Introduction to Real Analysis

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Subsection 1

Functions, Direct Images, Inverse Images

Image of a Set Under a Function

- A function $f : X \to Y$ acts on points of X to produce points of Y.
- It is useful to let f also act on subsets of X to produce subsets of Y and vice versa (even if f does not have an inverse function).
- If A is a subset of X we can let f act on all of the elements of A. This action results in a set of elements of Y, i.e., a subset of Y, denoted $f(A)$ and called the **image** (or **direct image**) of A under f. In symbols,

$$f(A) = \{y \in Y : y = f(x), \text{ for some } x \in A\} = \{f(x) : x \in A\}.$$

- Note that, if A is a singleton, say $A = \{a\}$, then $f(A)$ is also a singleton: $f(\{a\}) = \{f(a)\}$.
- More generally, if $x_1, \ldots, x_n$ is any finite list of elements of X, then

$$f(\{x_1, \ldots, x_n\}) = \{f(x_1), \ldots, f(x_n)\}.$$

Inverse Image of a Set Under a Function

- In the reverse direction (from Y to X), if B is a subset of Y, we consider the elements x of X that are mapped by f into B, i.e., such that $f(x) \in B$.
- The set of all such elements x (there may not be any!) forms a subset of X (possibly empty), called the **inverse image** of B under f and denoted $f^{-1}(B)$. In symbols,

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

- Example: Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2$. Then:
  - $f(\{2\}) = \{4\},\ f(\{-2, 2\}) = \{4\}$,
  - $f^{-1}(\{4\}) = \{-2, 2\}$,
  - $f([0, 2]) = [0, 4] = f([-1, 2])$,
  - $f^{-1}([0, 4]) = [-2, 2]$,
  - $f([0, +\infty)) = [0, +\infty)$.

Examples of Image and Inverse Image of a Set

- Example: Let f be the sine function, i.e., define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin x$. Then:
  - $f(\pi) = 0$,
  - $f^{-1}(\{0\}) = \{n\pi : n \in \mathbb{Z}\}$,
  - $f^{-1}(\{\pi\}) = \emptyset$,
  - $f([-\frac{\pi}{2}, \frac{\pi}{2}]) = [-1, 1] = f(\mathbb{R})$,
  - $f^{-1}([0, 1]) = \bigcup_{n \in \mathbb{Z}} [2n\pi, (2n+1)\pi]$.

Properties of Images and Inverse Images

Theorem

Let $f : X \to Y$ be any function.

(1) For subsets A_1, A_2 of X, $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$;

(1') For subsets B_1, B_2 of Y, $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$;

(2) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$, for all subsets A_1, A_2 of X;

(2') $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$, for all subsets B_1, B_2 of Y;

(3) $f(f^{-1}(B)) \subseteq B$, for every subset B of Y;

(3') $f^{-1}(f(A)) \supseteq A$, for every subset A of X;

(4') $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$, for all subsets B_1, B_2 of Y;

(5') $f^{-1}(Y - B) = X - f^{-1}(B)$, for every subset B of Y.

- There are no (4) and (5) since the obvious formulas that come to mind are, in general, false.

Proof of the Theorem

(1) Assuming $y \in f(A_1)$, we have to show that $y \in f(A_2)$. By assumption, $y = f(x)$, for some $x \in A_1$. But $A_1 \subseteq A_2$, so x also belongs to A_2, thus, $y = f(x) \in f(A_2)$.

(1') If $x \in f^{-1}(B_1)$, then $f(x) \in B_1 \subseteq B_2$, so $f(x) \in B_2$, i.e., $x \in f^{-1}(B_2)$.

(2) For a point y in Y, $y \in f(A_1 \cup A_2) \Leftrightarrow y = f(x)$, for some x in $A_1 \cup A_2$, $\Leftrightarrow y = f(x)$, for some x in A_1 or in A_2, $\Leftrightarrow y \in f(A_1)$ or $y \in f(A_2)$ $\Leftrightarrow y \in f(A_1) \cup f(A_2)$.

(2') For a point x in X, $x \in f^{-1}(B_1 \cup B_2) \Leftrightarrow f(x) \in B_1 \cup B_2 \Leftrightarrow f(x) \in B_1$ or $f(x) \in B_2 \Leftrightarrow x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2) \Leftrightarrow x \in f^{-1}(B_1) \cup f^{-1}(B_2)$.

(3) If $x \in f^{-1}(B)$, then $f(x) \in B$. Thus, $f(f^{-1}(B)) \subseteq B$.

(3') If $x \in A$, then $f(x) \in f(A)$, so $x \in f^{-1}(f(A))$. Thus, $A \subseteq f^{-1}(f(A))$.

(4') For x in X, $x \in f^{-1}(B_1 \cap B_2) \Leftrightarrow f(x) \in B_1 \cap B_2 \Leftrightarrow f(x) \in B_1$ and $f(x) \in B_2 \Leftrightarrow x \in f^{-1}(B_1)$ and $x \in f^{-1}(B_2) \Leftrightarrow x \in f^{-1}(B_1) \cap f^{-1}(B_2)$.

(5') For a point x in X, $x \in f^{-1}(Y - B) \Leftrightarrow f(x) \in Y - B \Leftrightarrow f(x) \notin B$ $\Leftrightarrow x \notin f^{-1}(B) \Leftrightarrow x \in X - f^{-1}(B)$.

Subsection 2

Continuity at a Point

Continuity at a Point

Definition (Continuity at a Point)

Let $f : S \to R$, where S is a subset of $\mathbb{R}$, and let $a \in S$. In other words, a is a point of the domain of a real-valued function of a real variable. We say that f is **continuous at** a if it has the following property:

$$x_n \in S, \; x_n \to a \Rightarrow f(x_n) \to f(a).$$

I.e., if (x_n) is any sequence in S converging to the point a of S, then $(f(x_n))$ converges to $f(a)$.
If f is not continuous at a, it is said to be **discontinuous at** a.

- Example: The identity function $\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ is continuous at every $a \in \mathbb{R}$.
- Example: If $f : \mathbb{R} \to \mathbb{R}$ is a constant function, say $f(x) = c$, for all $x \in \mathbb{R}$, then f is continuous at every $a \in \mathbb{R}$.

More Examples on the Continuity at a Point

- Example: The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x \leq 0 \end{cases}$ is discontinuous at $a = 0$.

  Consider the sequence $x_n = \frac{1}{n}$.
- Example: The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \notin \mathbb{Q} \end{cases}$ is discontinuous at every $a \in \mathbb{R}$.
  - If a is rational, consider $x_n = a + \frac{1}{n}\sqrt{2}$.
  - If a is irrational, let x_n be a rational number with $a < x_n < a + \frac{1}{n}$.

Delta-Epsilon Characterization of Continuity

Theorem

Let $a \in S \subseteq \mathbb{R}$ and $f : S \to \mathbb{R}$. The following conditions on f are equivalent:

(a) f is continuous at a;

(b) For every $\epsilon > 0$, there exists a $\delta > 0$, such that $x \in S$, $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$.

- (b)⇒(a): Let $x_n \in S$, $x_n \to a$. We have to show that $f(x_n) \to f(a)$. Let $\epsilon > 0$. We want N, such that $n \geq N \Rightarrow |f(x_n) - f(a)| < \epsilon$. Choose $\delta > 0$ as in (b), then choose N, so that $n \geq N \Rightarrow |x_n - a| < \delta$ (possible because $x_n \to a$). By (b), $n \geq N \Rightarrow |f(x_n) - f(a)| < \epsilon$.
- ¬(b)⇒¬(a): Assume not (b). There exists an $\epsilon > 0$, such that, for every $\delta > 0$, the implication in (b) fails. Thus, for all $\delta > 0$, exists $x \in S$, such that $|x - a| < \delta$ and $|f(x) - f(a)| > \epsilon$. For each $n \in \mathbb{P}$, choose $x_n \in S$ so that $|x_n - a| < \frac{1}{n}$ and $|f(x_n) - f(a)| > \epsilon$. Then (x_n) in S converges to a, but $(f(x_n))$ does not converge to $f(a)$.

Characterization of Continuity in $\mathbb{R}$

Theorem

If $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$, the following conditions are equivalent:

(a) f is continuous at a;

(b) For every neighborhood V of $f(a)$, $f^{-1}(V)$ is a neighborhood of a.

- (a)$\Rightarrow$(b): Let V be a neighborhood of $f(a)$. Then, there is an $\epsilon > 0$, such that $(f(a) - \epsilon, f(a) + \epsilon) \subseteq V$. Since f is continuous, by the preceding theorem, there exists a $\delta > 0$, such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$. I.e., $x \in (a - \delta, a + \delta) \Rightarrow f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$. Thus, $f((a - \delta, a + \delta)) \subseteq (f(a) - \epsilon, f(a) + \epsilon) \subseteq V$, whence $(a - \delta, a + \delta) \subseteq f^{-1}(V)$, i.e., $f^{-1}(V)$ is a neighborhood of a.
- (b)$\Rightarrow$(a): We verify the criterion: Given any $\epsilon > 0$, we seek a $\delta > 0$. Since $V = (f(a) - \epsilon, f(a) + \epsilon)$ is a neighborhood of $f(a)$, by hypothesis $f^{-1}(V)$ is a neighborhood of a. So there exists a $\delta > 0$, such that $(a - \delta, a + \delta) \subseteq f^{-1}(V)$. This inclusion means that $x \in (a - \delta, a + \delta) \Rightarrow f(x) \in V$, i.e., $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$.

Subsection 3

Algebra of Continuity

Algebraic Combinations of Continuous Functions

Theorem

Suppose $a \in S \subseteq \mathbb{R}$ and $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$ and c be any real number. If f and g are continuous at a, then so are the functions $f + g$, fg and cf.

- The functions in question are defined on S by the formulas $(f+g)(x) = f(x) + g(x)$, $(fg)(x) = f(x)g(x)$ and $(cf)(x) = cf(x)$. If $x_n \in S$ and $x_n \to a$, then $f(x_n) \to f(a)$ and $g(x_n) \to g(a)$, by the assumptions on f and g. Therefore

  $$(f+g)(x_n) = f(x_n) + g(x_n) \to f(a) + g(a) = (f+g)(a).$$

  This shows that $f + g$ is continuous at a.

  The proofs for fg and cf are similar.
- Note cf is the special case of fg when g is the constant function equal to c.

Polynomial Functions and Quotient Rule

Corollary (Polynomial Functions)

Every polynomial function $p : \mathbb{R} \to \mathbb{R}$ is continuous at every point of $\mathbb{R}$.

- Say $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_rx^r$, $x \in \mathbb{R}$, where the coefficients $a_0, a_1, \ldots, a_r$ are fixed real numbers. If $u : \mathbb{R} \to \mathbb{R}$ is the identity function $u(x) = x$, then p is a linear combination of powers of u: $p = a_0 \cdot 1 + a_1u + a_2u^2 + \cdots + a_ru^r$. Since u is continuous, so are its powers. Therefore, so is any linear combination of them.

Theorem (Quotient Rule)

Assume that f and g are continuous at a and that g is not zero at any point of S. Then $\frac{f}{g}$ is also continuous at a.

- $\frac{f}{g} : S \to \mathbb{R}$ is given by $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$, $x \in S$. Note that $\frac{f}{g}$ is defined on the subset $T = \{x \in S : g(x) \neq 0\}$ of S. In the present case, $T = S$. If $x_n \in S$ and $x_n \to a$, then $(\frac{f}{g})(x_n) = \frac{f(x_n)}{g(x_n)} \to \frac{f(a)}{g(a)} = (\frac{f}{g})(a)$.

Rational Functions

- Suppose $p : \mathbb{R} \to \mathbb{R}$ and $q : \mathbb{R} \to \mathbb{R}$ are polynomial functions, q not the zero polynomial. Let $F = \{x \in \mathbb{R} : q(x) = 0\}$, which is a finite set (possibly empty):

  By the factor theorem of elementary algebra, $q(c) = 0$ if and only if the linear polynomial $x - c$ is a factor of q, i.e., $q(x) = (x - c)q_1(x)$, for a suitable polynomial q_1 and for all $x \in \mathbb{R}$. Thus, every root of q splits off a linear factor, so the degree of q puts an upper bound on the number of roots.

  Let $r = \frac{p}{q}$ be the quotient function (called a **rational function**), defined on the set $S = \mathbb{R} - F$ by the formula $r(x) = \frac{p(x)}{q(x)}, x \in S$. If $f = p \restriction S$ and $g = q \restriction S$ are the restrictions of p and q to S, it is clear that f and g are continuous at every point of S, whence so is $r = \frac{f}{g}$.

Subsection 4

Continuous Functions

Continuous Functions

Definition (Continuous Function)

Suppose $f : S \to \mathbb{R}$, where S is a subset of $\mathbb{R}$. f is said to be a **continuous function** (or **continuous mapping**) if it is continuous at every $a \in S$.

- Example: The polynomial and rational functions discussed in the preceding section are important examples of continuous functions.
- Example: An example not covered by these is the function $x \mapsto |x|$.
- Example: The function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is continuous.

  For a sequence (x_n) in $[0, \infty)$, (x_n) is null if and only if $(\sqrt{x_n})$ is null. This assures continuity at 0.

  If $x > 0$ and $x_n \to x$, we substitute x_n for y in the inequality $|y - x| = |(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})| \geq |\sqrt{y} - \sqrt{x}|\sqrt{x}$, which holds for all $x > 0$, $y > 0$.

Algebraic Properties and Mapping Open/Closed Sets

Theorem

Let S be a nonempty subset of $\mathbb{R}$.

- If $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$ are continuous functions and c is any real number, then the functions $f + g$, fg and cf are also continuous.
- If, moreover, g is not zero at any point of S, then $\frac{f}{g}$ is also continuous.

- The proof follows by the theorems on continuity at a point.

Theorem

For a function $f : \mathbb{R} \to \mathbb{R}$, the following conditions are equivalent:

(a) f is continuous;

(b) U open $\Rightarrow f^{-1}(U)$ open;

(c) A closed $\Rightarrow f^{-1}(A)$ closed.

Proof of the Theorem

- (a)⇒(c): Suppose f is continuous at every point of $\mathbb{R}$. Let A be a closed subset of $\mathbb{R}$. Assuming $x_n \in f^{-1}(A)$ and $x_n \to x \in \mathbb{R}$, we have to show that $x \in f^{-1}(A)$. Since f is continuous, $f(x_n) \to f(x)$. But $f(x_n) \in A$ and A is closed, so $f(x) \in A$, i.e., $x \in f^{-1}(A)$.
- (c)⇒(b): If U is an open set, its complement U^c is closed. Thus $f^{-1}(U^c)$ is closed by (c). Then $f^{-1}(U^c) = (f^{-1}(U))^c$ shows that $f^{-1}(U)$ is the complement of a closed set, whence $f^{-1}(U)$ is open.
- (b)⇒(a): Given any $a \in \mathbb{R}$, we have to show that f is continuous at a. Let $\epsilon > 0$. We seek $\delta > 0$, such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$, i.e., $x \in (a - \delta, a + \delta) \Rightarrow f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$. Equivalently, $(a - \delta, a + \delta) \subseteq f^{-1}((f(a) - \epsilon, f(a) + \epsilon))$. The interval $U = (f(a) - \epsilon, f(a) + \epsilon)$ is an open set, so $f^{-1}(U)$ is open by (b). Obviously, $f(a) \in U$, so $a \in f^{-1}(U)$. Thus, $f^{-1}(U)$ is a neighborhood of a. Therefore, there exists $\delta > 0$, such that $(a - \delta, a + \delta) \subseteq f^{-1}(U)$.

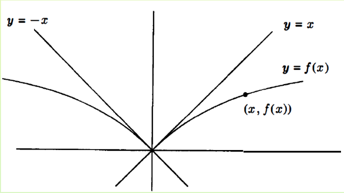
Subsection 5

One-Sided Continuity

One-Sided Behavior

- In discussing functions f defined on an interval $[a, b]$, behavior at the endpoints requires some special treatment (for example, the point a can only be approached from the right).
- Another reason for considering "one-sided" behavior is that "two-sided" behavior can be discussed by breaking it up into "left-behavior" and "right-behavior".
- Example: The function f whose graph is shown on the right has a tangent line problem at the origin:

  As x approaches 0 from the right, the slope of the chord joining $(0, 0)$ and $(x, f(x))$ approaches 1. For x approaching 0 from the left, the slope of the chord approaches -1. The function fails to have a well-defined "slope" at $(0, 0)$ because the "left slope" and "right slope" are different.



Right and Left Neighborhoods

Definition (Right and Left Neighborhood)

Let $a \in N \subseteq \mathbb{R}$.

- We say that N is a **right neighborhood** of a if there exists an $r > 0$, such that $[a, a + r] \subseteq N$.
- If there exists an $s > 0$, such that $[a - s, a] \subseteq N$, then N is called a **left neighborhood** of a.

Thus N is a neighborhood of a in the previous sense if and only if it is both a left neighborhood and a right neighborhood of a.

- Example: If $a < b$, then $[a, b]$ is a right neighborhood of a, a left neighborhood of b, and a neighborhood of each point $x \in (a, b)$.
- If M and N are right neighborhoods of a, then so is $M \cap N$.
- If M is a right neighborhood of a and $M \subseteq N$, then N is also a right neighborhood of a.

Right and Left Continuity

Definition (Right and Left Continuous)

Let $a \in S \subseteq \mathbb{R}$, $f : S \to \mathbb{R}$. We say that f is **right continuous** at a if:

(i) S is a right neighborhood of a;

(ii) If (x_n) is a sequence in S, such that $x_n > a$ and $x_n \to a$, then $f(x_n) \to f(a)$, in symbols,

$$x_n \in S,\ x_n > a,\ x_n \to a \ \Rightarrow \ f(x_n) \to f(a).$$

- **Left continuity** is defined dually, i.e., with "right" replaced by "left" and "$x_n > a$" by "$x_n < a$".
- Suppose $a \in S \subseteq \mathbb{R}$, $f : S \to \mathbb{R}$. Let $T = -S = \{-x : x \in S\}$, and define $g : T \to \mathbb{R}$ by $g(x) = f(-x)$. Then f is left continuous at a if and only if g is right continuous at $-a$.

Characterization of Right Continuity

Theorem

Suppose $f : S \to \mathbb{R}$ and S is a right neighborhood of a. The following conditions on f are equivalent:

(a) f is right continuous at a;

(b) For every $\epsilon > 0$, there exists a $\delta > 0$, such that $a < x < a + \delta \Rightarrow |f(x) - f(a)| < \epsilon$;

(c) V a neighborhood of $f(a) \Rightarrow f^{-1}(V)$ is a right neighborhood of a.

- (a)$\Rightarrow$(b): We prove $\neg$(b)$\Rightarrow \neg$(a). Condition (b) says that for every $\epsilon > 0$, there exists a "successful" $\delta > 0$. Its negation asserts that there exists an $\epsilon_0 > 0$, for which every $\delta > 0$ "fails". In particular, for each $n \in \mathbb{P}$, $\delta = \frac{1}{n}$ fails, so there exists a point $x_n \in S$, with $a < x_n < a + \frac{1}{n}$, such that $|f(x_n) - f(a)| \geq \epsilon_0$. Then $x_n > a$ and $x_n \to a$ but $(f(x_n))$ does not converge to $f(a)$, so f is not right continuous at a.

Characterization of Right Continuity (Cont'd)

- (b)⇒(c): If V is a neighborhood of $f(a)$, there is an $\epsilon > 0$, such that $(f(a) - \epsilon, f(a) + \epsilon) \subseteq V$. Choose $\delta > 0$ as in (b). By the implication in (b), $f((a, a + \delta)) \subseteq (f(a) - \epsilon, f(a) + \epsilon) \subseteq V$. Also $f(a) \in V$, so $f([a, a + \delta)) \subseteq V$. Thus, $[a, a + \delta) \subseteq f^{-1}(V)$, whence $f^{-1}(V)$ is a right neighborhood of a.
- (c)⇒(a): Assuming $x_n \in S$, $x_n > a$, $x_n \to a$, we have to show that $f(x_n) \to f(a)$. Let $\epsilon > 0$. We must show that $|f(x_n) - f(a)| < \epsilon$ ultimately. Since $V = (f(a) - \epsilon, f(a) + \epsilon)$ is a neighborhood of $f(a)$, by hypothesis $f^{-1}(V)$ is a right neighborhood of a, so there is a $\delta > 0$, such that $[a, a + \delta) \subseteq f^{-1}(V)$. Ultimately $x_n \in [a, a + \delta)$, whence $f(x_n) \in V$, i.e., $|f(x_n) - f(a)| < \epsilon$.

Continuity and One-Sided Continuity

Theorem

If $a \in S \subseteq \mathbb{R}$ and $f : S \to \mathbb{R}$, the following conditions are equivalent:

(a) f is both left and right continuous at a;

(b) S is a neighborhood of a and f is continuous at a.

- (a)⇒(b): By the definition of "one-sided continuity", S is both a left and right neighborhood of a, hence is a neighborhood of a. If V is a neighborhood of $f(a)$, then $f^{-1}(V)$ is both a left neighborhood and a right neighborhood of a. So it is a neighborhood of a. In particular, if $\epsilon > 0$ and $V = (f(a) - \epsilon, f(a) + \epsilon)$, then there exists a $\delta > 0$, such that $(a - \delta, a + \delta) \subseteq f^{-1}(V)$, i.e., $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$. Thus, f is continuous at a.
- (b)⇒(a): By assumption, S is a neighborhood of a, and $x_n \in S$, $x_n \to a$ imply $f(x_n) \to f(a)$. In particular, S is a right neighborhood of a and $x_n \in S$, $x_n > a$, $x_n \to a$ imply $f(x_n) \to f(a)$. Thus, f is right continuous at a. Similarly, f is left continuous at a.

Continuity and One-Sided Continuity on $[a, b]$

Corollary

If $f : [a, b] \to \mathbb{R}$, $a < b$, then the following conditions are equivalent:

(a) f is continuous;

(b) f is right continuous at a, left continuous at b, and both left and right continuous at each point of the open interval (a, b).

- In Part (b):
  - The first statement means that f is continuous at a;
  - The second statement means that f is continuous at b;
  - The third statement mean that f is continuous at every point of (a, b).

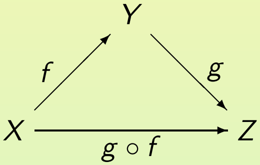
  Thus, (b) holds if and only if f is continuous (on $[a, b]$).

Subsection 6

Composition

Composition of Functions

- The composition of functions - one function followed by another - is familiar from calculus.
- We are given functions $f : X \to Y$ and $g : Y \to Z$, where the final set for f is the initial set for g:



For $x \in X$, the correspondence

$$x \stackrel{f}{\mapsto} f(x) \stackrel{g}{\mapsto} g(f(x))$$

produces a function $X \to Z$, called the **composite** of g and f and denoted $g \circ f$ (verbalized "g circle f"). The defining formula for $g \circ f : X \to Z$ is

$$(g \circ f)(x) = g(f(x)), \text{ for all } x \in X.$$

Examples

- The simplest situation of all is where $X = Y = Z$. E.g., if $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are the functions $f(x) = x^2 + 5$ and $g(y) = y^3$, then

$$(g \circ f)(x) = g(f(x)) = (f(x))^3 = (x^2 + 5)^3.$$

  Thus $h = g \circ f$ is the function $h(x) = (x^2 + 5)^3$, for all $x \in \mathbb{R}$.

- More general is the case when $f : X \to Y$, $g : U \to V$ and $f(X) \subseteq U$. If $x \in X$, then $f(x) \in f(X) \subseteq U$, so $g(f(x))$ makes sense. Thus, $g \circ f : X \to V$ can be defined by the same formula

$$(g \circ f)(x) = g(f(x)), \text{ for all } x \in X.$$

Composition and Continuity

Theorem (Composition and Continuity)

Suppose $f : S \to \mathbb{R}$, $g : T \to \mathbb{R}$, where S and T are subsets of $\mathbb{R}$, such that $f(S) \subseteq T$, and let $a \in S$. If f is continuous at a, and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

- If $x_n \in S$, $x_n \to a$ then $f(x_n) \to f(a)$ (because f is continuous at a). Thus, $g(f(x_n)) \to g(f(a))$ (because g is continuous at $f(a)$), i.e., $(g \circ f)(x_n) \to (g \circ f)(a)$.

Corollary

Suppose $f : S \to \mathbb{R}$, $g : T \to \mathbb{R}$, where S and T are subsets of $\mathbb{R}$, such that $f(S) \subseteq T$. If f and g are continuous functions, then so is $g \circ f$.

- Note, it suffices that f be continuous and that g be continuous at every point of the range $f(S)$ of f.

Composition: Most General Case

- Given two functions $f : X \to Y$ and $g : U \to V$, consider
$$A = \{x \in X : f(x) \in U\}.$$
A function $g \circ f : A \to V$ can be defined by the formula
$$(g \circ f)(x) = g(f(x)), \text{ for all } x \in A.$$
- In principle, one can compose any two functions, but the result may be disappointing.
- Example: If f and g are the functions
$$\begin{array}{ll} f : \mathbb{R} \to \mathbb{R}, & f(x) = 0, \text{ for all } x \in \mathbb{R}, \\ g : \mathbb{R} - \{0\} \to \mathbb{R}, & g(x) = \frac{1}{x}, \text{ for all } x \neq 0. \end{array}$$
The formula for $g \circ f$ is $(g \circ f)(x) = g(f(x)) = \frac{1}{f(x)} = \frac{1}{0}$. However, the domain of $g \circ f$ is the empty set.