Introduction to Real Analysis

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LSSU Math 421

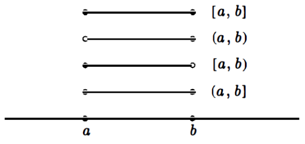
1. Special Subsets of $\mathbb{R}$
   - Intervals
   - Closed Sets
   - Open Sets, Neighborhoods
   - Finite and Infinite Sets
   - Heine-Borel Covering Theorem

Subsection 1

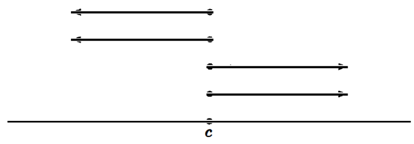
Intervals

Intervals in $\mathbb{R}$

- There are nine kinds of subsets of $\mathbb{R}$ that are called **intervals**.
  - First, there are $[a, b], (a, b), [a, b), (a, b]$.



  - Next, for each real number c there are the four "half-lines" $\{x \in \mathbb{R} : x \leq c\}$, $\{x \in \mathbb{R} : x < c\}$, $\{x \in \mathbb{R} : x \geq c\}$, and $\{x \in \mathbb{R} : x > c\}$.



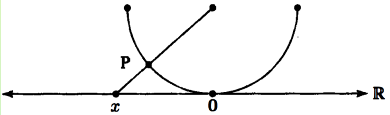
  - Finally, $\mathbb{R}$ itself is regarded as an interval (extending indefinitely in both directions).

The Symbols $\pm\infty$

Definition (The Symbols $\pm\infty$)

For every real number x, we write $x < +\infty$ and $x > -\infty$, or, concisely, $-\infty < x < +\infty$. We think of $+\infty$ (read "plus infinity") as a symbol that stands to the right of every point of the real line, and $-\infty$ ("minus infinity") as a symbol that stands to the left of every point of the line. Finally, we write $-\infty < +\infty$.

- A new set $\mathbb{R} \cup \{-\infty, +\infty\}$ has been created, by adjoining to $\mathbb{R}$ two new elements and specifying the order relations between the new elements $-\infty$ and $+\infty$ and the old ones (those in $\mathbb{R}$).
- A natural correspondence between real numbers x and points P of a semicircle presents the "points at $\pm\infty$" as the endpoints of the semicircle:

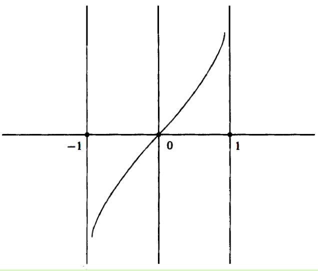


Another Illustration of the Extension

- A computationally simpler explanation uses the function $f:(-1,1)\to\mathbb{R}$, defined by

$$f(x)=\frac{x}{1-|x|}.$$

It is an order-preserving bijection.



- It can be extended (in an order-preserving way) to the closed interval $[-1,1]$ by assigning the values $\pm\infty$ to the endpoints ± 1.

Formal Definition of Intervals

Definition (Unbounded Intervals)

For any real number c, we write

$$\begin{aligned}
[c,+\infty) &= \{x \in \mathbb{R} : c \leq x < +\infty\} = \{x \in \mathbb{R} : x \geq c\} \\
(c,+\infty) &= \{x \in \mathbb{R} : c < x < +\infty\} = \{x \in \mathbb{R} : x > c\} \\
(-\infty,c] &= \{x \in \mathbb{R} : -\infty < x \leq c\} = \{x \in \mathbb{R} : x \leq c\} \\
(-\infty,c) &= \{x \in \mathbb{R} : -\infty < x < c\} = \{x \in \mathbb{R} : x < c\} \\
(-\infty,+\infty) &= \{x \in \mathbb{R} : -\infty < x < +\infty\} = \mathbb{R}.
\end{aligned}$$

- When $+\infty$ or $-\infty$ (neither of which is a real number) is used as an "endpoint" of an interval of $\mathbb{R}$, it is always absent from the interval.

Definition (Intervals)

An **interval** of $\mathbb{R}$ is a subset of $\mathbb{R}$ of one of the following 9 types: $[a,b], (a,b), [a,b), (a,b], [c,+\infty), (c,+\infty), (-\infty,c], (-\infty,c), (-\infty,+\infty) = \mathbb{R}$, where a,b,c are real numbers and $a \leq b$. In particular, the empty set $\emptyset = (a,a) = [a,a) = (a,a]$ and singletons $\{a\} = [a,a]$ qualify as intervals.

Convexity

- The intervals of $\mathbb{R}$ are characterized by **convexity**:

Theorem

Let A be a nonempty subset of $\mathbb{R}$. The following conditions are equivalent:

(a) A is an interval;

(b) For every pair of points in A, the segment joining them is contained in A; i.e., for all $x, y \in A$, $x \leq y \Rightarrow [x, y] \subseteq A$.

- (a)⇒(b) is obvious.
- (b)⇒(a): There are four cases, according as A is bounded above (or not) and bounded below (or not).
  1. *A bounded below, but not bounded above*: Let $a = \inf A$. We will show that $A = (a, +\infty)$ or $A = [a, +\infty)$. It suffices to show that $(a, +\infty) \subseteq A \subseteq [a, +\infty)$. The second inclusion is immediate from the definition of a. Assuming $r \in (a, +\infty)$, we have to show that $r \in A$. Since $r > a = \text{GLB}A$, there exists $x \in A$, such that $a < x < r$. Since A is not bounded above, there exists $y \in A$, such that $y > r$. Thus $x < r < y$ with $x, y \in A$, so $r \in [x, y] \subseteq A$, by the hypothesis.

  2.-4. Similar arguments.

Intersection of a Family of Intervals

Corollary

If $\mathcal{S}$ is any set of intervals and $J = \bigcap \mathcal{S}$ is their intersection, then J is an interval (possibly empty).

- By definition, J is the set of all real numbers common to all of the intervals belonging to $\mathcal{S}$, i.e.,

$$J = \{r \in \mathbb{R} : r \in I, \text{for every } I \in \mathcal{S}\}.$$

Assuming J nonempty, it will suffice to verify convexity. Suppose $x, y \in J$, $x \leq y$. Then $x, y \in I$, for every $I \in \mathcal{S}$. By the theorem, $[x, y] \subseteq I$, for all $I \in \mathcal{S}$, and, therefore, $[x, y] \subseteq J$.

Subsection 2

Closed Sets

Qualitative Differences Between Types of Intervals

- The most important subsets of $\mathbb{R}$ for calculus are the intervals.
- There are differences among intervals, some important, others not:
  - The difference between $(0,1)$ and $(0,5)$ is only a matter of scale; otherwise, the inequalities defining them are qualitatively the same.
  - The intervals $(0,1)$ and $(0,+\infty)$ are different in kind, since one is bounded and the other not.
  - By contrast, the intervals $I=(0,1)$ and $J=[0,1]$ prove to have dramatically different properties.
    - The crux is that the endpoints 0, 1 of I can be approximated as closely as we like by points of I but they are not themselves points of I. More precisely, the endpoints of I, though not in I, are limits of convergent sequences whose terms are in I.
    - On the other hand, if a convergent sequence has its terms in J then its limit must also be in J.

Closed Sets of $\mathbb{R}$

Definition (Closed Sets)

A set A of real numbers is said to be a **closed subset** of $\mathbb{R}$ (or to be a **closed set** in $\mathbb{R}$) if, whenever a convergent sequence has all of its terms in A, the limit of the sequence must also be in A, i.e., if $x_n \to x$ and $x_n \in A$, for all n, then necessarily $x \in A$. (One cannot "escape" from a closed set by means of a convergent sequence!) In symbols,

$$\left.\begin{array}{c}(\forall n)(x_n \in A) \\ x \in \mathbb{R} \\ x_n \to x\end{array}\right\} \Rightarrow x \in A.$$

- Note that the empty subset $\emptyset$ of $\mathbb{R}$ is closed.

Examples of Closed Sets

- $\mathbb{R}$ is a closed subset of $\mathbb{R}$ (there is nowhere else for the limit to go!).
- Every singleton $\{a\}$, for $a \in \mathbb{R}$, is closed (the constant sequence $x_n = a$ converges to a).
- For every real number c, the intervals $[c, +\infty)$ and $(-\infty, c]$ are closed sets.

  If $x_n \to x$ and $x_n \geq c$, for all n, then $x \geq c$.

  Caution: These are closed sets and they are intervals, but they are not closed intervals. The term "closed interval" is reserved for intervals of the form $[a, b]$.

Some Properties of Closed Sets

Lemma

If A is closed in $\mathbb{R}$, $x_n \to x$ in $\mathbb{R}$, and $x_n \in A$ frequently, then $x \in A$.

- By assumption, there is a subsequence (x_{n_k}) with $x_{n_k} \in A$, for all k. Since $x_{n_k} \to x$, $x \in A$, by the definition of a closed set.

Theorem

(i) $\emptyset$ and $\mathbb{R}$ are closed sets in $\mathbb{R}$.
(ii) If A and B are closed sets in $\mathbb{R}$, then so is their union $A \cup B$.
(iii) If $\mathcal{S}$ is any set of closed sets in $\mathbb{R}$, then $\bigcap \mathcal{S}$ is also a closed set.

(i) Both have been noted.
(ii) Suppose $x_n \in A \cup B$, for all n and $x_n \to x$. If $x_n \in A$ frequently, then $x \in A$ by the lemma. The alternative is that $x_n \in B$ ultimately, in which case $x \in B$, again by the lemma. Either way, $x \in A \cup B$.
(iii) Let $B = \bigcap \mathcal{S} = \{x \in \mathbb{R} : x \in A, \text{for all } A \in \mathcal{S}\}$. Suppose $x_n \to x$ and $x_n \in B$, for all n. For each $A \in \mathcal{S}$, $x_n \in A$, for all n, whence $x \in A$. Thus $x \in A$, for all $A \in \mathcal{S}$, and, therefore, $x \in \bigcap \mathcal{S} = B$.

Some Applications of the Properties

- Every closed interval

$$[a, b] = (-\infty, b] \cap [a, +\infty)$$

  is a closed set.

  It is the intersection of two closed sets.

- If $A_1, \ldots, A_r$ is a finite list of closed sets in $\mathbb{R}$, then their union $A_1 \cup \cdots \cup A_r$ is also a closed set.

  By Induction on r.

- Every finite subset $A = \{a_1, \ldots, a_r\}$ of $\mathbb{R}$ is a closed set.

  $A = \{a_1\} \cup \cdots \cup \{a_r\}$ is closed since each singleton is closed and a finite union of closed sets is also closed.

Set of all Limits

- If A is a closed set, we know where the limits of its convergent sequences are.
- On the other hand, the set $A = (0, 1]$ contains a convergent sequence - for example $x_n = \frac{1}{n}$ - whose limit is not in A.
- For an arbitrary subset A of $\mathbb{R}$ we may contemplate the set $\overline{A}$ of all real numbers that are limits of sequences whose terms are in A.
- Regardless of the status of A , $\overline{A}$ is always a closed set.

Characterizing the Set of all Limits

Theorem

Let A be any subset of $\mathbb{R}$ and let

$$\overline{A} = \{x \in \mathbb{R} : a_n \to x \text{ for some sequence } (a_n) \text{ in } A\}.$$

Then $\overline{A}$ is the smallest closed set containing A, i.e.,

(1) $\overline{A} \supseteq A$,

(2) $\overline{A}$ is a closed set,

(3) if B is a closed set with $B \supseteq A$, then $B \supseteq \overline{A}$.

Moreover,

(4) A is closed $\Leftrightarrow \overline{A} = A$;

(5) $\overline{A}$ is the set of all real numbers that can be approximated as closely as we like by elements of A, i.e.,

$$x \in \overline{A} \Leftrightarrow (\forall \epsilon > 0)(\exists a \in A)(|x - a| < \epsilon).$$

Proof of the Theorem

(1) If $a \in A$, let $a_n = a$, for all n. Then $a_n \to a$, and so $a \in \overline{A}$.

(3) Assuming B is a closed set with $A \subseteq B$, we have to show that $\overline{A} \subseteq B$. Let $x \in \overline{A}$, say $a_n \to x$, with $a_n \in A$, for all n. Then $a_n \in B$, for all n (because $A \subseteq B$), therefore $x \in B$ (because B is closed).

(4) To say that A is closed means that $\overline{A} \subseteq A$. Since $A \subseteq \overline{A}$ automatically, the condition $\overline{A} \subseteq A$ is equivalent to $\overline{A} = A$.

(5) ($\Leftarrow$): For each positive integer n let $\epsilon = \frac{1}{n}$ and choose $a_n \in A$, such that $|x - a_n| < \frac{1}{n}$. Then $a_n \to x$ so $x \in \overline{A}$.

($\Rightarrow$): Let $x \in \overline{A}$, say $a_n \to x$, with $a_n \in A$, for all n. If $\epsilon > 0$, then $|x - a_n| < \epsilon$, for some n (in fact, ultimately!).

(2) Assuming $x_n \to x$, with $x_n \in \overline{A}$, for all n, we have to show that $x \in \overline{A}$. We apply the criterion of (5): If $\epsilon > 0$, choose n so that $|x - x_n| < \frac{\epsilon}{2}$; for this n, choose $a \in A$, so that $|x_n - a| < \frac{\epsilon}{2}$. Then $|x - a| < \epsilon$ by the triangle inequality.

Closure of a Set

Definition (Closure)

$\overline{A}$ is called the **closure** of A in $\mathbb{R}$.
Alternatively, the points of $\overline{A}$ are said to be **adherent** to A, and $\overline{A}$ is called the **adherence** of A.

Subsection 3

Open Sets, Neighborhoods

Complement of $\overline{A}$

- According to the preceding theorem, the meaning of $x \in \overline{A}$ is that for every $\epsilon > 0$ the interval $(x - \epsilon, x + \epsilon)$ intersects A, i.e.,

$$(\forall \epsilon > 0)((x - \epsilon, x + \epsilon) \cap A \neq \emptyset).$$

- The meaning of $x \notin \overline{A}$ is the negation of the preceding condition: there exists an $\epsilon > 0$, for which the interval $(x - \epsilon, x + \epsilon)$ is disjoint from A, i.e.,

$$(\exists \epsilon > 0)((x - \epsilon, x + \epsilon) \cap A = \emptyset),$$

  or, denoting by $A^c = \mathbb{R} - A$, the **complement** of A in $\mathbb{R}$,

$$(\exists \epsilon > 0)((x - \epsilon, x + \epsilon) \subseteq A^c).$$

  Not only does x belong to A^c, but there is a little "buffer zone" about x that remains in A^c - informally, all points "sufficiently close to x" are in A^c.

Interior Points

Definition (Interior Point)

A point $x \in \mathbb{R}$ is said to be **interior** to a subset A of $\mathbb{R}$ if there exists an $r > 0$, such that $(x - r, x + r) \subseteq A$, i.e., such that $|y - x| < r \Rightarrow y \in A$.
If x is interior to A, one also says that A is a **neighborhood** of x.
The set of all interior points of A (there may not be any!) is called the **interior of** A, denoted A°:

$$A^\circ = \{x \in \mathbb{R} : x \text{ is interior to } A\}.$$

Thus, A° is the set of all points of A of which A is a neighborhood.

- Example: $\mathbb{Q}$ has no interior points (i.e., it has empty interior), because every open interval contains an irrational number.
- Example: Assuming $a < b$, the point a belongs to $[a, b)$ but not to its interior; the interior of $[a, b]$ is (a, b).

Relation Between Closure and Interior

Theorem

If A is any subset of $\mathbb{R}$, then $x \notin \overline{A} \Leftrightarrow x \in (A^c)^\circ$. Thus, $(\overline{A})^c = (A^c)^\circ$.

- Thus, the passage from A to its closure $\overline{A}$ is the composite of three operations:
  - take complement,
  - then take interior,
  - then take complement again.
- More precisely, we have

$$\overline{A} = ((A^c)^\circ)^c \quad \text{and} \quad A^\circ = (\overline{A^c})^c.$$

Open Sets

- In general, $A \subseteq \overline{A}$. Equality is a special event (A closed).
- In general, $A^\circ \subseteq A$. Again, equality is a special event:

Definition (Open Set)

A subset A of $\mathbb{R}$ is called an **open set** if every point of A is an interior point, i.e.,

$$(\forall x \in A)(\exists \epsilon > 0)((x - \epsilon, x + \epsilon) \subseteq A).$$

Equivalently, A is a neighborhood of each of its points.

- Intuitively, for every point of an open set A, there is a buffer zone about the point - whose size may depend on the point - that is also contained in A.
- To say that A is open means that $A \subseteq A^\circ$; Since $A \supseteq A^\circ$ automatically, an equivalent condition is that $A = A^\circ$, i.e., A is equal to its interior.

Characterization of Open Sets

Theorem

For a subset A of $\mathbb{R}$,

(i) A is open $\Leftrightarrow$ A^c is closed;

(ii) A is closed $\Leftrightarrow$ A^c is open.

(i) In general, $A^\circ = (\overline{A^c})^c$, so the following conditions are equivalent:
- A open,
- $A = A^\circ$,
- $A = (\overline{A^c})^c$,
- $A^c = \overline{A^c}$,
- A^c closed.

(ii) Apply (i) with A replaced by A^c.

Two Corollaries

Corollary

For every subset A of $\mathbb{R}$, A° is the largest open subset of A.

- Since $A^\circ = (\overline{A^c})^c$ is the complement of a closed set, it is open. On the other hand, if U is any open set with $U \subseteq A$, then $U \subseteq A^\circ$: If $x \in U$, then x is interior to U, so it is obviously interior to A as well.

Corollary

Let $A \subseteq \mathbb{R}$, $x \in \mathbb{R}$. The following conditions are equivalent:

(a) A is a neighborhood of x;

(b) There exists an open set U, such that $x \in U \subseteq A$.

- (a)$\Rightarrow$(b): $U = A^\circ$ fills the bill.
- (b)$\Rightarrow$(a): Since $x \in U$ and U is open, U is a neighborhood of x. Therefore so is its superset A.

Properties of Open Sets

Theorem

(i) $\emptyset$ and $\mathbb{R}$ are open sets in $\mathbb{R}$.

(ii) If A and B are open sets in $\mathbb{R}$, then so is their intersection $A \cap B$.

(iii) If $\mathcal{S}$ is any set of open sets in $\mathbb{R}$, then $\bigcup \mathcal{S}$ is also an open set.

(i) $\emptyset = \mathbb{R}^c$ and $\mathbb{R} = \emptyset^c$ are complements of closed sets and so are open.

(ii) Assuming $x \in A \cap B$, we have to show that x is interior to $A \cap B$. Since $x \in A$ and A is open, there is an $r > 0$, with $(x - r, x + r) \subseteq A$. Similarly, there is an $s > 0$, with $(x - s, x + s) \subseteq B$. If t is the smaller of r and s, then $(x - t, x + t) \subseteq A \cap B$, so x is interior to $A \cap B$.

(iii) Let $B = \bigcup \mathcal{S}$. If $x \in B$, then $x \in A$, for some $A \in \mathcal{S}$. By assumption, A is open, so there is an $r > 0$, with $(x - r, x + r) \subseteq A$. Since $A \subseteq B$, it follows that $(x - r, x + r) \subseteq B$, thus x is interior to B.

- Note that the proof of Part (ii) shows that if A and B are neighborhoods of x, then so is $A \cap B$.

Subsection 4

Finite and Infinite Sets

Finite and Infinite Sets

Definition (Finite and Infinite Sets)

A nonempty set A is said to be **finite** if there exist a positive integer r and a surjection $\{1,\ldots,r\} \to A$.
Convention: The empty set $\emptyset$ is finite.
A set is said to be **infinite** if it is not finite.

- If $\sigma : \{1,\ldots,r\} \to A$ is a surjection and one writes $x_i = \sigma(i)$, for $i = 1,\ldots,r$, then $A = \sigma(\{1,\ldots,r\}) = \{x_1,\ldots,x_r\}$.
- We also say that $x_1,\ldots,x_r$ is a finite list of elements.
- Example: For each positive integer r, the set $\{1,\ldots,r\}$ is finite. The identity mapping $\{1,\ldots,r\} \to \{1,\ldots,r\}$ is a surjection.
- Example: The set $\mathbb{P}$ of all positive integers is infinite. We show that there does not exist a surjection $\{1,\ldots,r\} \to \mathbb{P}$ for any r. Assuming $r \in \mathbb{P}$ and $\varphi : \{1,\ldots,r\} \to \mathbb{P}$. Let $n = 1 + \varphi(1) + \cdots + \varphi(r)$. Then $\varphi(i) < n$, for all $i = 1,\ldots,r$, whence n is not in the range of φ.

Properties of Finite Sets

Theorem

If $f : X \to Y$ is any function and A is a finite subset of X, then $f(A)$ is a finite subset of Y.

- If $\sigma : \{1, \ldots, r\} \to A$ is surjective, then $i \mapsto f(\sigma(i))$ is a surjection $\{1, \ldots, r\} \to f(A)$.

Theorem

If $A_1, \ldots, A_m$ is a finite list of finite subsets of a set, then $A_1 \cup \cdots \cup A_m$ is also finite.

- For each $j = 1, \ldots, m$, there is a positive integer r_j and a surjection $\sigma_j : \{1, \ldots, r_j\} \to A_j$. Let $r = r_1 + \cdots + r_m$. We will construct a surjection $\sigma : \{1, \ldots, r\} \to A_1 \cup \cdots \cup A_m$. The elements of $\{1, \ldots, r\}$ can be organized, in ascending order, as a union of m subsets:

Properties of Finite Sets (Cont'd)

$$\begin{aligned} \{1,\ldots,r\} &= \{1,\ldots,r_1\} \cup \{r_1+1,\ldots,r_1+r_2\} \cup \cdots \\ &\quad \cup \{r_1+\cdots+r_{m-1}+1,\ldots,r_1+\cdots+r_{m-1}+r_m\} \\ &= B_1 \cup B_2 \cup \cdots \cup B_m, \end{aligned}$$

where the sets B_j are pairwise disjoint. For each $j = 1,\ldots,m$ the formula $\theta_j(i) = r_1 + \cdots + r_{j-1} + i$ defines a bijection $\theta_j : \{1,\ldots,r_j\} \to B_j$. Define $\sigma : \{1,\ldots,r\} \to A_1 \cup \cdots \cup A_m$ as follows: If $k \in \{1,\ldots,r\}$ then $k \in B_j$, for a unique $j \in \{1,\ldots,m\}$, so $k = \theta_j(i)$ for a unique $i \in \{1,\ldots,r_j\}$. Define $\sigma(k) = \sigma_j(i) = \sigma_j(\theta_j^{-1}(k))$. In other words, σ is the unique mapping on $\{1,\ldots,r\}$ that agrees with $\sigma_j \circ \theta_j^{-1}$ on B_j. It remains to show that σ is surjective:

$$\begin{aligned} \sigma(\{1,\ldots,r\}) &= \sigma(B_1 \cup \cdots \cup B_m) \\ &= \sigma(B_1) \cup \cdots \cup \sigma(B_m) \\ &= \sigma_1(\theta_1^{-1}(B_1)) \cup \cdots \cup \sigma_m(\theta_m^{-1}(B_m)) \\ &= \sigma_1(\{1,\ldots,r_1\}) \cup \cdots \cup \sigma_m(\{1,\ldots,r_m\}) \\ &= A_1 \cup \cdots \cup A_m. \end{aligned}$$

Finding Increasing Sequences

Lemma

If $A \subseteq \mathbb{P}$ and A is infinite, then there exists a strictly increasing mapping $\varphi : \mathbb{P} \to A$. In particular, φ is injective and $\varphi(n) \geq n$, for all $n \in \mathbb{P}$.

- Apart from notation, it is the same to show that there exists a sequence (a_n) in A such that $m < n \Rightarrow a_m < a_n$. Define a_n recursively as follows: Since A is not finite, it is nonempty. Let a_1 be the smallest element of A. Then $A \neq \{a_1\}$ (because $\{a_1\}$ is finite and A is not), so $A - \{a_1\} \neq \emptyset$. Let a_2 be the smallest element of $A - \{a_1\}$. Then $a_2 > a_1$ and $A \neq \{a_1, a_2\}$ (because $\{a_1, a_2\}$ is finite), so $A - \{a_1, a_2\}$ has a smallest element a_3, and $a_3 > a_2$. Assuming $a_1, \ldots, a_n$ already defined, let a_{n+1} be the smallest element of $A - \{a_1, \ldots, a_n\}$. The function $\varphi : \mathbb{P} \to A$ defined by $\varphi(n) = a_n$ is strictly increasing. By induction, $\varphi(n) \geq n$, for all n:
  - For, $\varphi(1) \geq 1$;
  - If $\varphi(k) \geq k$, then $\varphi(k+1) > \varphi(k) \geq k$, therefore, $\varphi(k+1) \geq k+1$.

Finite Subsets of Finite Sets

Theorem

Every subset of a finite set is finite.

- Suppose F is finite and $B \subseteq F$. By assumption, there exists a surjection $\sigma : \{1, \ldots, r\} \to F$, for some positive integer r. Let $A = \sigma^{-1}(B)$ be the inverse image of B under σ. Then $\sigma(A) = B$ (because σ is surjective). So it will suffice to show that A is finite. We have $A \subseteq \{1, \ldots, r\}$. If A were infinite, by the lemma, there would exist a mapping $\varphi : \mathbb{P} \to A$, such that $\varphi(n) \geq n$, for all $n \in \mathbb{P}$. But then $n \leq \varphi(n) \leq r$, for all n, which is absurd for $n = r + 1$.

Consequences for Infinite Sets

Corollary

Every superset of an infinite set is infinite.

- Suppose $B \supseteq A$. By the theorem, B finite implies A finite. By the contrapositive, A not finite implies B not finite.

Corollary

If $\varphi : \mathbb{P} \to A$ is injective, then A is infinite.

- If $B = \varphi(\mathbb{P})$ is the range of φ, then φ defines a bijection $\mathbb{P} \to B$. Let $\psi : B \to \mathbb{P}$ be the inverse of this bijection. Since $\psi(B) = \mathbb{P}$ and $\mathbb{P}$ is infinite, B cannot be finite. But $B \subseteq A$, so A cannot be finite either.

Characterization of Infinite Sets

- The property appearing in the last Corollary characterizes infinite sets:

Theorem

A set A is infinite if and only if there exists an injection $\mathbb{P} \to A$.

- The "if" part is the Corollary.

  Conversely, assuming A infinite, we have to produce a sequence (a_n) in A, such that $m \neq n \Rightarrow a_m \neq a_n$: "Construct" a_n recursively as follows: Since A is infinite, it is nonempty. Choose $a_1 \in A$. Then $A \neq \{a_1\}$ (because $\{a_1\}$ is finite), so $A - \{a_1\} \neq \emptyset$. Choose $a_2 \in A - \{a_1\}$. Assuming $a_1, \ldots, a_n$ already chosen, $A \neq \{a_1, \ldots, a_n\}$. Choose $a_{n+1} \in A - \{a_1, \ldots, a_n\}$.

Subsection 5

Heine-Borel Covering Theorem

Coverings, Open Covering and Subcoverings

Definition (Covering)

Let $A \subseteq \mathbb{R}$ and let $\mathcal{C}$ be a set of subsets of $\mathbb{R}$.

- If each point of A belongs to some set in $\mathcal{C}$, we say that $\mathcal{C}$ is a **covering** of A (or that $\mathcal{C}$ **covers** A). In symbols,

$$(\forall x \in A)(\exists C \in \mathcal{C})(x \in C).$$

  More concisely, $A \subseteq \bigcup \mathcal{C}$.
  - If, moreover, every set in $\mathcal{C}$ is an open subset of $\mathbb{R}$, then $\mathcal{C}$ is said to be an **open covering** of A.
  - If a covering $\mathcal{C}$ of A consists of only a finite number of sets, it is called a **finite covering**.
- If $\mathcal{C}$ is a covering of A and if $\mathcal{D} \subseteq \mathcal{C}$ is such that $\mathcal{D}$ is also a covering of A, then $\mathcal{D}$ is referred to as a **subcovering** (it is a subset of $\mathcal{C}$ and still a covering of A).

Examples

- Suppose A consists of the terms of a convergent sequence and its limit, i.e., $A = \{x\} \cup \{x_n : n \in \mathbb{P}\}$, where $x_n \to x$. If $\mathcal{C}$ is an open covering of A, then A is covered by finitely many of the sets in $\mathcal{C}$:
  The limit x belongs to one of the sets in $\mathcal{C}$, say $x \in U \in \mathcal{C}$. Since U is open, there is an $\epsilon > 0$, with $(x - \epsilon, x + \epsilon) \subseteq U$. It follows that $x_n \in U$ ultimately, say for $n > N$. Each of the terms x_i, $i = 1, \ldots, N$, belongs to some $U_i \in \mathcal{C}$, so A is covered by the sets $U, U_1, \ldots, U_N$.
  In the preceding terminology, every open covering of A admits a finite subcovering.
- Let A be the open interval $(2, 5)$ and let $\mathcal{C}$ be the set of all open intervals $(2 + \frac{1}{n}, 5 - \frac{1}{n})$, $n \in \mathbb{P}$. Then $\mathcal{C}$ is an open covering of A, but no finite set of elements of $\mathcal{C}$ can cover A.
  Each element of $\mathcal{C}$ is a proper subset of A. Moreover, among any finite set of elements of $\mathcal{C}$, one of them contains all the others. Thus, $\mathcal{C}$ is an open covering of A that admits no finite subcovering.

Heine-Borel Theorem

Theorem (Heine-Borel Theorem)

If $[a, b]$ is a closed interval in $\mathbb{R}$ and $\mathcal{C}$ is an open covering of $[a, b]$, then $[a, b]$ is covered by a finite number of the sets in $\mathcal{C}$.

- Let S be the set of all $x \in [a, b]$, such that the closed interval $[a, x]$ is covered by finitely many sets of $\mathcal{C}$. At least $a \in S$, because $[a, a] = \{a\}$ and a belongs to some set in $\mathcal{C}$. We will show that $b \in S$. At any rate, S is nonempty and bounded. Let

$$m = \sup S.$$

Since $S \subseteq [a, b]$, we have $a \leq m \leq b$. The strategy of the proof is to show that:

(1) $m \in S$;
(2) $m = b$.

Proof of the Heine-Borel Theorem

(1) Since $m \in [a, b] \subseteq \bigcup \mathcal{C}$, there is a $V \in \mathcal{C}$, such that $m \in V$. Since V is open, $[m - \epsilon, m + \epsilon] \subseteq V$, for some $\epsilon > 0$. Note that we can take ϵ to be as small as we like. Since $m - \epsilon < m$ and m is the least upper bound of S, there exists $x \in S$, with $m - \epsilon < x \leq m$. From $x \in S$, we know that the interval $[a, x]$ is covered by finitely many sets in $\mathcal{C}$, say $[a, x] \subseteq U_1 \cup \cdots \cup U_r$. On the other hand, $[x, m] \subseteq [m - \epsilon, m + \epsilon] \subseteq V$, so $[a, m] = [a, x] \cup [x, m]$ is covered by the sets $V, U_1, \ldots, U_r$ of $\mathcal{C}$. This proves that $m \in S$,

and a little more: $[a, m + \epsilon] \subseteq V \cup U_1 \cup \cdots \cup U_r$, whence $m + \epsilon > b$, because $m + \epsilon \leq b$ would imply that $m + \epsilon \in S$, contrary to the fact that every element of S is $\leq m$.

(2) The preceding argument shows that $b - m < \epsilon$ and the argument is valid with ϵ replaced by any positive number smaller than ϵ. It follows that $b - m \leq 0$. Thus $b \leq m$. Since, already $m \leq b$, we get $b = m \in S$.

Compact Sets

Definition (Compact Set)

A subset A of $\mathbb{R}$ is said to be **compact** if every open covering of A admits a finite subcovering.

Theorem (Characterization of Compact Sets)

For a subset A of $\mathbb{R}$, the following conditions are equivalent:

(a) A is compact;

(b) A is bounded and closed.

- (a)⇒(b): Suppose A is compact. The open intervals $(-n, n)$, $n \in \mathbb{P}$, have union $\mathbb{R}$, so they certainly cover A. By hypothesis, a finite number of them suffice to cover A, which means that $A \subseteq (-m, m)$, for some m. Consequently A is bounded. To show that A is closed, we need only show that $\overline{A} \subseteq A$, equivalently, $A^c \subseteq (\overline{A})^c$. Assuming $x \notin A$, we show that $x \notin \overline{A}$.

Compact Sets (Cont'd)

- Assuming $x \notin A$, we show that $x \notin \overline{A}$. We must find a neighborhood V of x such that $V \cap A = \emptyset$. If $a \in A$, then $x \neq a$ (because $x \notin A$), so there exist open intervals U_a, V_a, such that $a \in U_a$, $x \in V_a$ and $U_a \cap V_a = \emptyset$. As a varies over A, the sets U_a form an open covering of A. Suppose $A \subseteq U_{a_1} \cup \cdots \cup U_{a_r}$. Let $U = U_{a_1} \cup \cdots \cup U_{a_r}$ and $V = V_{a_1} \cap \cdots \cap V_{a_r}$. Then $A \subseteq U$ and V is a neighborhood of x. If $y \in U_{a_j}$, then $y \notin V_{a_j}$, whence $y \notin V$. It follows that $V \cap U = \emptyset$ (V misses every term in the formula for U, so it misses their union), and, consequently, $V \cap A = \emptyset$.
- (b)$\Rightarrow$(a): Assume that A is bounded and closed and that $\mathcal{C}$ is an open covering of A. By hypothesis, the set $V = \mathbb{R} - A$ is open and A is contained in some closed interval, say $A \subseteq [a, b]$. We apply the Heine-Borel theorem to $[a, b]$: The points of $[a, b]$ that are in A are covered by $\mathcal{C}$ and what is left, $[a, b] - A$, is contained in V. We thus have an open covering of $[a, b]$: the sets in $\mathcal{C}$, helped out by V. It follows that $[a, b] \subseteq V \cup U_1 \cup \cdots \cup U_r$ for suitable $U_1, \ldots, U_r$ in $\mathcal{C}$.

Compact Sets (Finishing the Proof)

- We showed $[a, b] \subseteq V \cup U_1 \cup \cdots \cup U_r$ for suitable $U_1, \ldots, U_r$ in $\mathcal{C}$. The set A is contained in $[a, b]$ but is disjoint from V, so $A \subseteq U_1 \cup \cdots \cup U_r$ is the desired finite subcovering.

Corollary

Every nonempty compact set $A \subseteq \mathbb{R}$ has a largest element and a smallest element.

- By the theorem, A is bounded and closed. Let $M = \sup A$ and choose a sequence (x_n) in A such that $x_n \to M$. Then $M \in A$ (because A is closed) and M is obviously the largest element of A. Similarly, $\inf A$ belongs to A and is its smallest element.