Introduction to Real Analysis

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LSSU Math 421
1 Continuous Functions on an Interval

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Subsection 1

Intermediate Value Theorem
Zero Values in a Closed Interval

**Lemma**

If \( f : [a, b] \to \mathbb{R} \) is a continuous function such that \( f(a)f(b) < 0 \), then there exists a point \( c \in (a, b) \) such that \( f(c) = 0 \).

We can suppose \( f(a) > 0 \) and \( f(b) < 0 \) (otherwise consider \(-f\)). The idea is that there are points \( x \) in \([a, b]\) (for example, \( x = a \)) for which \( f(x) \geq 0 \), and \( b \) is not one of them. The “last” such point \( x \) is a likely candidate for \( c \).

The set \( A = \{ x \in [a, b] : f(x) \geq 0 \} \) is nonempty (because \( a \in A \)) and bounded. It is also closed: If \( x_n \in A \) and \( x_n \to x \), then \( x \in [a, b] \) and \( f(x_n) \to f(x) \) by the continuity of \( f \). Since \( f(x_n) \geq 0 \) for all \( n \), \( f(x) \geq 0 \). Thus \( x \in A \).

Let \( c \) be the largest element of \( A \). In particular, \( f(c) \geq 0 \), whence \( c \neq b \), and, thus, \( c < b \). If \( c < x < b \), then \( x \not\in A \) (all elements of \( A \) are \( \leq c \)), so \( f(x) < 0 \). Choose a sequence \( (x_n) \), with \( c < x_n < b \) and \( x_n \to c \). Then \( f(c) = \lim f(x_n) \leq 0 \), and, therefore, \( f(c) = 0 \).
Intermediate Value Theorem

**Theorem.** (Intermediate Value Theorem)

If \( I \) is an interval in \( \mathbb{R} \) and \( f : I \to \mathbb{R} \) is continuous, then \( f(I) \) is also an interval.

Assuming \( r, s \in f(I), \ r < s \), it will suffice to show that \([r, s] \subseteq f(I)\).

Let \( r < k < s \). We seek \( c \in I \), such that \( f(c) = k \). (The theorem says that, if \( r \) and \( s \) are values of \( f \), then so is every number between \( r \) and \( s \).) By assumption, \( r = f(a) \) and \( s = f(b) \), for suitable points \( a, b \) of \( I \). Since \( r \neq s \), also \( a \neq b \). Let \( J \) be the closed interval with endpoints \( a \) and \( b \). Since \( I \) is an interval, \( J \subseteq I \). Define \( g : J \to \mathbb{R} \) by the formula \( g(x) = f(x) - k, \ x \in J \). Since \( f \) is continuous, so is \( g \).

Note that \( g(a) = f(a) - k = r - k < 0 \), \( g(b) = f(b) - k = s - k > 0 \).

By the lemma, there exists a point \( c \in J \), such that \( g(c) = 0 \). Thus \( c \in I \) and \( f(c) - k = 0 \), whence \( k = f(c) \in f(I) \).
Consequences of the Theorem

Corollary

Let $I$ be an interval in $\mathbb{R}$ and $f : I \rightarrow \mathbb{R}$ continuous on $I$. If $f$ is not zero at any point of $I$, then either $f(x) > 0$, for all $x \in I$, or $f(x) < 0$, for all $x \in I$.

The alternative is that $f(a) < 0$ and $f(b) > 0$ for suitable points $a, b$ of $I$. Then $0 \in f(I)$ by the theorem, contrary to the hypothesis on $f$.

Corollary

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and $I$ is any interval in $\mathbb{R}$, then $f(I)$ is also an interval.

Apply the theorem to $f \upharpoonright_I : I \rightarrow \mathbb{R}$ (the restriction of $f$ to $I$).
Subsection 2

$n$-th Roots
Bijectivity of $f(x) = x^n$ on $[0, +\infty)$

- The Dedekind cut technique used to construct square roots, can be adapted to higher-order roots, but the Intermediate Value Theorem provides an efficient shortcut:

**Theorem**

If $n$ is a positive integer and $f : [0, +\infty) \to [0, +\infty)$ is the function defined by $f(x) = x^n$, then $f$ is bijective.

- We have proved, based on the order axioms of an ordered field, that $f(a) = f(b) \Rightarrow a = b$. So $f$ is injective. Write $I = [0, +\infty)$. Then $f : I \to I$ and it remains to show that $f$ is surjective, i.e., that $f(I) = I$:

  Since $f$ is continuous, its range $f(I)$ is an interval. From $f(0) = 0$ we have $0 \in f(I)$. An easy induction argument shows that $f(k) \geq k$, for every positive integer $k$. It follows that $[0, k] \subseteq f(I)$ for all $k \in \mathbb{P}$, whence (Archimedes) $[0, +\infty) \subseteq f(I)$. Thus, $I \subseteq f(I) \subseteq I$. 
**n-th Roots**

**Definition (n-th Root)**

If $x \geq 0$ and $n$ is a positive integer, the unique $y \geq 0$ such that $y^n = x$ is called the $n$-th root of $x$, written $\sqrt[n]{x}$ (or $x^{1/n}$).

**Corollary**

If $n$ is an odd positive integer and $g : \mathbb{R} \to \mathbb{R}$ is the function defined by $g(x) = x^n$, then $g$ is bijective.

- We know that $g(\mathbb{R})$ is an interval, and $g(\mathbb{R})$ contains $[0, +\infty)$, by the preceding theorem. Since $g(-x) = -g(x)$ (because $n$ is odd), we get that $g(\mathbb{R})$ also contains $(-\infty, 0]$, and, thus, is equal to $\mathbb{R}$. Injectivity follows from the theorem since $x$ and $g(x)$ have the same sign.

**Definition (n-th Root)**

If $x \in \mathbb{R}$ and $n \in \mathbb{P}$ is odd, the unique real number $y$ such that $y^n = x$ is called the $n$-th root of $x$, written $\sqrt[n]{x}$ (or $x^{1/n}$). Of course, when $x \geq 0$, this is consistent with the preceding definition.
Continuous Functions on a Closed Interval

Subsection 3

Continuous Functions on a Closed Interval
Theorem

If \( f : [a, b] \rightarrow \mathbb{R} \) is continuous, then the range of \( f \) is a closed interval.

- Write \( I = [a, b] \). We know that \( f(I) \) is an interval. We need only show that \( f(I) \) is: (i) bounded; (ii) a closed set.
  
  (i) The claim is that \( \{ |f(x)| : x \in I \} \) is bounded above. Assume to the contrary. For each positive integer \( n \), choose \( x_n \in I \), such that \( |f(x_n)| > n \). It is clear that no subsequence of \( (f(x_n)) \) is bounded. However, \( (x_n) \) is bounded, so it has a convergent subsequence (Bolzano-Weierstraß), say \( x_{n_k} \rightarrow x \). Then, since \( I \) is closed, \( x \in I \), and \( f(x_{n_k}) \rightarrow f(x) \). In particular, \( (f(x_{n_k})) \) is bounded, a contradiction.

(ii) Suppose \( y_n \in f(I), y_n \rightarrow y \). We have to show that \( y \in f(I) \). Say \( y_n = f(x_n), x_n \in I \). Passing to a subsequence, we can suppose \( x_n \rightarrow x \in \mathbb{R} \). As in the proof of (i), \( x \in I \) and \( f(x_n) \rightarrow f(x) \), i.e., \( y_n \rightarrow f(x) \). But \( y_n \rightarrow y \), so \( y = f(x) \in f(I) \).
Continuous Functions on a Closed Interval

Consequences of Closed Image

Corollary (Weierstraß)

If \( f : [a, b] \rightarrow \mathbb{R} \) is continuous, then \( f \) takes on a smallest value and a largest value.

- By the theorem, \( f([a, b]) = [m, M] \), for suitable \( m \) and \( M \). Thus, if \( m = f(c) \) and \( M = f(d) \), then \( f(c) \leq f(x) \leq f(d) \) for all \( x \in [a, b] \).
- The continuous function \( (0, 1] \rightarrow \mathbb{R} \) defined by \( x \mapsto x \) is \( > 0 \) at every point of its domain, but it has values as near to 0 as we like. On a closed interval, that cannot happen:

Corollary

If \( f : [a, b] \rightarrow \mathbb{R} \) is continuous and \( f(x) > 0 \) for all \( x \in [a, b] \), then there exists an \( m > 0 \) such that \( f(x) \geq m \), for all \( x \in [a, b] \).

- By the theorem, \( f([a, b]) = [m, M] \), for some \( m \) and \( M \). If \( m = f(c) \) and \( M = f(d) \), we have \( f(x) \geq m = f(c) > 0 \), for all \( x \in [a, b] \).
Bounded and Unbounded Functions

Definition (Bounded and Unbounded Functions)

A real-valued function \( f : X \to \mathbb{R} \) is said to be **bounded** if its range \( f(X) \) is a bounded subset of \( \mathbb{R} \), i.e., if there exists a real number \( M > 0 \) such that \( |f(x)| \leq M \), for all \( x \in X \).

\( f \) is said to be **unbounded** if it is not bounded.

- **Example:** Every continuous real-valued function on a closed interval is bounded.

- **Example:** The continuous function \( f : (0, 1] \to \mathbb{R} \) defined by \( f(x) = \frac{1}{x} \) is unbounded.
Subsection 4

Monotonic Continuous Functions
Increasing and Decreasing Functions

Definition (Increasing and Decreasing Functions)

Let $S \subseteq \mathbb{R}$ (in the most important examples, $S$ is an interval). A function $f : S \rightarrow \mathbb{R}$ is said to be:

- **increasing**, if $x < y \Rightarrow f(x) \leq f(y)$,
- **strictly increasing**, if $x < y \Rightarrow f(x) < f(y)$,
- **decreasing**, if $x < y \Rightarrow f(x) \geq f(y)$,
- **strictly decreasing**, if $x < y \Rightarrow f(x) > f(y)$,

where it is understood that $x$ and $y$ are in the domain $S$ of $f$.

If $f$ is either increasing or decreasing, it is said to be **monotone**.

A function is **strictly monotone** if it is strictly increasing or strictly decreasing.
Examples

- Let $n$ be a positive integer. The function $f : [0, +\infty) \to [0, +\infty)$, $f(x) = x^n$ is strictly increasing.
- The function $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^{2n-1}$ is also strictly increasing.
- The function $\mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x^2$ is neither increasing nor decreasing.
- The function $(0, +\infty) \to (0, +\infty)$ defined by $x \mapsto \frac{1}{x}$ is strictly decreasing.
- Every constant function is increasing and decreasing, but not strictly. Conversely, if a function is both increasing and decreasing, then it is a constant function.
- The functions $\log : (0, +\infty) \to \mathbb{R}$ and $\exp : \mathbb{R} \to (0, +\infty)$ are both strictly increasing (here the base is $e$).
If $f : S \to \mathbb{R}$ is strictly monotone, it is obvious that $f$ is injective.

**Theorem**

If $f : [a, b] \to \mathbb{R}$ is continuous and injective, then $f$ is strictly monotone.

If $a = b$, there is nothing to prove. If $a < b$, then $f(a) \neq f(b)$, by injectivity. We suppose $f(a) < f(b)$ (if not, consider $-f$). We show that $f$ is then strictly increasing.

**Claim:** If $a < x < b$ then $f(a) < f(x) < f(b)$.

Assume to the contrary that $f(x) \leq f(a)$ or $f(x) \geq f(b)$, i.e., by injectivity, $f(x) < f(a)$ or $f(x) > f(b)$.

- In the first case, $f(x) < f(a) < f(b)$. Thus $k = f(a)$ is intermediate to the values of $f \upharpoonright_{[x,b]}$ at the endpoints of $[x,b]$. The IVT yields a point $t \in (x, b)$ with $f(t) = k = f(a)$, contrary to injectivity.
- In the second case, $f(a) < f(b) < f(x)$. An application of the IVT to $f \upharpoonright_{[a,x]}$ yields a point $t \in (a, x)$, with $f(t) = f(b)$, again contradicting injectivity.
Assuming now that $a < c < d < b$, we have to show that $f(c) < f(d)$.

- If $a = c$ and $d = b$, there is nothing to prove.
- If $a = c < d < b$, then $f(c) < f(d)$ by the claim.
- If $a < c < d = b$, we proceed similarly.
- If $a < c < d < b$, then $f(a) < f(c) < f(b)$ by the claim applied to $a < c < b$. But then $f(c) < f(d) < f(b)$ by the claim applied to $c < d < b$ and the function $f |_{[c,b]}$. 
Corollary

If $I$ is an interval and $f : I \to \mathbb{R}$ is continuous and injective, then $f$ is strictly monotone.

If $I$ is a singleton there is nothing to prove. Otherwise, let $r, s \in I$, with $r < s$. Since $f$ is injective, $f(r) \neq f(s)$. We can suppose $f(r) < f(s)$ (if not, consider $-f$).

Claim: We assert that $f$ is strictly increasing.

Given $c, d \in I$, $c < d$, we must show that $f(c) < f(d)$. Let $J = [a, b]$ be a closed subinterval of $I$ that contains all four points $r, s, c, d$. For example, $a = \min \{r, c\}$, $b = \max \{s, d\}$ will do. From the theorem, we know that $f \upharpoonright J$ is either strictly increasing or strictly decreasing. Since $f(r) < f(s)$, it must be the former, whence $f(c) < f(d)$.
Subsection 5

Inverse Function Theorem
Lemma

If \( f : [a, b] \rightarrow [c, d] \) is bijective and continuous then the inverse function \( f^{-1} : [c, d] \rightarrow [a, b] \) is also continuous.

Assuming \( y_n \rightarrow y \) in \([c, d]\), we must show that \( f^{-1}(y_n) \rightarrow f^{-1}(y)\). Let \( x_n = f^{-1}(y_n) \), \( x = f^{-1}(y) \) and assume to the contrary that \((x_n)\) does not converge to \(x\). Then, there exists an \( \epsilon > 0 \), such that \(|x_n - x| \geq \epsilon\) frequently. Passing to a subsequence, we can suppose that \(|x_n - x| \geq \epsilon\), for all \(n\). Since \((x_n)\) is bounded, some subsequence is convergent (Bolzano-Weierstraß), say \(x_{n_k} \rightarrow t\). Then \(t \in [a, b]\) and \(f(x_{n_k}) \rightarrow f(t)\) by continuity. But \(f(x_{n_k}) = y_{n_k} \rightarrow y\), so \(y = f(t)\), \(t = f^{-1}(y) = x\). Thus, \(x_{n_k} \rightarrow x\), contrary to \(|x_{n_k} - x| \geq \epsilon\), for all \(k\).
The Inverse Function Theorem

Theorem (Inverse Function Theorem)

Let \( I \) be an interval in \( \mathbb{R} \), \( f : I \to \mathbb{R} \) continuous and injective. Let \( J = f(I) \) (an interval), so that \( f : I \to J \) is continuous and bijective. Then \( f^{-1} : J \to I \) is also continuous.

- By the preceding Corollary, we know that \( f \) is monotone. We can suppose that \( f \) is increasing (if not, consider \(-f\)). Suppose \( y_n \to y \) in \( J \). Writing \( x_n = f^{-1}(y_n) \), \( x = f^{-1}(y) \), we have to show that \( x_n \to x \).

The set \( A = \{y\} \cup \{y_n : n \in \mathbb{P}\} \) is compact. So it has a smallest element \( c \) and a largest element \( d \). Then \( A \subseteq [c, d] \subseteq J \). Say \( c = f(a) \), \( d = f(b) \). Since \( f \) is increasing, \( a \leq b \) and \( f([a, b]) = [f(a), f(b)] = [c, d] \), so \( x_n \to x \) follows from applying the lemma to the restriction \( f \restriction_{[a, b]} : [a, b] \to [c, d] \).

- **Example:** If \( n \in \mathbb{P} \) and \( I = [0, +\infty) \), then the function \( I \to I \), \( x \mapsto n\sqrt{x} \) is continuous: It is the inverse of a continuous bijection. If \( n \) is odd, then the function \( \mathbb{R} \to \mathbb{R} \), \( x \mapsto n\sqrt{x} \) is continuous.
Subsection 6

Uniform Continuity
Uniform Continuity Theorem

**Theorem**
Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous. Given any $\epsilon > 0$, there exists a $\delta > 0$, such that $x, y \in [a, b]$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

- This is not just a restatement of the definition of continuity; there is a subtle difference:
  - To say that $f : S \rightarrow \mathbb{R}$ is continuous means that for each $y \in S$ and $\epsilon > 0$, there is a $\delta > 0$ (depending in general on both $y$ and $\epsilon$) such that $x \in S$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.
  - The theorem ensures that when the domain of $f$ is a closed interval, the choice of $\delta$ can be made to depend on $\epsilon$ alone. Informally speaking, $\delta$ works “uniformly well” at all points of the domain.
Proof of the Uniform Continuity Theorem

Theorem

Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Given any $\epsilon > 0$, there exists a $\delta > 0$, such that $x, y \in [a, b]$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

Let $\epsilon > 0$. We seek a $\delta > 0$, for which the stated implication is valid. Assume to the contrary that no such $\delta$ exists. In particular, for each $n \in \mathbb{N}$, the choice $\delta = \frac{1}{n}$ fails to validate the implication, so there is a pair of points $x_n, y_n$ in $[a, b]$, such that $|x_n - y_n| < \frac{1}{n}$, but $|f(x_n) - f(y_n)| \geq \epsilon$. For a suitable subsequence, $x_{n_k} \to x \in [a, b]$ (Bolzano-Weierstraß). Then $y_{n_k} = x_{n_k} - (x_{n_k} - y_{n_k})$ and $x_{n_k} - y_{n_k} \to 0$ show that also $y_{n_k} \to x$. By continuity, $f(x_{n_k}) \to f(x)$ and $f(y_{n_k}) \to f(x)$, so $f(x_{n_k}) - f(y_{n_k}) \to 0$, contrary to $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon$. 
Uniformly Continuous Functions

**Definition (Uniformly Continuous Function)**

Let $S$ be a subset of $\mathbb{R}$. A function $f : S \to \mathbb{R}$ is said to be **uniformly continuous** (on $S$) if, for every $\epsilon > 0$, there exists a $\delta > 0$, such that $x, y \in S$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

- Uniform continuity implies continuity, but the converse is false:
- **Example**: The function $f : (0, 2] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is continuous but not uniformly continuous. Looking at the graph of $f$, we see that “the nearer $y$ is to 0, the steeper the “slope” of the graph”. This suggests that for a particular $\epsilon$, the nearer $y$ is to 0, the smaller $\delta$ will have to be taken.

**Formal Argument**: Assume $f$ is uniformly continuous. In particular, for $\epsilon = 1$, there is a $\delta > 0$ (which we can suppose to be $< 1$) for which $x, y \in (0, 2], |x - y| \leq \delta \Rightarrow \left| \frac{1}{x} - \frac{1}{y} \right| < 1$, i.e., $|x - y| < xy$.

Thus, $x, y \in (0, 2], |x - y| = \delta \Rightarrow \delta < xy$. If $y_n = \frac{1}{n}$ and $x_n = \delta + \frac{1}{n}$, then $|x_n - y_n| = \delta < x_n y_n \to 0$, which is absurd.