion for Convergence

Theorem

For a sequence (a_n) in $\mathbb{R}$, the following conditions are equivalent:

(a) (a_n) is convergent;

(b) (a_n) is bounded and $\liminf a_n = \limsup a_n$.

For such a sequence, $\lim a_n = \liminf a_n = \limsup a_n$.

- (a)⇒(b): If $a_n \to a$, then (a_n) is bounded and our problem is to show that $c = b = a$. Let $\epsilon > 0$. Choose an index N, such that $|a_n - a| \leq \epsilon$, for all $n \geq N$. Then, for all $n \geq N$, $-\epsilon \leq a_n - a \leq \epsilon$, i.e., $a - \epsilon \leq a_n \leq a + \epsilon$. This shows that $A_N \subseteq [a - \epsilon, a + \epsilon]$. Consequently $a - \epsilon \leq c_N \leq b_N \leq a + \epsilon$. But $c_N \leq c \leq b \leq b_N$. Thus, $a - \epsilon \leq c \leq b \leq a + \epsilon$. In particular, $a - \frac{1}{n} \leq c \leq b \leq a + \frac{1}{n}$, for every positive integer n. Since $\frac{1}{n} \to 0$, it follows that $a \leq c \leq b \leq a$, i.e., $a = c = b$.

Criterion for Convergence (Cont'd)

- (b)⇒(a): Assuming (a_n) is bounded, define b and c by

$$c = \sup c_n = \sup\{c_n : n \in \mathbb{P}\}, \quad b = \inf b_n = \inf\{b_n : n \in \mathbb{P}\}.$$

Let $\epsilon > 0$. Since $b = \text{GLB} b_n$ and $b + \epsilon > b$, $b + \epsilon$ cannot be a lower bound for the b_n. Thus, $b + \epsilon$ is not $\leq$ every b_n, i.e., $b + \epsilon > b_N$, for some N. Hence, $b + \epsilon > \sup\{a_n : n \geq N\}$. Then $n \geq N$ implies $a_n < b + \epsilon$. We have shown that $(\forall \epsilon > 0)(a_n < b + \epsilon$ ultimately$)$.
A similar argument shows that $(\forall \epsilon > 0)(c - \epsilon < a_n$ ultimately$)$.
Combining, we have $(\forall \epsilon > 0)(c - \epsilon < a_n < b + \epsilon$ ultimately$)$. It follows that, if $c = b$ and a denotes the common value of c and b, then $(\forall \epsilon > 0)(|a_n - a| < \epsilon$ ultimately$)$, i.e., $a_n \to a$.

- Remark: For any bounded sequence (a_n), $b_n \uparrow b$ and $-c_n \uparrow -c$, whence $b_n - c_n \uparrow b - c$. Thus, the theorem says:
A sequence (a_n) in $\mathbb{R}$ is convergent if and only if it is bounded and $b_n - c_n \uparrow 0$.

Limits of Subsequences

- We showed that every bounded sequence (a_n) has a convergent subsequence: In fact, there are subsequences converging to c and to b, and these numbers are, respectively, the smallest and largest possible limits for convergent subsequences.

Theorem

Let (a_n) be a bounded sequence in $\mathbb{R}$ and let

$$S = \{x \in \mathbb{R} : a_{n_k} \to x, \text{ for some subsequence } (a_{n_k})\}.$$

Let $c = \liminf a_n$ and $b = \limsup a_n$. Then $\{c, b\} \subseteq S \subseteq [c, b]$. Thus, c is the smallest element of S and b is the largest.

- The first inclusion asserts that each of c and b is the limit of a suitable subsequence of (a_n): Let $\epsilon > 0$. We showed that $a_n < b + \epsilon$ ultimately. Also, $a_n > b - \epsilon$ frequently, since the alternative $a_n \leq b - \epsilon$ ultimately, say for $n \geq N$, would imply that $b_N \leq b - \epsilon < b$, contrary to $b \leq b_N$.

Limits of Subsequences (Cont'd)

Thus, we get $(\forall \epsilon > 0)(b - \epsilon < a_n < b + \epsilon$ frequently).

- With $\epsilon = 1$, choose n_1, such that $b - 1 < a_{n_1} < b + 1$.
- With $\epsilon = \frac{1}{2}$, choose $n_2 > n_1$, sch that $b - \frac{1}{2} < a_{n_2} < b + \frac{1}{2}$.
- Continuing, construct (a_{n_k}), such that $|a_{n_k} - b| < \frac{1}{k}$, for all k.

Then $a_{n_k} \to b$, whence $b \in S$.

The proof that $c \in S$ is similar.

- To prove the second inclusion, assuming $a_{n_k} \to x$, we have to show that $c \leq x \leq b$. Given any $\epsilon > 0$, $a_n < b + \epsilon$, for all sufficiently large n. Therefore $a_{n_k} < b + \epsilon$, for all sufficiently large k. This implies that $x \leq b + \epsilon$. Since $\epsilon > 0$ is arbitrary, $x \leq b$.

  The proof that $c \leq x$ is similar.