Introduction to Real Analysis

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LSSU Math 421
1 Special Subsets of $\mathbb{R}$

- Intervals
- Closed Sets
- Open Sets, Neighborhoods
- Finite and Infinite Sets
- Heine-Borel Covering Theorem
Subsection 1

Intervals
Intervals in $\mathbb{R}$

- There are nine kinds of subsets of $\mathbb{R}$ that are called **intervals**.
  - First, there are $[a, b], (a, b), [a, b), (a, b]$.

- Next, for each real number $c$ there are the four “half-lines”
  - $\{x \in \mathbb{R} : x \leq c\}$,
  - $\{x \in \mathbb{R} : x < c\}$,
  - $\{x \in \mathbb{R} : x \geq c\}$, and
  - $\{x \in \mathbb{R} : x > c\}$.

- Finally, $\mathbb{R}$ itself is regarded as an interval (extending indefinitely in both directions).

The Symbols \(\pm\infty\)

**Definition (The Symbols \(\pm\infty\))**

For every real number \(x\), we write \(x < +\infty\) and \(x > -\infty\), or, concisely, \(-\infty < x < +\infty\). We think of \(+\infty\) (read “plus infinity”) as a symbol that stands to the right of every point of the real line, and \(-\infty\) (“minus infinity”) as a symbol that stands to the left of every point of the line. Finally, we write \(-\infty < +\infty\).

- A new set \(\mathbb{R} \cup \{-\infty, +\infty\}\) has been created, by adjoining to \(\mathbb{R}\) two new elements and specifying the order relations between the new elements \(-\infty\) and \(+\infty\) and the old ones (those in \(\mathbb{R}\)).
- A natural correspondence between real numbers \(x\) and points \(P\) of a semicircle presents the “points at \(\pm\infty\)” as the endpoints of the semicircle:
Another Illustration of the Extension

- A computationally simpler explanation uses the function $f : (-1, 1) \rightarrow \mathbb{R}$, defined by

$$f(x) = \frac{x}{1 - |x|}.$$

It is an order-preserving bijection.

- It can be extended (in an order-preserving way) to the closed interval $[-1, 1]$ by assigning the values $\pm \infty$ to the endpoints $\pm 1$. 

![Graph of the function $f(x) = \frac{x}{1 - |x|}$]

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Formal Definition of Intervals

Definition (Unbounded Intervals)

For any real number \( c \), we write
\[
\begin{align*}
[c, +\infty) &= \{x \in \mathbb{R} : c \leq x < +\infty\} = \{x \in \mathbb{R} : x \geq c\} \\
(c, +\infty) &= \{x \in \mathbb{R} : c < x < +\infty\} = \{x \in \mathbb{R} : x > c\} \\
(-\infty, c] &= \{x \in \mathbb{R} : -\infty < x \leq c\} = \{x \in \mathbb{R} : x \leq c\} \\
(-\infty, c) &= \{x \in \mathbb{R} : -\infty < x < c\} = \{x \in \mathbb{R} : x < c\} \\
(-\infty, +\infty) &= \{x \in \mathbb{R} : -\infty < x < +\infty\} = \mathbb{R}.
\end{align*}
\]

When \(+\infty\) or \(-\infty\) (neither of which is a real number) is used as an “endpoint” of an interval of \( \mathbb{R} \), it is always absent from the interval.

Definition (Intervals)

An **interval** of \( \mathbb{R} \) is a subset of \( \mathbb{R} \) of one of the following 9 types:
\[
[a, b], (a, b), [a, b), (a, b], [c, +\infty), (c, +\infty), (-\infty, c], (-\infty, c), (-\infty, +\infty)
\]
\( = \mathbb{R} \), where \( a, b, c \) are real numbers and \( a \leq b \). In particular, the empty set \( \emptyset = (a, a) = [a, a) = (a, a] \) and singletons \{a\} = [a, a] qualify as intervals.
Convexity

The intervals of \( \mathbb{R} \) are characterized by **convexity**:

**Theorem**

Let \( A \) be a nonempty subset of \( \mathbb{R} \). The following conditions are equivalent:

(a) \( A \) is an interval;

(b) For every pair of points in \( A \), the segment joining them is contained in \( A \); i.e., for all \( x, y \in A \), \( x \leq y \Rightarrow [x, y] \subseteq A \).

- \((a) \Rightarrow (b)\) is obvious.
- \((b) \Rightarrow (a)\): There are four cases, according as \( A \) is bounded above (or not) and bounded below (or not).

1. **A bounded below, but not bounded above**: Let \( a = \inf A \). We will show that \( A = (a, +\infty) \) or \( A = [a, +\infty) \). It suffices to show that \((a, +\infty) \subseteq A \subseteq [a, +\infty) \). The second inclusion is immediate from the definition of \( a \). Assuming \( r \in (a, +\infty) \), we have to show that \( r \in A \). Since \( r > a = \text{GLBA} \), there exists \( x \in A \), such that \( a < x < r \). Since \( A \) is not bounded above, there exists \( y \in A \), such that \( y > r \). Thus \( x < r < y \) with \( x, y \in A \), so \( r \in [x, y] \subseteq A \), by the hypothesis.

2.-4. Similar arguments.
Corollary

If \( S \) is any set of intervals and \( J = \bigcap S \) is their intersection, then \( J \) is an interval (possibly empty).

- By definition, \( J \) is the set of all real numbers common to all of the intervals belonging to \( S \), i.e.,

\[
J = \{ r \in \mathbb{R} : r \in I, \text{ for every } I \in S \}.
\]

Assuming \( J \) nonempty, it will suffice to verify convexity. Suppose \( x, y \in J, \ x \leq y \). Then \( x, y \in I \), for every \( I \in S \). By the theorem, \( [x, y] \subseteq I \), for all \( I \in S \), and, therefore, \( [x, y] \subseteq J \).
Subsection 2

Closed Sets
The most important subsets of \( \mathbb{R} \) for calculus are the intervals.

There are differences among intervals, some important, others not:

- The difference between \((0, 1)\) and \((0, 5)\) is only a matter of scale; otherwise, the inequalities defining them are qualitatively the same.
- The intervals \((0, 1)\) and \((0, +\infty)\) are different in kind, since one is bounded and the other not.
- By contrast, the intervals \(I = (0, 1)\) and \(J = [0, 1]\) prove to have dramatically different properties.
  - The crux is that the endpoints 0, 1 of \(I\) can be approximated as closely as we like by points of \(I\) but they are not themselves points of \(I\). More precisely, the endpoints of \(I\), though not in \(I\), are limits of convergent sequences whose terms are in \(I\).
  - On the other hand, if a convergent sequence has its terms in \(J\) then its limit must also be in \(J\).
Closed Sets of $\mathbb{R}$

**Definition (Closed Sets)**

A set $A$ of real numbers is said to be a **closed subset** of $\mathbb{R}$ (or to be a **closed set** in $\mathbb{R}$) if, whenever a convergent sequence has all of its terms in $A$, the limit of the sequence must also be in $A$, i.e., if $x_n \to x$ and $x_n \in A$, for all $n$, then necessarily $x \in A$. (One cannot “escape” from a closed set by means of a convergent sequence!) In symbols,

$$
\left\{ \begin{align*}
(\forall n) (x_n \in A) \\
x \in \mathbb{R} \\
x_n \to x
\end{align*} \right\} \implies x \in A.
$$

- Note that the empty subset $\emptyset$ of $\mathbb{R}$ is closed.
Examples of Closed Sets

- $\mathbb{R}$ is a closed subset of $\mathbb{R}$ (there is nowhere else for the limit to go!).
- Every singleton $\{a\}$, for $a \in \mathbb{R}$, is closed (the constant sequence $x_n = a$ converges to $a$).
- For every real number $c$, the intervals $[c, +\infty)$ and $(-\infty, c]$ are closed sets.

If $x_n \to x$ and $x_n \geq c$, for all $n$, then $x \geq c$.

**Caution**: These are closed sets and they are intervals, but they are not closed intervals. The term “closed interval” is reserved for intervals of the form $[a, b]$. 
Some Properties of Closed Sets

**Lemma**

If $A$ is closed in $\mathbb{R}$, $x_n \to x$ in $\mathbb{R}$, and $x_n \in A$ frequently, then $x \in A$.

- By assumption, there is a subsequence $(x_{n_k})$ with $x_{n_k} \in A$, for all $k$. Since $x_{n_k} \to x$, $x \in A$, by the definition of a closed set.

**Theorem**

(i) $\emptyset$ and $\mathbb{R}$ are closed sets in $\mathbb{R}$.

(ii) If $A$ and $B$ are closed sets in $\mathbb{R}$, then so is their union $A \cup B$.

(iii) If $S$ is any set of closed sets in $\mathbb{R}$, then $\bigcap S$ is also a closed set.

(i) Both have been noted.

(ii) Suppose $x_n \in A \cup B$, for all $n$ and $x_n \to x$. If $x_n \in A$ frequently, then $x \in A$ by the lemma. The alternative is that $x_n \in B$ ultimately, in which case $x \in B$, again by the lemma. Either way, $x \in A \cup B$.

(iii) Let $B = \bigcap S = \{x \in \mathbb{R} : x \in A, \text{ for all } A \in S\}$. Suppose $x_n \to x$ and $x_n \in B$, for all $n$. For each $A \in S$, $x_n \in A$, for all $n$, whence $x \in A$. Thus $x \in A$, for all $A \in S$, and, therefore, $x \in \bigcap S = B$. 
Some Applications of the Properties

- Every closed interval

\[ [a, b] = (-\infty, b] \cap [a, +\infty) \]

is a closed set.

It is the intersection of two closed sets.

- If \( A_1, \ldots, A_r \) is a finite list of closed sets in \( \mathbb{R} \), then their union \( A_1 \cup \cdots \cup A_r \) is also a closed set.

By Induction on \( r \).

- Every finite subset \( A = \{a_1, \ldots, a_r\} \) of \( \mathbb{R} \) is a closed set.

\( A = \{a_1\} \cup \cdots \cup \{a_r\} \) is closed since each singleton is closed and a finite union of closed sets is also closed.
Set of all Limits

- If $A$ is a closed set, we know where the limits of its convergent sequences are.
- On the other hand, the set $A = (0, 1]$ contains a convergent sequence - for example $x_n = \frac{1}{n}$ - whose limit is not in $A$.
- For an arbitrary subset $A$ of $\mathbb{R}$ we may contemplate the set $\overline{A}$ of all real numbers that are limits of sequences whose terms are in $A$.
- Regardless of the status of $A$, $\overline{A}$ is always a closed set.
Characterizing the Set of all Limits

**Theorem**

Let $A$ be any subset of $\mathbb{R}$ and let

$$\overline{A} = \{ x \in \mathbb{R} : a_n \to x \text{ for some sequence } (a_n) \text{ in } A \}.$$ 

Then $\overline{A}$ is the smallest closed set containing $A$, i.e.,

1. $\overline{A} \supseteq A$,
2. $\overline{A}$ is a closed set,
3. if $B$ is a closed set with $B \supseteq A$, then $B \supseteq \overline{A}$.

Moreover,

4. $A$ is closed $\iff \overline{A} = A$;
5. $\overline{A}$ is the set of all real numbers that can be approximated as closely as we like by elements of $A$, i.e.,

$$x \in \overline{A} \iff (\forall \epsilon > 0) (\exists a \in A)(|x - a| < \epsilon).$$
Proof of the Theorem

(1) If $a \in A$, let $a_n = a$, for all $n$. Then $a_n \to a$, and so $a \in \overline{A}$.

(3) Assuming $B$ is a closed set with $A \subseteq B$, we have to show that $\overline{A} \subseteq B$. Let $x \in \overline{A}$, say $a_n \to x$, with $a_n \in A$, for all $n$. Then $a_n \in B$, for all $n$ (because $A \subseteq B$), therefore $x \in B$ (because $B$ is closed).

(4) To say that $A$ is closed means that $\overline{A} \subseteq A$. Since $A \subseteq \overline{A}$ automatically, the condition $\overline{A} \subseteq A$ is equivalent to $\overline{A} = A$.

(5) ($\Leftarrow$): For each positive integer $n$ let $\epsilon = \frac{1}{n}$ and choose $a_n \in A$, such that $|x - a_n| < \frac{1}{n}$. Then $a_n \to x$ so $x \in \overline{A}$.

($\Rightarrow$): Let $x \in \overline{A}$, say $a_n \to x$, with $a_n \in A$, for all $n$. If $\epsilon > 0$, then $|x - a_n| < \epsilon$, for some $n$ (in fact, ultimately!).

(2) Assuming $x_n \to x$, with $x_n \in \overline{A}$, for all $n$, we have to show that $x \in \overline{A}$. We apply the criterion of (5): If $\epsilon > 0$, choose $n$ so that $|x - x_n| < \frac{\epsilon}{2}$; for this $n$, choose $a \in A$, so that $|x_n - a| < \frac{\epsilon}{2}$. Then $|x - a| < \epsilon$ by the triangle inequality.
Closure of a Set

Definition (Closure)

$\overline{A}$ is called the **closure** of $A$ in $\mathbb{R}$.
Alternatively, the points of $\overline{A}$ are said to be **adherent** to $A$, and $\overline{A}$ is called the **adherence** of $A$. 

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Subsection 3

Open Sets, Neighborhoods
Complement of $A$

According to the preceding theorem, the meaning of $x \in \overline{A}$ is that for every $\epsilon > 0$ the interval $(x - \epsilon, x + \epsilon)$ intersects $A$, i.e.,

$$(\forall \epsilon > 0)((x - \epsilon, x + \epsilon) \cap A \neq \emptyset).$$

The meaning of $x \notin \overline{A}$ is the negation of the preceding condition: there exists an $\epsilon > 0$, for which the interval $(x - \epsilon, x + \epsilon)$ is disjoint from $A$, i.e.,

$$(\exists \epsilon > 0)((x - \epsilon, x + \epsilon) \cap A = \emptyset),$$

or, denoting by $A^c = \mathbb{R} - A$, the complement of $A$ in $\mathbb{R}$,

$$(\exists \epsilon > 0)((x - \epsilon, x + \epsilon) \subseteq A^c).$$

Not only does $x$ belong to $A^c$, but there is a little “buffer zone” about $x$ that remains in $A^c$ - informally, all points “sufficiently close to $x$” are in $A^c$. 
Interior Points

Definition (Interior Point)

A point $x \in \mathbb{R}$ is said to be \textbf{interior} to a subset $A$ of $\mathbb{R}$ if there exists an $r > 0$, such that $(x - r, x + r) \subseteq A$, i.e., such that $|y - x| < r \Rightarrow y \in A$. If $x$ is interior to $A$, one also says that $A$ is a \textbf{neighborhood} of $x$. The set of all interior points of $A$ (there may not be any!) is called the \textbf{interior of $A$}, denoted $A^\circ$:

$$A^\circ = \{x \in \mathbb{R} : x \text{ is interior to } A\}.$$  

Thus, $A^\circ$ is the set of all points of $A$ of which $A$ is a neighborhood.

- \textbf{Example}: $\mathbb{Q}$ has no interior points (i.e., it has empty interior), because every open interval contains an irrational number.
- \textbf{Example}: Assuming $a < b$, the point $a$ belongs to $[a, b)$ but not to its interior; the interior of $[a, b]$ is $(a, b)$.
Relation Between Closure and Interior

Theorem

If $A$ is any subset of $\mathbb{R}$, then $x \notin \overline{A} \iff x \in (A^c)^\circ$. Thus, $(\overline{A})^c = (A^c)^\circ$.

Thus, the passage from $A$ to its closure $\overline{A}$ is the composite of three operations:
- take complement,
- then take interior,
- then take complement again.

More precisely, we have

$$\overline{A} = ((A^c)^\circ)^c \quad \text{and} \quad A^\circ = (\overline{A}^c)^c.$$
Open Sets

- In general, $A \subseteq \overline{A}$. Equality is a special event ($A$ closed).
- In general, $A^\circ \subseteq A$. Again, equality is a special event:

**Definition (Open Set)**

A subset $A$ of $\mathbb{R}$ is called an **open set** if every point of $A$ is an interior point, i.e.,

$$\forall x \in A \exists \epsilon > 0 ((x - \epsilon, x + \epsilon) \subseteq A).$$

Equivalently, $A$ is a neighborhood of each of its points.

- Intuitively, for every point of an open set $A$, there is a buffer zone about the point - whose size may depend on the point - that is also contained in $A$.
- To say that $A$ is open means that $A \subseteq A^\circ$; Since $A \supseteq A^\circ$ automatically, an equivalent condition is that $A = A^\circ$, i.e., $A$ is equal to its interior.
Characterization of Open Sets

**Theorem**

For a subset \( A \) of \( \mathbb{R} \),

(i) \( A \) is open ⇔ \( A^c \) is closed;
(ii) \( A \) is closed ⇔ \( A^c \) is open.

(i) In general, \( A^o = (\overline{A^c})^c \), so the following conditions are equivalent:
- \( A \) open,
- \( A = A^o \),
- \( A = (\overline{A^c})^c \),
- \( A^c = \overline{A^c} \),
- \( A^c \) closed.

(ii) Apply (i) with \( A \) replaced by \( A^c \).
Two Corollaries

**Corollary**

For every subset $A$ of $\mathbb{R}$, $A^\circ$ is the largest open subset of $A$.

- Since $A^\circ = (\overline{A})^c$ is the complement of a closed set, it is open. On the other hand, if $U$ is any open set with $U \subseteq A$, then $U \subseteq A^\circ$: If $x \in U$, then $x$ is interior to $U$, so it is obviously interior to $A$ as well.

**Corollary**

Let $A \subseteq \mathbb{R}$, $x \in \mathbb{R}$. The following conditions are equivalent:

(a) $A$ is a neighborhood of $x$;

(b) There exists an open set $U$, such that $x \in U \subseteq A$.

- $(a) \Rightarrow (b)$: $U = A^\circ$ fills the bill.
- $(b) \Rightarrow (a)$: Since $x \in U$ and $U$ is open, $U$ is a neighborhood of $x$. Therefore so is its superset $A$. 
Properties of Open Sets

Theorem

(i) $\emptyset$ and $\mathbb{R}$ are open sets in $\mathbb{R}$.

(ii) If $A$ and $B$ are open sets in $\mathbb{R}$, then so is their intersection $A \cap B$.

(iii) If $S$ is any set of open sets in $\mathbb{R}$, then $\bigcup S$ is also an open set.

(i) $\emptyset = \mathbb{R}^c$ and $\mathbb{R} = \emptyset^c$ are complements of closed sets and so are open.

(ii) Assuming $x \in A \cap B$, we have to show that $x$ is interior to $A \cap B$. Since $x \in A$ and $A$ is open, there is an $r > 0$, with $(x - r, x + r) \subseteq A$. Similarly, there is an $s > 0$, with $(x - s, x + s) \subseteq B$. If $t$ is the smaller of $r$ and $s$, then $(x - t, x + t) \subseteq A \cap B$, so $x$ is interior to $A \cap B$.

(iii) Let $B = \bigcup S$. If $x \in B$, then $x \in A$, for some $A \in S$. By assumption, $A$ is open, so there is an $r > 0$, with $(x - r, x + r) \subseteq A$. Since $A \subseteq B$, it follows that $(x - r, x + r) \subseteq B$, thus $x$ is interior to $B$.

Note that the proof of Part (ii) shows that if $A$ and $B$ are neighborhoods of $x$, then so is $A \cap B$. 
Finite and Infinite Sets

Definition (Finite and Infinite Sets)

A nonempty set $A$ is said to be **finite** if there exist a positive integer $r$ and a surjection $\{1, \ldots, r\} \to A$.

**Convention:** The empty set $\emptyset$ is finite.

A set is said to be **infinite** if it is not finite.

- If $\sigma : \{1, \ldots, r\} \to A$ is a surjection and one writes $x_i = \sigma(i)$, for $i = 1, \ldots, r$, then $A = \sigma(\{1, \ldots, r\}) = \{x_1, \ldots, x_r\}$.
- We also say that $x_1, \ldots, x_r$ is a finite list of elements.
- **Example:** For each positive integer $r$, the set $\{1, \ldots, r\}$ is finite. The identity mapping $\{1, \ldots, r\} \to \{1, \ldots, r\}$ is a surjection.
- **Example:** The set $\mathbb{P}$ of all positive integers is infinite. We show that there does not exist a surjection $\{1, \ldots, r\} \to \mathbb{P}$ for any $r$. Assuming $r \in \mathbb{P}$ and $\varphi : \{1, \ldots, r\} \to \mathbb{P}$. Let $n = 1 + \varphi(1) + \cdots + \varphi(r)$. Then $\varphi(i) < n$, for all $i = 1, \ldots, r$, whence $n$ is not in the range of $\varphi$. 
Properties of Finite Sets

**Theorem**

If $f : X \to Y$ is any function and $A$ is a finite subset of $X$, then $f(A)$ is a finite subset of $Y$.

- If $\sigma : \{1, \ldots, r\} \to A$ is surjective, then $i \mapsto f(\sigma(i))$ is a surjection $\{1, \ldots, r\} \to f(A)$.

**Theorem**

If $A_1, \ldots, A_m$ is a finite list of finite subsets of a set, then $A_1 \cup \cdots \cup A_m$ is also finite.

- For each $j = 1, \ldots, m$, there is a positive integer $r_j$ and a surjection $\sigma_j : \{1, \ldots, r_j\} \to A_j$. Let $r = r_1 + \cdots + r_m$. We will construct a surjection $\sigma : \{1, \ldots, r\} \to A_1 \cup \cdots \cup A_m$. The elements of $\{1, \ldots, r\}$ can be organized, in ascending order, as a union of $m$ subsets:
Properties of Finite Sets (Cont’d)

\[ \{1, \ldots, r\} = \{1, \ldots, r_1\} \cup \{r_1 + 1, \ldots, r_1 + r_2\} \cup \cdots \]
\[ \cup \{r_1 + \cdots + r_{m-1} + 1, \ldots, r_1 + \cdots + r_{m-1} + r_m\} \]
\[ = B_1 \cup B_2 \cup \cdots \cup B_m, \]

where the sets \( B_j \) are pairwise disjoint. For each \( j = 1, \ldots, m \) the formula \( \theta_j(i) = r_1 + \cdots + r_{j-1} + i \) defines a bijection \( \theta_j : \{1, \ldots, r_j\} \to B_j \). Define \( \sigma : \{1, \ldots, r\} \to A_1 \cup \cdots \cup A_m \) as follows: If \( k \in \{1, \ldots, r\} \) then \( k \in B_j \), for a unique \( j \in \{1, \ldots, m\} \), so \( k = \theta_j(i) \) for a unique \( i \in \{1, \ldots, r_j\} \).

Define \( \sigma(k) = \sigma_j(i) = \sigma_j(\theta_j^{-1}(k)) \). In other words, \( \sigma \) is the unique mapping on \( \{1, \ldots, r\} \) that agrees with \( \sigma_j \circ \theta_j^{-1} \) on \( B_j \). It remains to show that \( \sigma \) is surjective:

\[ \sigma(\{1, \ldots, r\}) = \sigma(B_1 \cup \cdots \cup B_m) \]
\[ = \sigma(B_1) \cup \cdots \cup \sigma(B_m) \]
\[ = \sigma_1(\theta_1^{-1}(B_1)) \cup \cdots \cup \sigma_m(\theta_m^{-1}(B_m)) \]
\[ = \sigma_1(\{1, \ldots, r_1\}) \cup \cdots \cup \sigma_m(\{1, \ldots, r_m\}) \]
\[ = A_1 \cup \cdots \cup A_m. \]
Finding Increasing Sequences

Lemma

If $A \subseteq \mathbb{P}$ and $A$ is infinite, then there exists a strictly increasing mapping $\varphi : \mathbb{P} \to A$. In particular, $\varphi$ is injective and $\varphi(n) \geq n$, for all $n \in \mathbb{P}$.

Apart from notation, it is the same to show that there exists a sequence $(a_n)$ in $A$ such that $m < n \Rightarrow a_m < a_n$. Define $a_n$ recursively as follows: Since $A$ is not finite, it is nonempty. Let $a_1$ be the smallest element of $A$. Then $A \neq \{a_1\}$ (because $\{a_1\}$ is finite and $A$ is not), so $A - \{a_1\} \neq \emptyset$. Let $a_2$ be the smallest element of $A - \{a_1\}$. Then $a_2 > a_1$ and $A \neq \{a_1, a_2\}$ (because $\{a_1, a_2\}$ is finite), so $A - \{a_1, a_2\}$ has a smallest element $a_3$, and $a_3 > a_2$. Assuming $a_1, \ldots, a_n$ already defined, let $a_{n+1}$ be the smallest element of $A - \{a_1, \ldots, a_n\}$. The function $\varphi : \mathbb{P} \to A$ defined by $\varphi(n) = a_n$ is strictly increasing. By induction, $\varphi(n) \geq n$, for all $n$:

- For, $\varphi(1) \geq 1$;
- If $\varphi(k) \geq k$, then $\varphi(k + 1) > \varphi(k) \geq k$, therefore, $\varphi(k + 1) \geq k + 1$. 
Theorem

Every subset of a finite set is finite.

Suppose $F$ is finite and $B \subseteq F$. By assumption, there exists a surjection $\sigma : \{1, \ldots, r\} \to F$, for some positive integer $r$. Let $A = \sigma^{-1}(B)$ be the inverse image of $B$ under $\sigma$. Then $\sigma(A) = B$ (because $\sigma$ is surjective). So it will suffice to show that $A$ is finite. We have $A \subseteq \{1, \ldots, r\}$. If $A$ were infinite, by the lemma, there would exist a mapping $\varphi : \mathbb{P} \to A$, such that $\varphi(n) \geq n$, for all $n \in \mathbb{P}$. But then $n \leq \varphi(n) \leq r$, for all $n$, which is absurd for $n = r + 1$. 
Consequences for Infinite Sets

**Corollary**

Every superset of an infinite set is infinite.

- Suppose $B \supseteq A$. By the theorem, $B$ finite implies $A$ finite. By the contrapositive, $A$ not finite implies $B$ not finite.

**Corollary**

If $\varphi : \mathbb{P} \rightarrow A$ is injective, then $A$ is infinite.

- If $B = \varphi(\mathbb{P})$ is the range of $\varphi$, then $\varphi$ defines a bijection $\mathbb{P} \rightarrow B$. Let $\psi : B \rightarrow \mathbb{P}$ be the inverse of this bijection. Since $\psi(B) = \mathbb{P}$ and $\mathbb{P}$ is infinite, $B$ cannot be finite. But $B \subseteq A$, so $A$ cannot be finite either.
The property appearing in the last Corollary characterizes infinite sets:

**Theorem**

A set $A$ is infinite if and only if there exists an injection $\mathbb{P} \rightarrow A$.

The “if” part is the Corollary.

Conversely, assuming $A$ infinite, we have to produce a sequence $(a_n)$ in $A$, such that $m \neq n \Rightarrow a_m \neq a_n$: “Construct” $a_n$ recursively as follows: Since $A$ is infinite, it is nonempty. Choose $a_1 \in A$. Then $A \neq \{a_1\}$ (because $\{a_1\}$ is finite), so $A - \{a_1\} \neq \emptyset$. Choose $a_2 \in A - \{a_1\}$. Assuming $a_1, \ldots, a_n$ already chosen, $A \neq \{a_1, \ldots, a_n\}$. Choose $a_{n+1} \in A - \{a_1, \ldots, a_n\}$. 

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Subsection 5

Heine-Borel Covering Theorem
Coverings, Open Coverings and Subcoverings

**Definition (Covering)**

Let $A \subseteq \mathbb{R}$ and let $C$ be a set of subsets of $\mathbb{R}$.

- If each point of $A$ belongs to some set in $C$, we say that $C$ is a **covering** of $A$ (or that $C$ **covers** $A$). In symbols,

  $$(\forall x \in A)(\exists C \in C)(x \in C).$$

  More concisely, $A \subseteq \bigcup C$.

- If, moreover, every set in $C$ is an open subset of $\mathbb{R}$, then $C$ is said to be an **open covering** of $A$.

- If a covering $C$ of $A$ consists of only a finite number of sets, it is called a **finite covering**.

- If $C$ is a covering of $A$ and if $D \subseteq C$ is such that $D$ is also a covering of $A$, then $D$ is referred to as a **subcovering** (it is a subset of $C$ and still a covering of $A$).
Examples

Suppose $A$ consists of the terms of a convergent sequence and its limit, i.e., $A = \{x\} \cup \{x_n : n \in \mathbb{P}\}$, where $x_n \to x$. If $C$ is an open covering of $A$, then $A$ is covered by finitely many of the sets in $C$: The limit $x$ belongs to one of the sets in $C$, say $x \in U \in C$. Since $U$ is open, there is an $\epsilon > 0$, with $(x - \epsilon, x + \epsilon) \subseteq U$. It follows that $x_n \in U$ ultimately, say for $n > N$. Each of the terms $x_i, i = 1, \ldots, N$, belongs to some $U_i \in C$, so $A$ is covered by the sets $U, U_1, \ldots, U_N$. In the preceding terminology, every open covering of $A$ admits a finite subcovering.

Let $A$ be the open interval $(2, 5)$ and let $C$ be the set of all open intervals $(2 + \frac{1}{n}, 5 - \frac{1}{n}), n \in \mathbb{P}$. Then $C$ is an open covering of $A$, but no finite set of elements of $C$ can cover $A$. Each element of $C$ is a proper subset of $A$. Moreover, among any finite set of elements of $C$, one of them contains all the others. Thus, $C$ is an open covering of $A$ that admits no finite subcovering.
Heine-Borel Theorem

Let $S$ be the set of all $x \in [a, b]$, such that the closed interval $[a, x]$ is covered by finitely many sets of $\mathcal{C}$. At least $a \in S$, because $[a, a] = \{a\}$ and $a$ belongs to some set in $\mathcal{C}$. We will show that $b \in S$. At any rate, $S$ is nonempty and bounded. Let

$$m = \sup S.$$ 

Since $S \subseteq [a, b]$, we have $a \leq m \leq b$. The strategy of the proof is to show that:

(1) $m \in S$;

(2) $m = b$. 

Theorem (Heine-Borel Theorem)

If $[a, b]$ is a closed interval in $\mathbb{R}$ and $\mathcal{C}$ is an open covering of $[a, b]$, then $[a, b]$ is covered by a finite number of the sets in $\mathcal{C}$. 

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Proof of the Heine-Borel Theorem

(1) Since \( m \in [a, b] \subseteq \bigcup \mathcal{C} \), there is a \( V \in \mathcal{C} \), such that \( m \in V \). Since \( V \) is open, \( [m - \varepsilon, m + \varepsilon] \subseteq V \), for some \( \varepsilon > 0 \). Note that we can take \( \varepsilon \) to be as small as we like. Since \( m - \varepsilon < m \) and \( m \) is the least upper bound of \( S \), there exists \( x \in S \), with \( m - \varepsilon < x \leq m \). From \( x \in S \), we know that the interval \([a, x]\) is covered by finitely many sets in \( \mathcal{C} \), say \([a, x] \subseteq U_1 \cup \cdots \cup U_r\). On the other hand, \([x, m] \subseteq [m - \varepsilon, m + \varepsilon] \subseteq V\), so \([a, m] = [a, x] \cup [x, m]\) is covered by the sets \( V, U_1, \ldots, U_r \) of \( \mathcal{C} \). This proves that \( m \in S \), and a little more: \([a, m + \varepsilon] \subseteq V \cup U_1 \cup \cdots \cup U_r\), whence \( m + \varepsilon > b \), because \( m + \varepsilon \leq b \) would imply that \( m + \varepsilon \in S \), contrary to the fact that every element of \( S \) is \( \leq m \).

(2) The preceding argument shows that \( b - m < \varepsilon \) and the argument is valid with \( \varepsilon \) replaced by any positive number smaller than \( \varepsilon \). It follows that \( b - m \leq 0 \). Thus \( b \leq m \). Since, already \( m \leq b \), we get \( b = m \in S \).
Compact Sets

Definition (Compact Set)

A subset $A$ of $\mathbb{R}$ is said to be **compact** if every open covering of $A$ admits a finite subcovering.

Theorem (Characterization of Compact Sets)

For a subset $A$ of $\mathbb{R}$, the following conditions are equivalent:

(a) $A$ is compact;

(b) $A$ is bounded and closed.

(a)$\implies$(b): Suppose $A$ is compact. The open intervals $(-n, n)$, $n \in \mathbb{P}$, have union $\mathbb{R}$, so they certainly cover $A$. By hypothesis, a finite number of them suffice to cover $A$, which means that $A \subseteq (-m, m)$, for some $m$. Consequently $A$ is bounded. To show that $A$ is closed, we need only show that $\overline{A} \subseteq A$, equivalently, $A^c \subseteq (\overline{A})^c$. Assuming $x \notin A$, we show that $x \notin \overline{A}$. 
Compact Sets (Cont’d)

- Assuming $x \notin A$, we show that $x \notin \overline{A}$. We must find a neighborhood $V$ of $x$ such that $V \cap A = \emptyset$. If $a \in A$, then $x \neq a$ (because $x \notin A$), so there exist open intervals $U_a, V_a$, such that $a \in U_a$, $x \in V_a$ and $U_a \cap V_a = \emptyset$. As $a$ varies over $A$, the sets $U_a$ form an open covering of $A$. Suppose $A \subseteq U_{a_1} \cup \cdots \cup U_{a_r}$. Let $U = U_{a_1} \cup \cdots \cup U_{a_r}$ and $V = V_{a_1} \cap \cdots \cap V_{a_r}$. Then $A \subseteq U$ and $V$ is a neighborhood of $x$. If $y \in U_{a_j}$, then $y \notin V_{a_j}$, whence $y \notin V$. It follows that $V \cap U = \emptyset$ ($V$ misses every term in the formula for $U$, so it misses their union), and, consequently, $V \cap A = \emptyset$.

- $(b) \Rightarrow (a)$: Assume that $A$ is bounded and closed and that $C$ is an open covering of $A$. By hypothesis, the set $V = \mathbb{R} - A$ is open and $A$ is contained in some closed interval, say $A \subseteq [a, b]$. We apply the Heine-Borel theorem to $[a, b]$: The points of $[a, b]$ that are in $A$ are covered by $C$ and what is left, $[a, b] - A$, is contained in $V$. We thus have an open covering of $[a, b]$: the sets in $C$, helped out by $V$. It follows that $[a, b] \subseteq V \cup U_1 \cup \cdots \cup U_r$ for suitable $U_1, \ldots, U_r$ in $C$. 

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We showed \([a, b] \subseteq V \cup U_1 \cup \cdots \cup U_r\) for suitable \(U_1, \ldots, U_r\) in \(C\).

The set \(A\) is contained in \([a, b]\) but is disjoint from \(V\), so \(A \subseteq U_1 \cup \cdots \cup U_r\) is the desired finite subcovering.

**Corollary**

Every nonempty compact set \(A \subseteq \mathbb{R}\) has a largest element and a smallest element.

By the theorem, \(A\) is bounded and closed. Let \(M = \sup A\) and choose a sequence \((x_n)\) in \(A\) such that \(x_n \to M\). Then \(M \in A\) (because \(A\) is closed) and \(M\) is obviously the largest element of \(A\). Similarly, \(\inf A\) belongs to \(A\) and is its smallest element.