Introduction to Real Analysis

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1. Limits of Functions
   - Deleted Neighborhoods
   - Limits
   - Limits and Continuity
   - ϵ, δ Characterization of Limits
   - Algebra of Limits

Subsection 1

Deleted Neighborhoods

Deleted Neighborhoods

- The idea of "deleted neighborhood" of a point c is to permit a variable x to approach c without ever having to be equal to c:

Definition (Deleted Neighborhood)

Let S be a subset of $\mathbb{R}$, c a real number, i.e., $S \subseteq \mathbb{R}$, $c \in \mathbb{R}$ (c need not belong to S, but it might). We say that:

- S is a **deleted right neighborhood** (**DRN**) of c if there is an $r > 0$, such that $(c, c + r) \subseteq S$ (that is, $c < x < c + r \Rightarrow x \in S$);
- S is a **deleted left neighborhood** (**DLN**) of c if there is an $r > 0$, such that $(c - r, c) \subseteq S$ (that is, $c - r < x < c \Rightarrow x \in S$);
- S is a **deleted neighborhood** (**DN**) of c if there is an $r > 0$, such that $(c - r, c) \cup (c, c + r) \subseteq S$ (that is, $0 < |x - c| < r \Rightarrow x \in S$).

Remarks on Deleted Neighborhoods

(i) S is a deleted neighborhood of c if and only if it is both a deleted left neighborhood and a deleted right neighborhood of c.

(ii) If S is a deleted neighborhood of c and if $S \subseteq T \subseteq \mathbb{R}$, then T is also a deleted neighborhood of c; In particular, every neighborhood of c is a deleted neighborhood of c and similarly for DRN's and DLN's.

(iii) If S and T are DRN's of c, then so is $S \cap T$ and similarly for DLN's.

(iv) S is a DRN of c if and only if $S \cup \{c\}$ is a right neighborhood of c and similarly for DLN's and DN's.

Examples

- If $a < b$, then the open interval (a, b) is a DRN of a, a DLN of b, and a neighborhood of every internal point.
- The same is true of the intervals $[a, b]$, $(a, b]$ and $[a, b)$.
- If $f : [a, b] \to \mathbb{R}$, $a < b$, and if $c \in [a, b]$, then the set $[a, b] - \{c\}$ is a DN of c if $c \in (a, b)$; a DRN of c if $c = a$; and a DLN of c if $c = b$.

  The function $g : [a, b] - \{c\} \to \mathbb{R}$ defined by $g(x) = \frac{f(x) - f(c)}{x - c}$ is familiar from elementary calculus.
- Remarks on the terminology:
  - A "deleted neighborhood" of c might contain c, in analogy with a "neighborhood" of c being permitted to contain points that are far away from c.
    - For neighborhoods, the points far from c are ignored;
    - For deleted neighborhoods, the presence of c - if it is present - is likewise ignored.
  - Regardless of terminology, the concept of a "deleted neighborhood" should encompass the ordinary neighborhoods of c.

Subsection 2

Limits

Limits

- Continuity of a function f at a point a means, informally, that $f(x)$ approaches $f(a)$ as x approaches a.
- In the theory of limits, f is permitted to be undefined at a provided that, as x approaches a, $f(x)$ approaches something:

Definition (Limit)

Let $f : S \to \mathbb{R}$, where S is a deleted neighborhood of $c \in \mathbb{R}$. We say that f **has a limit at** c if there exists a real number L, such that

$$\left.\begin{array}{c} x_n \in S \\ x_n \neq c \\ x_n \to c \end{array}\right\} \Rightarrow f(x_n) \to L.$$

- Such a number L is unique and is called the **limit** of f at c, written $\lim_{x \to c} f(x) = L$ or, for emphasis, $\lim_{\substack{x \to c \\ x \neq c}} f(x) = L$.
- The statement that f has a limit at c is also expressed by saying that "$\lim_{x \to c} f(x)$ exists".

Examples

- If S is a neighborhood of c and $f : S \to \mathbb{R}$ is continuous at c, then f has limit $f(c)$ at c:
$$\lim_{x \to c} f(x) = f(c).$$
Indeed, $f(x_n) \to f(c)$, for every sequence (x_n) in S, such that $x_n \to c$ (and in particular for those with $x_n \neq c$, for all n).
- If $f : \mathbb{R} - \{3\} \to \mathbb{R}$ is the function defined by $f(x) = x^2$ for $x \neq 3$, then $\lim_{x \to 3} f(x) = 9$. The same is true for the function $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} 2x + 3, & \text{if } x < 3 \\ x^2, & \text{if } x > 3 \\ 1, & \text{if } x = 3 \end{cases}$$
and for $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 2x + 3, & \text{if } x \text{ rational} \\ x^2, & \text{if } x \text{ irrational} \end{cases}$

One-Sided Limits

Definition (One-Sided Limit)

Let $f : S \to \mathbb{R}$, where S is a deleted right neighborhood of $c \in \mathbb{R}$. We say that f **has a right limit at** c if there exists a real number L, such that

$$\left.\begin{array}{l} x_n \in S \\ x_n > c \\ x_n \to c \end{array}\right\} \Rightarrow f(x_n) \to L.$$

- Such a number L is unique and is called the **right limit** of f at c, written $\lim\limits_{\substack{x \to c \\ x > c}} f(x) = L$ or $\lim\limits_{x \to c^+} f(x) = L$, or, concisely, $f(c+) = L$.
- The statement that f has a right limit at c is also expressed by saying that "$f(c+)$ exists".
- Left limits (when they exist) are defined similarly: S is assumed to be a DLN of c and we require $x_n < c$. The symbols $\lim\limits_{\substack{x \to c \\ x < c}} f(x)$, $\lim\limits_{x \to c^-} f(x)$, $f(c-)$ denote the left limit of f at c (when it exists).

Examples

- If S is a right neighborhood of c and $f : S \to \mathbb{R}$ is right continuous at c, then f has right limit $f(c)$ at c.
- Similarly with "right" replaced by "left".
- If $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 1, & \text{if } x < 0 \\ 2, & \text{if } x = 0 \\ 3, & \text{if } x > 0 \end{cases},$$

then, at the point 0, f has left limit 1 and right limit 3; f does not have a limit at 0 (applying f to the sequence $x_n = \frac{(-1)^n}{n}$ produces a divergent sequence).

Limits and One-Sided Limits

Theorem

Let $f : S \to \mathbb{R}$, where $S \subseteq \mathbb{R}$, and let $c \in \mathbb{R}$. The following conditions are equivalent:

(a) f has a limit at c;

(b) $f(c-)$ and $f(c+)$ exist and are equal.

When f has a limit L at c, necessarily $L = f(c-) = f(c+)$.

- (a)$\Rightarrow$(b): By assumption, S is a deleted neighborhood of c, so it is also a DLN and a DRN. If f has limit L at c, then $f(x_n) \to L$, for every sequence in S with $x_n \to c$ and $x_n \neq c$. This is true, in particular, when $x_n < c$, for all n, and when $x_n > c$, for all n. Thus, $f(c-)$ and $f(c+)$ exist and are equal to L.

Limits and One-Sided Limits (Cont'd)

- (b)$\Rightarrow$(a): By assumption, S is a DLN and a DRN of c, so it is a DN of c. Write L for the common value of $f(c-)$ and $f(c+)$. If (x_n) is a sequence in S, with $x_n \to c$ and $x_n \neq c$, for all n, then either
  (i) $x_n < c$ ultimately, or
  (ii) $x_n > c$ ultimately, or
  (iii) $x_n < c$ frequently and $x_n > c$ frequently.

  In cases (i) and (ii) it is clear that $f(x_n) \to L$.
  In case (iii), let (x_{n_k}) be the subsequence with $x_{n_k} < c$ and (x_{m_j}) the subsequence with $x_{m_j} > c$. Then $f(x_{n_k}) \to f(c-) = L$ and $f(x_{m_j}) \to f(c+) = L$, whence $f(x_n) \to L$.

Existence of One-Sided Limits

Theorem

If $f : (a, b) \to \mathbb{R}$ is a bounded monotone function, then f has a right limit at every point of $[a, b)$ and a left limit at every point of $(a, b]$.

- It suffices, for example, to show that $f(a+)$ exists. We can suppose f is increasing (if not, consider $-f$). Let $L = \inf \{f(x) : a < x < b\}$ (recall that f is bounded).
  Claim: f has right limit L at a, i.e.. $f(a+) = L$.
  Assuming $a < x_n < b$ and $x_n \to a$, we have to show that $f(x_n) \to L$. Let $\epsilon > 0$. By the definition of L (as a greatest lower bound), there exists $c \in (a, b)$, such that $L \leq f(c) < L + \epsilon$. Ultimately $a < x_n < c$, whence $L \leq f(x_n) \leq f(c) < L + \epsilon$, so $|f(x_n) - L| < \epsilon$. We have shown that, for every $\epsilon > 0$, $|f(x_n) - L| < \epsilon$ ultimately, i.e., $f(x_n) \to L$.
- Remark: A function $f : [a, b] \to \mathbb{R}$ is said to be **regulated** if f has a right limit at every point of $[a, b)$ and a left limit at every point of $(a, b]$. Thus, every monotone function $f : [a, b] \to \mathbb{R}$ is regulated.

Subsection 3

Limits and Continuity

Continuity and Limit Equal to Value

- For a function defined on a neighborhood of a point, continuity at the point means the same thing as having a limit equal to the value:

Theorem

Let $f : S \to \mathbb{R}$, where S is a neighborhood of $c \in \mathbb{R}$. The following conditions on f are equivalent:

(a) f is continuous at c;

(b) $\exists \lim_{x \to c} f(x) = f(c)$;

- (a)$\Rightarrow$(b): This has already been seen.
- (b)$\Rightarrow$(a): Assuming $x_n \in S$, $x_n \to c$, we must show $f(x_n) \to f(c)$. This is obvious if $x_n = c$ ultimately, and, if $x_n \neq c$ ultimately, then it is immediate from (b). The remaining case, that $x_n = c$ frequently and $x_n \neq c$ frequently, follows from applying the preceding two cases to the appropriate subsequences.
- "right" version: If S is a right neighborhood of c and $f : S \to \mathbb{R}$, then f is right continuous at c if and only if $\exists f(c+) = f(c)$.

Continuous Extendability

Corollary

Let S be a deleted neighborhood of $c \in \mathbb{R}$, $f : S \to \mathbb{R}$. The following conditions on f are equivalent:

(a) f has a limit at c;

(b) There exists a function $F : S \cup \{c\} \to \mathbb{R}$, such that F is continuous at c and $F(x) = f(x)$, for all $x \in S - \{c\}$.

Necessarily, $F(c) = \lim_{x\to c} f(x)$.

- Recall that c may or may not belong to S. If $c \in S$, then $S \cup \{c\} = S$. If $c \notin S$, then $S - \{c\} = S$.
  - (a)⇒(b): Say f has limit $L \in \mathbb{R}$ at c. Define $F : S \cup \{c\} \to \mathbb{R}$ by $F(x) = \begin{cases} L, & \text{if } x = c \\ f(x), & \text{if } x \in S - \{c\} \end{cases}$. Note that, if $c \notin S$, we are extending f to $S \cup \{c\}$. If $c \in S$ and $f(c) \neq L$, we are redefining f at c. Finally, if $c \in S$ and $f(c) = L$, nothing has changed.

Continuous Extendability (Cont'd)

- (a)⇒(b): We defined $F(x) = \begin{cases} L, & \text{if } x = c \\ f(x), & \text{if } x \in S - \{c\} \end{cases}$.
  If $x_n \in S \cup \{c\}$, $x_n \neq c$, $x_n \to c$, then $x_n \in S - \{c\}$ and $F(x_n) = f(x_n) \to L = F(c)$. Thus, F is continuous at c.
- (b)⇒(a): Assume F has the properties in (b). If $x_n \in S$, $x_n \neq c$, $x_n \to c$, then $f(x_n) = F(x_n) \to F(c)$ by the continuity of F at c. This shows that f has limit $F(c)$ at c.
- Remark: There are "one-sided" versions of the theorem.

  For example, if S is a deleted right neighborhood of c and $f : S \to \mathbb{R}$, then f has a right limit at c if and only if there exists a function $F : S \cup \{c\} \to \mathbb{R}$, such that F is right continuous at c and $F(x) = f(x)$, for all $x \in S - \{c\}$; necessarily, $F(c) = f(c+)$.

A Corollary of Continuous Extendability

Corollary

Let $f : [a, b] \to \mathbb{R}$, $a < b$. The following conditions are equivalent:

(a) f is continuous on $[a, b]$;

(b) $\exists f(a+) = f(a)$, $\exists f(b-) = f(b)$ and, for every $c \in (a, b)$ $\exists \lim_{x \to c} f(x) = f(c)$.

- Condition (b) says that f is right continuous at a, left continuous at b, and continuous at every internal point c. I.e., f is continuous at every point of $[a, b]$. Condition (a) says the same thing.

Subsection 4

ϵ, δ Characterization of Limits

ϵ, δ Characterization of Limits

Theorem

Let $f : S \to \mathbb{R}$, where S is a deleted neighborhood of $c \in \mathbb{R}$, and let $L \in \mathbb{R}$. The following conditions are equivalent:

(a) $\exists \lim_{x \to c} f(x) = L$;

(b) For every $\epsilon > 0$, there exists a $\delta > 0$, such that $x \in S, 0 < |x - c| < \delta$ $\Rightarrow |f(x) - L| < \epsilon$.

- By hypothesis, there exists a $\delta > 0$, such that $0 < |x - c| < \delta \Rightarrow x \in S$. In particular, $f(x)$ is defined for such x. So the problem in (b) is to assure that, in addition, $|f(x) - L| < \epsilon$. Let $F : S \cup \{c\} \to \mathbb{R}$ be the function such that $F(c) = L$ and $F(x) = f(x)$, $x \in S - \{c\}$. Condition (b) then says that for every $\epsilon > 0$, there exists a $\delta > 0$, such that $x \in S, 0 < |x - c| < \delta \Rightarrow |F(x) - F(c)| < \epsilon$, equivalently, $x \in S \cup \{c\}$, $|x - c| < \delta \Rightarrow |F(x) - F(c)| < \epsilon$. Thus, Condition (b) is equivalent to the continuity of F at c, which is equivalent to (a).

One-Sided Versions

- There are, also, "one-sided" versions of the preceding theorem.

  For example, let $f : S \to \mathbb{R}$, where S is a deleted right neighborhood of $c \in \mathbb{R}$, and let $L \in \mathbb{R}$. In order that f have a right limit at c equal to L, it is necessary and sufficient that, for every $\epsilon > 0$, there exist a $\delta > 0$, such that

$$x \in S,\ c < x < c + \delta \ \Rightarrow\ |f(x) - L| < \epsilon.$$

Subsection 5

Algebra of Limits

Algebra of Limits

- The "algebra of continuity" translates into an "algebra of limits":

Theorem (Algebra of Limits)

Let S be a deleted neighborhood of $c \in \mathbb{R}$, and suppose $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$ have limits at c, say $\lim_{x\to c} f(x) = L$, $\lim_{x\to c} g(x) = M$. Then the functions $f + g$, fg and af, $a \in \mathbb{R}$, also have limits at c, and:

- $\lim_{x\to c}(f + g)(x) = L + M$;
- $\lim_{x\to c}(af)(x) = aL$;
- $\lim_{x\to c}(fg)(x) = LM$.
- If, moreover, $M \neq 0$, then $\frac{f}{g}$ is defined on a deleted neighborhood of c and $\lim_{x\to c}(\frac{f}{g})(x) = \frac{L}{M}$.

- Let $F : S \cup \{c\} \to \mathbb{R}$, $G : S \cup \{c\} \to \mathbb{R}$ be the functions such that $F(c) = L$, $F(x) = f(x)$, $x \in S - \{c\}$, $G(c) = M$, $G(x) = g(x)$, $x \in S - \{c\}$. F and G are continuous at c.

Algebra of Limits (Cont'd)

- F and G are continuous at c. Therefore, so is $F+G$. Moreover, $(F+G)(c) = F(c) + G(c) = L + M$ and $(F+G)(x) = f(x) + g(x) = (f+g)(x)$, for $x \in S - \{c\}$. Therefore, $f+g$ has a limit at c equal to $L+M$.

  The proofs for af and fg are similar.

  Finally, suppose $M \neq 0$. With $\epsilon = \frac{1}{2}|M|$, choose $\delta > 0$, so that $0 < |x-c| < \delta \Rightarrow x \in S$ and $|g(x) - M| < \frac{1}{2}|M|$. In particular, $0 < |x-c| < \delta \Rightarrow g(x) \neq 0$. Restricting the functions f and g to the deleted neighborhood $(c-\delta, c) \cup (c, c+\delta)$ of c, we can suppose that g is never 0 on S. Then $\frac{F}{G}$ is continuous at c and $(\frac{F}{G})(x) = \frac{f(x)}{g(x)} = (\frac{f}{g})(x)$, for $x \in S - \{c\}$. Thus, $\frac{f}{g}$ has limit $\frac{F(c)}{G(c)} = \frac{L}{M}$ at c.

One-Sided Versions

- Once more, there are "one-sided" versions of the theorem.

  For example, if S is a deleted right neighborhood of c and the functions $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$ have right limits at c, then the functions $f + g$, af, $a \in \mathbb{R}$, and fg have right limits at c, and

$$\begin{aligned} (f+g)(c+) &= f(c+) + g(c+), \\ (af)(c+) &= af(c+), \\ (fg)(c+) &= f(c+)g(c+). \end{aligned}$$

  If, moreover, $g(c+) \neq 0$, then $\frac{f}{g}$ is defined on a deleted right neighborhood of c and $(\frac{f}{g})(c+) = \frac{f(c+)}{g(c+)}$.