Introduction to Real Analysis

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Subsection 1

Intermediate Value Theorem

Zero Values in a Closed Interval

Lemma

If $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f(a)f(b) < 0$, then there exists a point $c \in (a, b)$ such that $f(c) = 0$.

- We can suppose $f(a) > 0$ and $f(b) < 0$ (otherwise consider $-f$). The idea is that there are points x in $[a, b]$ (for example, $x = a$) for which $f(x) \geq 0$, and b is not one of them. The "last" such point x is a likely candidate for c.

  The set $A = \{x \in [a, b] : f(x) \geq 0\}$ is nonempty (because $a \in A$) and bounded. It is also closed: If $x_n \in A$ and $x_n \to x$, then $x \in [a, b]$ and $f(x_n) \to f(x)$ by the continuity of f. Since $f(x_n) \geq 0$ for all n, $f(x) \geq 0$. Thus $x \in A$.

  Let c be the largest element of A. In particular, $f(c) \geq 0$, whence $c \neq b$, and, thus, $c < b$. If $c < x < b$, then $x \notin A$ (all elements of A are $\leq c$), so $f(x) < 0$. Choose a sequence (x_n), with $c < x_n < b$ and $x_n \to c$. Then $f(c) = \lim f(x_n) \leq 0$, and, therefore, $f(c) = 0$.

Intermediate Value Theorem

Theorem. (Intermediate Value Theorem)

If I is an internal in $\mathbb{R}$ and $f : I \to \mathbb{R}$ is continuous, then $f(I)$ is also an interval.

- Assuming $r, s \in f(I)$, $r < s$, it will suffice to show that $[r, s] \subseteq f(I)$. Let $r < k < s$. We seek $c \in I$, such that $f(c) = k$. (The theorem says that, if r and s are values of f, then so is every number between r and s.) By assumption, $r = f(a)$ and $s = f(b)$, for suitable points a, b of I. Since $r \neq s$, also $a \neq b$. Let J be the closed interval with endpoints a and b. Since I is an interval, $J \subseteq I$. Define $g : J \to \mathbb{R}$ by the formula $g(x) = f(x) - k$, $x \in J$. Since f is continuous, so is g. Note that $g(a) = f(a) - k = r - k < 0$, $g(b) = f(b) - k = s - k > 0$. By the lemma, there exists a point $c \in J$, such that $g(c) = 0$. Thus $c \in I$ and $f(c) - k = 0$, whence $k = f(c) \in f(I)$.

Consequences of the Theorem

Corollary

Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ continuous on I. If f is not zero at any point of I, then either $f(x) > 0$, for all $x \in I$, or $f(x) < 0$, for all $x \in I$.

- The alternative is that $f(a) < 0$ and $f(b) > 0$ for suitable points a, b of I. Then $0 \in f(I)$ by the theorem, contrary to the hypothesis on f.

Corollary

If $f : \mathbb{R} \to \mathbb{R}$ is continuous and I is any interval in $\mathbb{R}$, then $f(I)$ is also an interval.

- Apply the theorem to $f \restriction_I : I \to \mathbb{R}$ (the restriction of f to I).

Subsection 2

n-th Roots

Bijectivity of $f(x) = x^n$ on $[0, +\infty)$

- The Dedekind cut technique used to construct square roots, can be adapted to higher-order roots, but the Intermediate Value Theorem provides an efficient shortcut:

Theorem

If n is a positive integer and $f : [0, +\infty) \to [0, +\infty)$ is the function defined by $f(x) = x^n$, then f is bijective.

- We have proved, based on the order axioms of an ordered field, that $f(a) = f(b) \Rightarrow a = b$. So f is injective. Write $I = [0, +\infty)$. Then $f : I \to I$ and it remains to show that f is surjective, i.e., that $f(I) = I$:

  Since f is continuous, its range $f(I)$ is an interval. From $f(0) = 0$ we have $0 \in f(I)$. An easy induction argument shows that $f(k) \geq k$, for every positive integer k. It follows that $[0, k] \subseteq f(I)$ for all $k \in \mathbb{P}$, whence (Archimedes) $[0, +\infty) \subseteq f(I)$. Thus, $I \subseteq f(I) \subseteq I$.

n-th Roots

Definition (n-th Root)

If $x \geq 0$ and n is a positive integer, the unique $y \geq 0$ such that $y^n = x$ is called the **n-th root** of x, written $\sqrt[n]{x}$ (or $x^{1/n}$).

Corollary

If n is an odd positive integer and $g : \mathbb{R} \to \mathbb{R}$ is the function defined by $g(x) = x^n$, then g is bijective.

- We know that $g(\mathbb{R})$ is an interval, and $g(\mathbb{R})$ contains $[0, +\infty)$, by the preceding theorem. Since $g(-x) = -g(x)$ (because n is odd), we get that $g(\mathbb{R})$ also contains $(-\infty, 0]$, and, thus, is equal to $\mathbb{R}$. Injectivity follows from the theorem since x and $g(x)$ have the same sign.

Definition (n-th Root)

If $x \in \mathbb{R}$ and $n \in \mathbb{P}$ is odd, the unique real number y such that $y^n = x$ is called the **n-th root** of x, written $\sqrt[n]{x}$ (or $x^{1/n}$). Of course, when $x \geq 0$, this is consistent with the preceding definition.

Subsection 3

Continuous Functions on a Closed Interval

Continuous Image of Closed Interval

Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous, then the range of f is a closed interval.

- Write $I = [a, b]$. We know that $f(I)$ is an interval. We need only show that $f(I)$ is: (i) bounded; (ii) a closed set.
  - (i) The claim is that $\{|f(x)| : x \in I\}$ is bounded above. Assume to the contrary. For each positive integer n, choose $x_n \in I$, such that $|f(x_n)| > n$. It is clear that no subsequence of $(f(x_n))$ is bounded. However, (x_n) is bounded, so it has a convergent subsequence (Bolzano-Weierstraß), say $x_{n_k} \to x$. Then, since I is closed, $x \in I$, and $f(x_{n_k}) \to f(x)$. In particular, $(f(x_{n_k}))$ is bounded, a contradiction.
  - (ii) Suppose $y_n \in f(I)$, $y_n \to y$. We have to show that $y \in f(I)$. Say $y_n = f(x_n)$, $x_n \in I$. Passing to a subsequence, we can suppose $x_n \to x \in \mathbb{R}$. As in the proof of (i), $x \in I$ and $f(x_n) \to f(x)$, i.e., $y_n \to f(x)$, But $y_n \to y$, so $y = f(x) \in f(I)$.

Consequences of Closed Image

Corollary (Weierstraß)

If $f : [a, b] \to \mathbb{R}$ is continuous, then f takes on a smallest value and a largest value.

- By the theorem, $f([a, b]) = [m, M]$, for suitable m and M. Thus, if $m = f(c)$ and $M = f(d)$, then $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$.
- The continuous function $(0, 1] \to \mathbb{R}$ defined by $x \mapsto x$ is > 0 at every point of its domain, but it has values as near to 0 as we like. On a closed interval, that cannot happen:

Corollary

If $f : [a, b] \to \mathbb{R}$ is continuous and $f(x) > 0$ for all $x \in [a, b]$, then there exists an $m > 0$ such that $f(x) \geq m$, for all $x \in [a, b]$.

- By the theorem, $f([a, b]) = [m, M]$, for some m and M. If $m = f(c)$ and $M = f(d)$, we have $f(x) \geq m = f(c) > 0$, for all $x \in [a, b]$.

Bounded and Unbounded Functions

Definition (Bounded and Unbounded Functions)

A real-valued function $f : X \to \mathbb{R}$ is said to be **bounded** if its range $f(X)$ is a bounded subset of $\mathbb{R}$, i.e., if there exists a real number $M > 0$ such that $|f(x)| \leq M$, for all $x \in X$.
f is said to be **unbounded** if it is not bounded.

- Example: Every continuous real-valued function on a closed interval is bounded.
- Example: The continuous function $f : (0,1] \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{x}$ is unbounded.

Subsection 4

Monotonic Continuous Functions

Increasing and Decreasing Functions

Definition (Increasing and Decreasing Functions)

Let $S \subseteq \mathbb{R}$ (in the most important examples, S is an interval). A function $f : S \to \mathbb{R}$ is said to be:

- **increasing**, if $x < y \Rightarrow f(x) \leq f(y)$,
- **strictly increasing**, if $x < y \Rightarrow f(x) < f(y)$,
- **decreasing**, if $x < y \Rightarrow f(x) \geq f(y)$,
- **strictly decreasing**, if $x < y \Rightarrow f(x) > f(y)$,

where it is understood that x and y are in the domain S of f.

If f is either increasing or decreasing, it is said to be **monotone**.

A function is **strictly monotone** if it is strictly increasing or strictly decreasing.

Examples

- Let n be a positive integer. The function $f:[0,+\infty)\to[0,+\infty)$, $f(x)=x^n$ is strictly increasing.
- The function $g:\mathbb{R}\to\mathbb{R}$, $g(x)=x^{2n-1}$ is also strictly increasing.
- The function $\mathbb{R}\to\mathbb{R}$ defined by $x\mapsto x^2$ is neither increasing nor decreasing.
- The function $(0,+\infty)\to(0,+\infty)$ defined by $x\mapsto\frac{1}{x}$ is strictly decreasing.
- Every constant function is increasing and decreasing, but not strictly. Conversely, if a function is both increasing and decreasing, then it is a constant function.
- The functions $\log:(0,+\infty)\to\mathbb{R}$ and $\exp:\mathbb{R}\to(0,+\infty)$ are both strictly increasing (here the base is e).

Continuity, Injectivity and Monotonicity in Closed Interval

- If $f : S \to \mathbb{R}$ is strictly monotone, it is obvious that f is injective.

Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous and injective, then f is strictly monotone.

- If $a = b$, there is nothing to prove. If $a < b$, then $f(a) \neq f(b)$, by injectivity. We suppose $f(a) < f(b)$ (if not, consider $-f$). We show that f is then strictly increasing.

  Claim: If $a < x < b$ then $f(a) < f(x) < f(b)$.
  Assume to the contrary that $f(x) \leq f(a)$ or $f(x) \geq f(b)$, i.e., by injectivity, $f(x) < f(a)$ or $f(x) > f(b)$.
  - In the first case, $f(x) < f(a) < f(b)$. Thus $k = f(a)$ is intermediate to the values of $f \restriction_{[x,b]}$ at the endpoints of $[x, b]$. The IVT yields a point $t \in (x, b)$ with $f(t) = k = f(a)$, contrary to injectivity.
  - In the second case, $f(a) < f(b) < f(x)$. An application of the IVT to $f \restriction_{[a,x]}$ yields a point $t \in (a, x)$, with $f(t) = f(b)$, again contradicting injectivity.

Continuity, Injectivity and Monotonicity (Cont'd)

- Assuming now that $a < c < d < b$, we have to show that $f(c) < f(d)$.
  - If $a = c$ and $d = b$, there is nothing to prove.
  - If $a = c < d < b$, then $f(c) < f(d)$ by the claim.
  - If $a < c < d = b$, we proceed similarly.
  - If $a < c < d < b$, then $f(a) < f(c) < f(b)$ by the claim applied to $a < c < b$. But then $f(c) < f(d) < f(b)$ by the claim applied to $c < d < b$ and the function $f \restriction_{[c,b]}$.

Continuity, Injectivity and Monotonicity in an Interval

Corollary

If I is an interval and $f : I \to \mathbb{R}$ is continuous and injective, then f is strictly monotone.

- If I is a singleton there is nothing to prove. Otherwise, let $r, s \in I$, with $r < s$. Since f is injective, $f(r) \neq f(s)$. We can suppose $f(r) < f(s)$ (if not, consider $-f$).

  Claim: We assert that f is strictly increasing.
  Given $c, d \in I$, $c < d$, we must show that $f(c) < f(d)$. Let $J = [a, b]$ be a closed subinterval of I that contains all four points r, s, c, d. For example, $a = \min\{r, c\}$, $b = \max\{s, d\}$ will do. From the theorem, we know that $f \restriction_J$ is either strictly increasing or strictly decreasing. Since $f(r) < f(s)$, it must be the former, whence $f(c) < f(d)$.

Subsection 5

Inverse Function Theorem

Continuity of the Inverse

Lemma

If $f : [a, b] \to [c, d]$ is bijective and continuous then the inverse function $f^{-1} : [c, d] \to [a, b]$ is also continuous.

- Assuming $y_n \to y$ in $[c, d]$, we must show that $f^{-1}(y_n) \to f^{-1}(y)$. Let $x_n = f^{-1}(y_n)$, $x = f^{-1}(y)$ and assume to the contrary that (x_n) does not converge to x. Then, there exists an $\epsilon > 0$, such that $|x_n - x| \geq \epsilon$ frequently. Passing to a subsequence, we can suppose that $|x_n - x| \geq \epsilon$, for all n. Since (x_n) is bounded, some subsequence is convergent (Bolzano-Weierstraß), say $x_{n_k} \to t$. Then $t \in [a, b]$ and $f(x_{n_k}) \to f(t)$ by continuity. But $f(x_{n_k}) = y_{n_k} \to y$, so $y = f(t)$, $t = f^{-1}(y) = x$. Thus, $x_{n_k} \to x$, contrary to $|x_{n_k} - x| \geq \epsilon$, for all k.

The Inverse Function Theorem

Theorem (Inverse Function Theorem)

Let I be an interval in $\mathbb{R}$, $f : I \to \mathbb{R}$ continuous and injective. Let $J = f(I)$ (an interval), so that $f : I \to J$ is continuous and bijective. Then $f^{-1} : J \to I$ is also continuous.

- By the preceding Corollary, we know that f is monotone. We can suppose that f is increasing (if not, consider $-f$). Suppose $y_n \to y$ in J. Writing $x_n = f^{-1}(y_n)$, $x = f^{-1}(y)$, we have to show that $x_n \to x$. The set $A = \{y\} \cup \{y_n : n \in \mathbb{P}\}$ is compact. So it has a smallest element c and a largest element d. Then $A \subseteq [c, d] \subseteq J$. Say $c = f(a)$, $d = f(b)$. Since f is increasing, $a \leq b$ and $f([a, b]) = [f(a), f(b)] = [c, d]$, so $x_n \to x$ follows from applying the lemma to the restriction $f \restriction_{[a,b]}: [a, b] \to [c, d]$.
- Example: If $n \in \mathbb{P}$ and $I = [0, +\infty)$, then the function $I \to I$, $x \mapsto \sqrt[n]{x}$ is continuous: It is the inverse of a continuous bijection. If n is odd, then the function $\mathbb{R} \to \mathbb{R}$, $x \mapsto \sqrt[n]{x}$ is continuous.

Subsection 6

Uniform Continuity

Uniform Continuity Theorem

Theorem

Suppose $f : [a, b] \to \mathbb{R}$ is continuous. Given any $\epsilon > 0$, there exists a $\delta > 0$, such that $x, y \in [a, b]$, $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.

- This is not just a restatement of the definition of continuity; there is a subtle difference:
  - To say that $f : S \to \mathbb{R}$ is continuous means that for each $y \in S$ and $\epsilon > 0$, there is a $\delta > 0$ (depending in general on both y and ϵ) such that $x \in S, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$.
  - The theorem ensures that when the domain of f is a closed interval, the choice of δ can be made to depend on ϵ alone.
    Informally speaking, δ works "uniformly well" at all points of the domain.

Proof of the Uniform Continuity Theorem

Theorem

Suppose $f:[a,b]\to\mathbb{R}$ is continuous. Given any $\epsilon>0$, there exists a $\delta>0$, such that $x,y\in[a,b]$, $|x-y|<\delta\Rightarrow|f(x)-f(y)|<\epsilon$.

- Let $\epsilon>0$. We seek a $\delta>0$, for which the stated implication is valid. Assume to the contrary that no such δ exists. In particular, for each $n\in\mathbb{P}$, the choice $\delta=\frac{1}{n}$ fails to validate the implication, so there is a pair of points x_n,y_n in $[a,b]$, such that $|x_n-y_n|<\frac{1}{n}$, but $|f(x_n)-f(y_n)|\geq\epsilon$. For a suitable subsequence, $x_{n_k}\to x\in[a,b]$ (Bolzano-Weierstraß). Then $y_{n_k}=x_{n_k}-(x_{n_k}-y_{n_k})$ and $x_{n_k}-y_{n_k}\to 0$ show that also $y_{n_k}\to x$. By continuity, $f(x_{n_k})\to f(x)$ and $f(y_{n_k})\to f(x)$, so $f(x_{n_k})-f(y_{n_k})\to 0$, contrary to $|f(x_{n_k})-f(y_{n_k})|\geq\epsilon$.

Uniformly Continuous Functions

Definition (Uniformly Continuous Function)

Let S be a subset of $\mathbb{R}$. A function $f : S \to \mathbb{R}$ is said to be **uniformly continuous** (**on** S) if, for every $\epsilon > 0$, there exists a $\delta > 0$, such that

$$x, y \in S, \quad |x - y| < \delta \ \Rightarrow \ |f(x) - f(y)| < \epsilon.$$

- Uniform continuity implies continuity, but the converse is false:
- Example: The function $f : (0, 2] \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is continuous but not uniformly continuous. Looking at the graph of f, we see that "the nearer y is to 0, the steeper the "slope" of the graph". This suggests that for a particular ϵ, the nearer y is to 0 , the smaller δ will have to be taken.

  Formal Argument: Assume f is uniformly continuous. In particular, for $\epsilon = 1$, there is a $\delta > 0$ (which we can suppose to be < 1) for which $x, y \in (0, 2], |x - y| \leq \delta \Rightarrow |\frac{1}{x} - \frac{1}{y}| < 1$, i.e., $|x - y| < xy$. Thus, $x, y \in (0, 2], |x - y| = \delta \Rightarrow \delta < xy$. If $y_n = \frac{1}{n}$ and $x_n = \delta + \frac{1}{n}$, then $|x_n - y_n| = \delta < x_n y_n \to 0$, which is absurd.