Introduction to Real Analysis

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

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Subsection 1

Differentiability

Differentiability

Definition (Differentiability)

Let S be a subset of $\mathbb{R}$, f a real-valued function defined on S, c a point of S, i.e., $c \in S \subseteq \mathbb{R}$ and $f : S \to \mathbb{R}$. Let $g : S - \{c\} \to \mathbb{R}$ be the function defined by the formula

$$g(x) = \frac{f(x) - f(c)}{x - c},$$

called a **difference-quotient function** associated with f. We say that:

- f is **differentiable at** c if S is a neighborhood of c and g has a limit at c;
- f is **right differentiable at** c if S is a right neighborhood of c and g has a right limit at c;
- f is **left differentiable at** c if S is a left neighborhood of c and g has a left limit at c.

Derivative

- When they exist, these limits are called the **derivative**, **right derivative** and **left derivative** of f at c, written

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c};$$

$$f'_r(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}; \quad f'_\ell(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}.$$

Theorem

Let $f : S \to \mathbb{R}$, where S is a neighborhood of $c \in \mathbb{R}$. The following conditions on f are equivalent:

(a) f is differentiable at c;

(b) f is both left and right differentiable at c, and $f'_\ell(c) = f'_r(c)$.

For such a function f, necessarily $f'(c) = f'_\ell(c) = f'_r(c)$.

- The equivalence is immediate from limit considerations.

δ, ϵ-Criterion

- Just as for general limits, there are sequential and ϵ, δ criteria for differentiability:

Theorem

Let $f : S \to \mathbb{R}$, where S is a neighborhood of $c \in \mathbb{R}$, and let $L \in \mathbb{R}$. The following conditions on f are equivalent:

(a) f is differentiable at c, with derivative L;

(b) For every $\epsilon > 0$, there exists a $\delta > 0$, such that, if $x \in S$ and $0 < |x - c| < \delta$, then $|f(x) - f(c) - L(x - c)| \leq \epsilon |x - c|$.

- The last inequality in (b) may be written $|g(x) - L| \leq \epsilon$, where g is the difference-quotient function. Thus, the theorem is immediate.
- Remark: There are "one-sided"' versions of the theorem: E.g., in the criterion for right differentiability, S is a right neighborhood of c and the condition on x in (b) is $c < x < c + \delta$.

Using the Extendability Criterion

Theorem

Let $f : S \to \mathbb{R}$, where S is a neighborhood of c. The following conditions on f are equivalent:

(a) f is differentiable at c;

(b) There exists a function $A : S \to \mathbb{R}$, such that A is continuous at c and $f(x) - f(c) = A(x)(x - c)$, for all $x \in S$.

A function A satisfying the conditions in (b) is unique, and $f'(c) = A(c)$.

- The equation in condition (b) is trivially satisfied for $x = c$. So the condition means that there exists a function $A : S \to \mathbb{R}$, such that A is continuous at c and $A(x) = g(x)$, for all $x \in S - \{c\}$. This is in turn equivalent to the existence of a limit for g at c, i.e., equivalent to condition (a), and the limit is necessarily equal to $A(c)$: $f'(c) = A(c)$.
- Remark: In a "one-sided" version, e.g., for right differentiability, S is a right neighborhood of c and A is required to be right continuous at c.

Differentiability and Continuity

Corollary

If $f : S \to \mathbb{R}$ is differentiable at c, then f is continuous at c.

- We have a function A continuous at c, such that

  $$f(x) = f(c) + A(x)(x - c), \text{ for all } x \in S,$$

  so f is continuous at c.

- Example: The function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is differentiable at every $c \in \mathbb{R} - \{0\}$, with $f'(c) = -\frac{1}{c^2}$.

  We can apply the last theorem to the identity

  $$\frac{1}{x} - \frac{1}{c} = -\frac{1}{cx}(x - c),$$

  citing the continuity of the function $A(x) = -\frac{1}{cx}$ at c.

Subsection 2

Algebra of Derivatives

Algebraic Laws of Derivatives

Theorem (Laws of Differentiation)

Let $f: S \to \mathbb{R}$, $g: S \to \mathbb{R}$, where S is a neighborhood of $c \in \mathbb{R}$.

- If f and g are differentiable at c, then so are $f+g$, af, $a \in \mathbb{R}$, and fg, and
$$\begin{aligned}(f+g)'(c) &= f'(c)+g'(c),\\ (af)'(c) &= af'(c),\\ (fg)'(c) &= f'(c)g(c)+f(c)g'(c).\end{aligned}$$
- If, moreover, $f(c) \neq 0$, then $\frac{1}{f}$ is differentiable at c and
$$\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{f(c)^2}.$$

Proof of the Sum Rule

- We know that there exist functions $A : S \to \mathbb{R}$, $B : S \to \mathbb{R}$, continuous at c, such that

$$f(x) - f(c) = A(x)(x - c) \quad \text{and} \quad g(x) - g(c) = B(x)(x - c),$$

for all $x \in S$. The function $A + B : S \to R$ is also continuous at c and, moreover, for all $x \in S$,

$$\begin{array}{rcl}(f+g)(x) - (f+g)(c) & = & [f(x) - f(c)] + [g(x) - g(c)] \\ & = & A(x)(x-c) + B(x)(x-c) \\ & = & (A+B)(x) \cdot (x-c).\end{array}$$

Thus, $f + g$ is differentiable at c, with derivative

$$(A + B)(c) = A(c) + B(c) = f'(c) + g'(c).$$

- The proof for af, $a \in \mathbb{R}$, is similar.

Proof of the Product Rule

- For all $x \in S$,

$$\begin{aligned} f(x)g(x) - f(c)g(c) &= [f(x) - f(c)]g(x) + f(c)[g(x) - g(c)] \\ &= A(x)(x - c)g(x) + f(c)B(x)(x - c). \end{aligned}$$

Thus,

$$(fg)(x) - (fg)(c) = [Ag + f(c)B](x) \cdot (x - c),$$

where $Ag + f(c)B$ is continuous at c. Therefore, fg is differentiable at c and

$$\begin{aligned} (fg)'(c) &= [Ag + f(c)B](c) \\ &= A(c)g(c) + f(c)B(c) \\ &= f'(c)g(c) + f(c)g'(c). \end{aligned}$$

Proof of the Reciprocal Rule

- Suppose, in addition, that $f(c) \neq 0$. Since f is continuous at c, $\frac{1}{f}$ is defined on a neighborhood T of c. For $x \in T$,

$$\begin{aligned} \frac{1}{f(x)} - \frac{1}{f(c)} &= -\frac{1}{f(c)f(x)}[f(x) - f(c)] \\ &= -\frac{1}{f(c)f(x)}[A(x)(x-c)] \\ &= -\frac{A(x)}{f(c)f(x)}(x-c). \end{aligned}$$

The function $B(x) = -\frac{A(x)}{f(c)f(x)}$ is continuous at c, whence $\frac{1}{f}$ is differentiable at c and

$$\left(\frac{1}{f}\right)'(c) = B(c) = -\frac{f'(c)}{f(c)^2}.$$

Subsection 3

Composition and the Chain Rule

The Chain Rule

Theorem (Chain Rule)

Let $f : S \to \mathbb{R}$, where S is a neighborhood of $c \in \mathbb{R}$, $g : T \to \mathbb{R}$, where T is a neighborhood of $f(c)$ and suppose that $f(S) \subseteq T$, so that the composite function $g \circ f : S \to \mathbb{R}$ is defined:

$$\begin{array}{ccccc} S & \xrightarrow{\ f\ } & T & \xrightarrow{\ g\ } & \mathbb{R} \\ c \in & & f(c) \in & & \end{array}$$

If f is differentiable at c, and g is differentiable at $f(c)$, then $g \circ f$ is differentiable at c and

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c).$$

- Write $h = g \circ f$. There exists a function $A : S \to \mathbb{R}$, continuous at c, such that $f(x) - f(c) = A(x)(x - c)$, for all $x \in S$. Similarly, there is a function $B : T \to \mathbb{R}$, continuous at $f(c)$, such that $g(y) - g(f(c)) = B(y)(y - f(c))$, for all $y \in T$.

The Chain Rule (Cont'd)

$$f(x) - f(c) = A(x)(x - c), \text{for all } x \in S,$$
$$g(y) - g(f(c)) = B(y)(y - f(c)), \text{ for all } y \in T.$$

If $x \in S$, then $f(x) \in T$. Putting $y = f(x)$, we get

$$\begin{aligned} g(f(x)) - g(f(c)) &= B(f(x))(f(x) - f(c)) \\ &= B(f(x))A(x)(x - c). \end{aligned}$$

Thus, $h(x) - h(c) = [(B \circ f)A](x) \cdot (x - c)$, for all $x \in S$. Since $(B \circ f)A$ is continuous at c, h is differentiable at c and

$$\begin{aligned} h'(c) &= [(B \circ f)A](c) \\ &= (B \circ f)(c) \cdot A(c) \\ &= B(f(c)) \cdot A(c) \\ &= g'(f(c)) \cdot f'(c). \end{aligned}$$

A Remark on One-Sided Versions

- There are partial "one-sided" versions of the theorem:

  E.g., assume S is a right neighborhood of c and T is a neighborhood of $f(c)$. If f is right differentiable at c, and g is differentiable at $f(c)$, then $g \circ f$ is right differentiable at c and

  $$(g \circ f)_r'(c) = g'(f(c)) \cdot f_r'(c).$$

  As in the proof of the theorem, A and f are right continuous at c, so $(B \circ f)A$ is also right continuous at c.

Subsection 4

Local Max and Min

Local Max and Min

Definition (Local Max and Min)

Let $f: S \to \mathbb{R}$, where S is a neighborhood of $c \in \mathbb{R}$. We say that:

- f has a **local maximum at** c if there exists a neighborhood V of c, with $V \subseteq S$, such that $f(x) \leq f(c)$, for all $x \in V$;
- f has a **local minimum at** c if there exists a neighborhood V of c, with $V \subseteq S$, such that $f(x) \geq f(c)$, for all $x \in V$ (in other words, $-f$ has a local maximum at c).

- Remark: A function $f: S \to \mathbb{R}$ is said to have a **maximum** (or **global maximum**) **at** $c \in S$, if $f(x) \leq f(c)$, for all $x \in S$ (here S need not be a neighborhood of c).
  If $f(x) \geq f(c)$, for all $x \in S$, then f is said to have a **minimum** (or **global minimum**) **at** c.
- Example: Every continuous function defined on a closed interval has a maximum and a minimum.

Signs of Right and Left Derivatives at Local Maxima

Local Max and Derivatives

Suppose $f : S \to \mathbb{R}$ has a local maximum at c. Then

(i) f right differentiable at $c \Rightarrow f'_r(c) \leq 0$;

(ii) f left differentiable at $c \Rightarrow f'_\ell(c) \geq 0$.

- It is implicit that S is a neighborhood of c in $\mathbb{R}$. Shrinking S, if necessary, we can suppose that $f(x) \leq f(c)$, for all $x \in S$.
  (i) Let (x_n) be a sequence in S, with $x_n > c$ and $x_n \to c$. By assumption, $\frac{f(x_n)-f(c)}{x_n-c} \to f'_r(c)$. But $f(x_n) - f(c) \leq 0$ and $x_n - c > 0$, so the fraction is ≤ 0, and, therefore, so is its limit.
  (ii) Assuming $x_n < c$, the numerator in the above difference quotient is ≤ 0 and the denominator is < 0.

Derivatives at Local Maxima

Theorem

Let $f : S \to \mathbb{R}$, where S is a neighborhood of $c \in \mathbb{R}$. If f has a local maximum or a local minimum at c, and if f is differentiable at c, then $f'(c) = 0$.

- If f has a local maximum at c then, by the lemma, $0 \leq f'_\ell(c) = f'(c) = f'_r(c) \leq 0$, so $f'(c) = 0$.

  If f has a local minimum at c, apply the preceding argument to $-f$.

Subsection 5

Mean Value Theorem

Rolle's Theorem

- We now focus on the interaction between a function and its derivative.

Theorem (Rolle's Theorem)

If $f : [a, b] \to \mathbb{R}$ is continuous, $a < b$, f is differentiable at every point of (a, b), and $f(a) = f(b)$, then there exists a point $c \in (a, b)$, such that $f'(c) = 0$.

- The range of f is a closed interval, say $f([a, b]) = [m, M]$.
  - If $m = M$ then f is constant and $f'(c) = 0$, for all $c \in (a, b)$.
  - Suppose $m < M$, say $m = f(c)$, $M = f(d)$. Since $f(a) = f(b)$ and $f(c) \neq f(d)$, not both of c and d can be endpoints of $[a, b]$. Thus, at least one of them must be an internal point. If, for example, $d \in (a, b)$, then f has a local maximum at d, so $f'(d) = 0$.

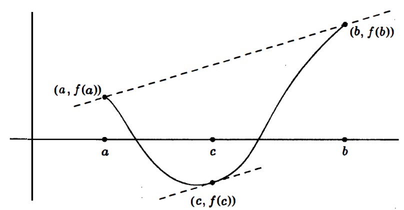
The Mean Value Theorem

Theorem. (Mean Value Theorem)

If $f : [a, b] \to \mathbb{R}$ is continuous, $a < b$, and f is differentiable at every point of (a, b), then there exists a point $c \in (a, b)$, such that

$$f(b) - f(a) = f'(c)(b - a).$$

- The theorem says there is an internal point at which the tangent line is parallel to the chord joining the endpoints:



Write $m = \frac{f(b)-f(a)}{b-a}$ for the slope of the chord. Its equation is $y = f(a) + m(x - a)$. Define $F : [a, b] \to \mathbb{R}$ by "subtracting" the line from the graph of f, i.e., $F(x) = f(x) - [f(a) + m(x - a)]$. Then F is continuous on $[a, b]$, differentiable on (a, b), $F(a) = F(b) = 0$. By Rolle's theorem, there is $c \in (a, b)$, such that $0 = F'(c) = f'(c) - m$.

Monotonicity and the Sign of the Derivative

Corollary

For a continuous function $f : [a, b] \to \mathbb{R}$ that is differentiable on (a, b), the following conditions are equivalent:

(a) f is increasing;

(b) $f'(x) \geq 0$, for all $x \in (a, b)$.

- (a)$\Rightarrow$(b): We consider $x_n \in (a, b)$, with $x_n \neq c$ and $x_n \to c$. Then $\frac{f(x)-f(c)}{x-c} \to f'(c)$. But, $\frac{f(x)-f(c)}{x-c} \geq 0$, for all $x \neq c$, whence $f'(c) \geq 0$.
- (b)$\Rightarrow$(a): Note that $f(a) \leq f(b)$: in fact, by the MVT, there exists $c \in (a, b)$, such that $f'(c) = \frac{f(b)-f(a)}{b-a} \geq 0$, whence, since $b - a > 0$, $f(b) - f(a) \geq 0$. More generally, if $a < x < y < b$, then $f(x) \leq f(y)$ (by applying the same argument to $f \restriction_{[x,y]}$). Thus, f is increasing.

Constant Functions and The Derivative

Corollary

If $f:[a,b]\to\mathbb{R}$ is continuous, differentiable on (a,b), and $f'(x)=0$, for all $x\in(a,b)$, then f is a constant function.

- By the preceding corollary, f is increasing. The hypotheses are also satisfied by $-f$, so f is also decreasing. Hence f is constant.