Introduction to Set Theory

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LSSU Math 400

Subsection 1

Introduction to Sets

Sets and Elements

- A **set** is any collection of objects.
- Objects from which a given set is composed are called **elements** or **members** of that set. We also say that they **belong to** that set.
- We are mainly concerned with sets of mathematical objects, such as numbers, points of space, functions, or sets.
- Since the first three (numbers, points and functions) may themselves be defined as sets with particular properties, in theoretical discussions we restrict attention to objects that are sets.
- In examples, however, we use various mathematical entities:
- Example: The following are sets of mathematical objects:
  (a) The set of all prime divisors of 324.
  (b) The set of all numbers divisible by 0.
  (c) The set of all continuous real-valued functions on the interval $[0, 1]$.
  (d) The set of all ellipses with major axis 5 and eccentricity 3.
  (e) The set of all sets whose elements are natural numbers less than 20.

Russell's Paradox

- Uncritical usage of "sets" remote from "everyday experience" may lead to contradictions.
- Consider for example the "set" R of all those sets which are not elements of themselves. I.e., R is a set of all sets x such that $x \notin x$ ($\in$ reads "belongs to," $\notin$ reads "does not belong to").
  We ask whether $R \in R$:
  - If $R \in R$, then R is not an element of itself (because no element of R belongs to itself), so $R \notin R$, a contradiction. Therefore, necessarily $R \notin R$.
  - But if $R \notin R$, then R is a set which is not an element of itself, and all such sets belong to R. We conclude that $R \in R$, a contradiction.

  We summarize the argument: Define R by:

$$x \in R \quad \text{if and only if} \quad x \notin x.$$

  Now consider $x = R$. By definition of R, $R \in R$ if and only if $R \notin R$, a contradiction.

On Russell's Paradox

- There is nothing wrong with R being a set of sets. Many sets whose elements are again sets are employed in mathematics and do not lead to contradictions.
- It is easy to give examples of elements of R: If x is the set of all natural numbers, then $x \notin x$ (the set of all natural numbers is not a natural number) and so $x \in R$.
- It is not so easy to give examples of sets which do not belong to R, but this is irrelevant.
- The argument would result in a contradiction even if there were sets which are elements of themselves. (A plausible candidate for a set which is an element of itself would be the "set of all sets" V; clearly $V \in V$.)
- The "set of all sets", however, leads to contradictions.

Search for a Resolution

- The goal is to develop set theory carefully to resolve this and similar contradictions.
- Having a set R defined as the set of all sets which are not elements of themselves leads to a contradiction.
- This can only mean that there is no set satisfying the definition of R.
- The argument proves that there exists no set whose members would be precisely the sets which are not elements of themselves.
- Lesson of Russell's Paradox: By merely defining a set we do not prove its existence.
- There are properties which do not define sets; that is, it is not possible to collect all objects with those properties into one set.
- One has to determine the properties which do define sets.
- No way on how to do this is known, and some results in logic (Incompleteness Theorems of Kurt Gödel) seem to indicate that a complete answer is not even possible.

Axiomatization of Set Theory

- We formulate some of the relatively simple properties of sets used by mathematicians as axioms.
- Then check that all theorems follow logically from the axioms.
- Since the axioms are obviously true and the theorems logically follow from them, the theorems are also true.
- We prove truths about sets which include, among other things, the basic properties of natural, rational, and real numbers, functions, orderings, etc., but, as far as is known, no contradictions.
- Experience has shown that all notions used in contemporary mathematics can be defined, and their mathematical properties derived, in this axiomatic system.
- In this sense, the axiomatic set theory serves as a satisfactory foundation for the other branches of mathematics.
- But not every true fact about sets can derived from the axioms we present, i.e., the axiomatic system is not complete in this sense.

Subsection 2

Properties

Discussion on Properties

- Some "properties" commonly considered in everyday life are so vague that they can hardly be admitted in a mathematical theory.
  - The "set of all the great twentieth century American novels". Is a twentieth century American novels "great"?
  - The "set of those natural numbers which could be written down in decimal notation" (by "could" we mean that someone could actually do it with paper and pencil).
    0 can be so written down. If number n can be written down, then number $n+1$ can also be written down. Therefore, by induction, every natural number n can be written down.
    But can $10^{10^{10}}$ be so written down?
    The problem, here, is caused by the vague meaning of "could".
- To avoid similar difficulties and "contradictions", we now describe explicitly what we mean by a property.

The Fundamental Concept of Membership

- The basic set-theoretic property is the membership property: "...is an element of...", which we denote by $\in$. So "$X \in Y$" reads "X is an element of Y" or "X is a member of Y" or "X belongs to Y".
- The letters X and Y in these expressions are variables; they stand for unspecified, arbitrary sets.
- The proposition "$X \in Y$" holds or does not hold depending on which sets are denoted by X and Y.
- We sometimes say "$X \in Y$" is a property of X and Y.
- Example: Similar principles apply in other branches of mathematics. For example, "m is less than n" is a property of numbers m and n. The letters m and n are variables denoting unspecified numbers. Some m and n have this property ("2 is less than 4" is true) but others do not ("3 is less than 2" is false).
- All other set-theoretic properties can be stated in terms of membership with the help of logical means: identity, logical connectives, and quantifiers.

Axiomatization of Identity

- Sometimes the same set is denoted by different variables in various contexts.
- We use the identity sign "=" to express that two variables denote the same set.
- So we write $X = Y$ if X is the same set as Y (X is identical with Y or X is equal to Y).
- We list some obvious facts about identity:
  (a) $X = X$. (X is identical with X.)
  (b) If $X = Y$, then $Y = X$. (If X and Y are identical, then Y and X are identical.)
  (c) If $X = Y$ and $Y = Z$, then $X = Z$. (If X is identical with Y and Y is identical with Z, then X is identical with Z.)
  (d) If $X = Y$ and $X \in Z$, then $Y \in Z$. (If X and Y are identical and X belongs to Z, then Y belongs to Z.)
  (e) If $X = Y$ and $Z \in X$, then $Z \in Y$. (If X and Y are identical and Z belongs to X, then Z belongs to Y.)

Logical Connectives

- Logical connectives can be used to construct more complicated properties from simpler ones.
- They are expressions like "not ...", "... and ...", "if ... then ..." and "... if and only if ...".
- Example:
  (a) "$X \in Y$ or $Y \in X$" is a property of X and Y.
  (b) "Not $X \in Y$ and not $Y \in X$" or, in more idiomatic English, "X is not an element of Y and Y is not an element of X" is also a property of X and Y.
  (c) "If $X = Y$, then $X \in Z$ if and only if $Y \in Z$" is a property of X, Y and Z.
  (d) "X is not an element of X" (or: "not $X \in X$") is a property of X.
- We write
  - $X \notin Y$ instead of "not $X \in Y$";
  - $X \neq Y$ instead of "not $X = Y$".

Quantifiers

- Quantifiers "for all" ("for every") and "there is" ("there exists") provide additional logical means.
- All mathematical facts can be expressed in this language consisting of equality, logical connectives and quantifiers. Moreover, this language does not allow vague expressions, like the ones causing paradoxes.
- Example:
  (a) "There exists $Y \in X$".
  (b) "For every $Y \in X$, there is Z such that $Z \in X$ and $Z \in Y$".
  (c) "There exists Z such that $Z \in X$ and $Z \notin Y$".
- Truth or falsity of (a) depends on the set X.
  - If X is the set of all American presidents after 1789, then (a) is true;
  - If X is the set of all American presidents before 1789, (a) is false.

  Thus, (a) is a property of X and (a) depends on the parameter X.
- Similarly, (b) is a property of X, and (c) is a property of X and Y.
- On the other hand, Y is not a parameter in (a), nor is (b) a property of Y or Z, or (c) a property of Z.

Properties, Parameters and Statements

- Instead of providing precise rules for determining which variables are parameters of a given property, we rely on an intuitive understanding.
- Example:
  (a) "$Y \in X$";
  (b) "There is $Y \in X$";
  (c) "For every X, there is $Y \in X$".

  (a) is a property of X and Y; it is true for some pairs of sets X, Y and false for others.
  (b) is a property of X (but not of Y).
  (c) has no parameters. (c) is, therefore, either true or false (it is in fact, false).
- Properties which have no parameters (and are, therefore, either true or false) are called statements.
- All mathematical theorems are (true) statements.

Notation for Properties and Parameters

- We sometimes wish to refer to an arbitrary, unspecified property.
- We use boldface capital letters to denote statements and properties and, if convenient, list some or all of their parameters in parentheses. So $\mathbf{A}(X)$ stands for any property of the parameter X.
- Example: In the preceding example, we could write
  - $\mathbf{E}(X, Y)$ for "$Y \in X$";
  - $\mathbf{A}(X)$ for "There is $Y \in X$".
- In general, $\mathbf{P}(X, Y, \ldots, Z)$ is a property whose truth or falsity depends on parameters $X, Y, \ldots, Z$ (and possibly others).

Defined Properties as Shorthands

- Recall that we adopted a restricted language, consisting of membership, equality, logical connectives and quantifiers.
- As more complicated theorems proved, it is practical to give names to various particular properties, i.e., to define new properties.
- A new symbol is then introduced to denote the property.
- We can view it as a shorthand for the explicit formulation.
- Example: The property of being a **subset** is defined by

  $X \subseteq Y$ if and only if, every element of X is an element of Y.

  "X is a subset of Y" ($X \subseteq Y$) is a property of X and Y. We can use it in more complicated formulations and, whenever desirable, replace $X \subseteq Y$ by its definition.

  "If $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$" is a shorthand for "If every element of X is an element of Y and every element of Y is an element of Z, then every element of X is an element of Z".

Defined Constants as Shorthands

- In principle, we could do mathematics without definitions, but it would be tedious and exceedingly clumsy.
- For another type of definition, consider the property $\mathbf{P}(X)$: "There exists no $Y \in X$". We will prove that:
  (a) There exists a set X such that $\mathbf{P}(X)$ (there exists a set X with no elements).
  (b) There exists at most one set X such that $\mathbf{P}(X)$, i.e., if $\mathbf{P}(X)$ and $\mathbf{P}(X')$, then $X = X'$ (if X bas no elements and X' has no elements, then X and X' are identical).

  (a) and (b) together express the fact that there is a unique set X with the property $\mathbf{P}(X)$. We can then give this set a name, say $\emptyset$ (the empty set) and use it as a shorthand in more complicated expressions. The full meaning of "$\emptyset \subseteq Z$" is then "`the set` X `which has no elements is a subset of` Z".
- We occasionally refer to $\emptyset$ as the constant defined by the property $\mathbf{P}$.

Defined Functions as Shorthands

- As a last example of a definition, consider the property $\mathbf{Q}(X, Y, Z)$ of X, Y and Z:

  "For every U, $U \in Z$ if and only if $U \in X$ and $U \in Y$".

  We see in the next section that:

  (a) For every X and Y there exists Z such that $\mathbf{Q}(X, Y, Z)$.
  (b) For every X and Y, if $\mathbf{Q}(X, Y, Z)$ and $\mathbf{Q}(X, Y, Z')$, then $Z = Z'$ (for every X and Y, there exists at most one Z such that $\mathbf{Q}(X, Y, Z)$).

  Conditions (a) and (b) (which have to be proved whenever this type of definition is used) guarantee that for every X and Y there is a unique set Z such that $\mathbf{Q}(X, Y, Z)$. We can then introduce a name, say $X \cap Y$, for this unique set Z and call $X \cap Y$ the intersection of X and Y. So $\mathbf{Q}(X, Y, X \cap Y)$ holds.
- We refer to $\cap$ as the operation/function defined by the property $\mathbf{Q}$.

Subsection 3

The Axioms

Axiom of Existence

- The first principle postulates that our "universe of discourse" is not void, i.e., that some sets exist.
- To be concrete, we postulate the existence of a specific set, namely the empty set.

The Axiom of Existence

There exists a set which has no elements.

- A set with no elements can be variously described intuitively:
  - The set of all U.S. Presidents before 1789;
  - The set of all real numbers x for which $x^2 = -1$, etc.
- All examples of this kind describe one and the same set, namely the empty, vacuous set. So, intuitively, there is only one empty set.
- The formal proof requires a second postulate.

The Axiom of Extensionality

- Our second postulate expresses the fact that each set is determined by its elements.
- Example:
  - X is the set consisting exactly of the numbers 2, 3, and 5.
  - Y is the set of all prime numbers greater than 1 and less than 7.
  - Z is the set of all solutions of the equation $x^3 - 10x^2 + 31x - 30 = 0$.

  Here $X = Y$, $X = Z$, and $Y = Z$, and we have three different descriptions of one and the same set.

The Axiom of Extensionality

If every element of X is an element of Y and every element of Y is an element of X, then $X = Y$.

- If two sets have the same elements, then they are identical.

Uniqueness of the Empty Set

Lemma (Uniqueness of the Empty Set)

There exists only one set with no elements.

- Assume that A and B are sets with no elements.
  - Every element of A is an element of B (since A has no elements, the statement "$a \in A$ implies $a \in B$" is an implication with a false antecedent, and thus automatically true).
  - Similarly, every element of B is an element of A (since B has no elements).

  Therefore, $A = B$, by the Axiom of Extensionality.

Definition (Empty Set)

The (unique) set with no elements is called the **empty set** and is denoted $\emptyset$.

- The definition of the constant $\emptyset$ is justified by the Axiom of Existence and the preceding lemma.

The Axiom Schema of Comprehension

- Since sets are collections of objects sharing some common property, we expect to have axioms expressing this fact.
- As demonstrated by the preceding paradoxes, not every property describes a set; properties "$X \notin X$" or "$X = X$" are typical examples.
- To be able to collect all objects having such a property into a set, we have to be able to perceive all sets.
- The difficulty is avoided if we postulate the existence of a set of all objects with a given property only if there already exists some set to which they all belong.

The Axiom Schema of Comprehension

Let $\mathbf{P}(x)$ be a property of x. For any set A, there is a set B such that $x \in B$ if and only if $x \in A$ and $\mathbf{P}(x)$.

- This is a schema of axioms, i.e., for each property $\mathbf{P}$, we have one axiom.

Instances of The Axiom Schema of Comprehension

- Example: If $\mathbf{P}(x)$ is "$x = x$", the axiom says:
  For any set A, there is a set B such that $x \in B$ if and only if $x \in A$ and $x = x$. (In this case, $B = A$.)
- Example: If $\mathbf{P}(x)$ is "$x \notin x$", the axiom postulates:
  For any set A, there is a set B such that $x \in B$ if and only if $x \in A$ and $x \notin x$.
- Although there are infinitely many axioms, this causes no problems, since it is easy to recognize whether a particular statement is or is not an axiom and since every proof uses only finitely many axioms.
- The property $\mathbf{P}(x)$ can depend on other parameters $p, \ldots, q$. The corresponding axiom then postulates that for any sets $p, \ldots, q$ and any A, there is a set B (depending on $p, \ldots, q$ and, of course, on A) consisting exactly of all those $x \in A$ for which $\mathbf{P}(x, p, \ldots, q)$.

Using Comprehension via Extensionality

Proposition (Existence of Intersection)

If P and Q are sets, then there is a set R such that $x \in R$ if and only if $x \in P$ and $x \in Q$.

- Consider the property $\mathbf{P}(x, Q)$ of x and Q: "$x \in Q$". Then, by the Comprehension Schema, for every Q and for every P there is a set R such that $x \in R$ if and only if $x \in P$ and $\mathbf{P}(x, Q)$, i.e., if and only if $x \in P$ and $x \in Q$. (Here P plays the role of A, Q is a parameter.)

Lemma (Uniqueness in Comprehension)

For all A, there is unique B such that $x \in B$ if and only if $x \in A$ and $\mathbf{P}(x)$.

- If B' is such that $x \in B'$ if and only if $x \in A$ and $P(x)$, then $x \in B$ if and only if $x \in B'$, so $B = B'$, by Extensionality.

Definition (Notation for Comprehension)

$\{x \in A : \mathbf{P}(x)\}$ the set of all $x \in A$ with the property $\mathbf{P}(x)$.

The Axiom of Pair

- The only set we proved to exist is the empty set; Moreover, applications of the Comprehension Schema to the empty set produce again the empty set: $\{x \in \emptyset : \mathbf{P}(x)\} = \emptyset$, regardless of the property $\mathbf{P}$.
- The next three principles postulate that some of the constructions frequently used in mathematics yield sets.

The Axiom of Pair

For any A and B, there is a set C such that $x \in C$ if and only if $x = A$ or $x = B$.

- So $A \in C$ and $B \in C$, and there are no other elements of C.
- The set C is unique.

Definition of Unordered Pair

We define the **unordered pair of A and B** as the set having exactly A and B as its elements and introduce the notation $\{A, B\}$ for the unordered pair of A and B. In particular, if $A = B$, we write $\{A\}$ instead of $\{A, A\}$.

Forming Unordered Pairs

- Example:
  (a) Set $A = \emptyset$ and $B = \emptyset$; then $\{\emptyset\} = \{\emptyset, \emptyset\}$ is a set for which $\emptyset \in \{\emptyset\}$, and if $x \in \{\emptyset\}$, then $x = \emptyset$. So $\{\emptyset\}$ has a unique element $\emptyset$.
  Note that $\{\emptyset\} \neq \emptyset$, since $\emptyset \in \{\emptyset\}$ but $\emptyset \notin \emptyset$.
  (b) Let $A = \emptyset$ and $B = \{\emptyset\}$; then $\emptyset \in \{\emptyset, \{\emptyset\}\}$ and $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$ and $\emptyset$ and $\{\emptyset\}$ are the only elements of $\{\emptyset, \{\emptyset\}\}$.
  Note, also, that $\emptyset \neq \{\emptyset, \{\emptyset\}\}$ and $\{\emptyset\} \neq \{\emptyset, \{\emptyset\}\}$.

The Axiom of Union

The Axiom of Union

For any set S, there exists a set U, such that $x \in U$ if and only if $x \in A$ for some $A \in S$.

- The set U is unique; it is called the **union of** S and denoted by $\bigcup S$.
- We say that S is a **system of sets** or a **collection of sets** when we want to stress that elements of S are sets (since all objects we consider are sets, this is always true and, thus, the expressions "set" and "system of sets" have the same meaning).
- The union of a system of sets S is then a set of precisely those x which belong to some set from the system S.
- Example:
  (a) Let $S = \{\emptyset, \{\emptyset\}\}$; $x \in \bigcup S$ if and only if $x \in A$ for some $A \in S$, i.e., if and only if $x \in \emptyset$ or $x \in \{\emptyset\}$. Therefore, $x \in \bigcup S$ if and only if $x = \emptyset$: $\bigcup S = \{\emptyset\}$.
  (b) $\bigcup \emptyset = \emptyset$.

Union of Two or More Sets

- Example: Let M and N be sets; $x \in \bigcup\{M, N\}$ if and only if $x \in M$ or $x \in N$. The set $\bigcup\{M, N\}$ is called the **union** of M and N and is denoted $M \cup N$.
- We introduced the familiar set-theoretic operation of union:
  - The Axiom of Pair and the Axiom of Union are necessary to define union of two sets;
  - The Axiom of Extensionality is needed to guarantee that it is unique.
- The union of two sets has the usual meaning:

$$x \in M \cup N \quad \text{if and only if} \quad x \in M \text{ or } x \in N.$$

- Example: $\{\{\emptyset\}\} \cup \{\emptyset, \{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$.
- The Axiom of Union is more powerful: it enables us to form unions of not just two, but of any, possibly infinite, collection of sets.
- If A, B and C are sets, we can prove the existence and uniqueness of the set P whose elements are exactly A, B and C. P is denoted $\{A, B, C\}$ and is called an **unordered triple of** A, B and C.

The Axiom of Power Set

Definition (Subset)

A is a **subset** of B if and only if every element of A belongs to B. In other words, A is a subset of B if, for every x, $x \in A$ implies $x \in B$.

- We write $A \subseteq B$ to denote that A is a subset of B.
- Example:
  (a) $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$ and $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}$.
  (b) $\emptyset \subseteq A$ and $A \subseteq A$, for every set A.
  (c) $\{x \in A : \mathbf{P}(x)\} \subseteq A$.
  (d) If $A \in S$, then $A \subseteq \bigcup S$.
- The next axiom postulates that all subsets of a given set can be collected into one set.

The Axiom of Power Set

For any set S, there exists a set P such that $X \in P$ if and only if $X \subseteq S$.

- Since the set P is again uniquely determined, we call the set of all subsets of S the **power set of** S and denote it by $\mathcal{P}(S)$.

Elements Satisfying a Property

- Example:
  (a) $\mathcal{P}(\emptyset) = \{\emptyset\}$.
  (b) $\mathcal{P}(\{a\}) = \{\emptyset, \{a\}\}$.
  (c) The elements of $\mathcal{P}(\{a, b\})$ are $\emptyset, \{a\}, \{b\}$, and $\{a, b\}$.
- Let $\mathbf{P}(x)$ be a property of x (and, possibly, of other parameters). If there is a set A such that, for all x, $\mathbf{P}(x)$ implies $x \in A$, then $\{x \in A : \mathbf{P}(x)\}$ exists, and, moreover, does not depend on A.
  I.e., if A' is another set such that for all x, $\mathbf{P}(x)$ implies $x \in A'$, then $\{x \in A' : \mathbf{P}(x)\} = \{x \in A : \mathbf{P}(x)\}$.
- We can now define $\{x : \mathbf{P}(x)\}$ to be the set $\{x \in A : \mathbf{P}(x)\}$, where A is any set for which $\mathbf{P}(x)$ implies $x \in A$ (since it does not matter which such set A we use):
  $\{x : \mathbf{P}(x)\}$ is the **set of all** x **with the property** $\mathbf{P}(x)$.
- It must be stressed that this notation can be used only after it has been proved that some A contains all x with the property $\mathbf{P}$.

Set of Elements Satisfying a Property

- Example:
  (a) $\{x : x \in P \text{ and } x \in Q\}$ exists.
  $\mathbf{P}(x, P, Q)$ is the property "$x \in P$ and $x \in Q$"; let $A = P$. Then $\mathbf{P}(x, P, Q)$ implies $x \in A$. Therefore, $\{x : x \in P \text{ and } x \in Q\} = \{x \in P : x \in P \text{ and } x \in Q\} = \{x \in P : x \in Q\}$.
  (b) $\{x : x = a \text{ or } x = b\}$ exists; for a proof put $A = \{a, b\}$; also show that $\{x : x = a \text{ or } x = b\} = \{a, b\}$.
  (c) $\{x : x \notin x\}$ does not exist (because of Russell's Paradox): thus in this instance the notation $\{x : \mathbf{P}(x)\}$ is inadmissible.
- Although our list of axioms is not complete, we postpone the introduction of the remaining postulates until they are needed.
  - We did not guarantee existence of infinite sets.
  - The Axiom Schema of Replacement is introduced later.
  - Towards the end we also introduce the Axiom of Choice.
- This axiomatic system was essentially formulated by Ernst Zermelo in 1908 and is often referred to as the **Zermelo-Fraenkel axiomatic system for set theory**.

Subsection 4

Elementary Operations on Sets

Subset ($\subseteq$) Relation Between Sets

- We may now introduce simple set-theoretic operations (union, intersection, difference, etc.) and prove some of their basic properties.
- The property of "being a subset", denoted $\subseteq$, is called **inclusion**.

Lemma

For all sets A, B, C:

(a) $A \subseteq A$.

(b) If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

(c) If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

- To verify (c) we have to prove: If $x \in A$, then $x \in C$. But if $x \in A$, then $x \in B$, since $A \subseteq B$. Now, $x \in B$ implies $x \in C$, since $B \subseteq C$. So $x \in A$ implies $x \in C$.
- If $A \subseteq B$ and $A \neq B$, we say that A is a **proper subset** of B (A is **properly contained in** B), and write $A \subset B$.
- $B \supseteq A$ means $A \subseteq B$ and $B \supset A$ means $A \subset B$.

Operations on Sets

Definition (Operations on Sets)

- The **intersection** of A and B, $A \cap B$, is the set of all x which belong to both A and B.
- The **union** of A and B, $A \cup B$, is the set of all x which belong to A or B (or both).
- The **difference** of A and B, $A - B$, is the set of all $x \in A$ which do not belong to B.
- The **symmetric difference** of A and B, $A \triangle B$, is defined by $A \triangle B = (A - B) \cup (B - A)$.

- Existence and uniqueness may be proved using the preceding axioms.
- As an example, we sketch these for $A - B$:
  - Existence: Since $x \in A - B$ implies $x \in A$, by Comprehension, there exists $\{x \in A : x \in A \text{ and } x \notin B\} = \{x : x \in A \text{ and } x \notin B\}$.
  - Uniqueness: If two sets are defined by the same property, they contain the same elements and, by Extensionality, they are equal.

Properties of Set-Theoretic Operations

Proposition

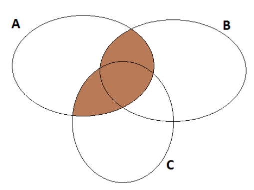
- (**Commutativity**) $A \cap B = B \cap A$, $A \cup B = B \cup A$;
- (**Associativity**) $(A \cap B) \cap C = A \cap (B \cap C)$,
  $(A \cup B) \cup C = A \cup (B \cup C)$;
  So we may write $A \cap B \cap C$ and $A \cup B \cup C$ unambiguously.
- (**Distributivity**) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
- (**DeMorgan Laws**) $C - (A \cap B) = (C - A) \cup (C - B)$,
  $C - (A \cup B) = (C - A) \cap (C - B)$;
- And for difference and symmetric difference:
  - $A \cap (B - C) = (A \cap B) - C$;
  - $A - B = \emptyset$ if and only if $A \subseteq B$;
  - $A \triangle A = \emptyset$;
  - $A \triangle B = B \triangle A$;
  - $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

Venn Diagrams and a Sample Proof

- Venn diagrams help in discovering set relationships.

  Recall the distributive law

  $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$



- To prove this, we have, to prove that the sets $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ have the same elements. This requires us to show:
  (a) Every element of $A \cap (B \cup C)$ belongs to $(A \cap B) \cup (A \cap C)$.
  Let $a \in A \cap (B \cup C)$. Then $a \in A$ and $a \in B \cup C$. Therefore, either $a \in B$ or $a \in C$. So $a \in A$ and $a \in B$ or $a \in A$ and $a \in C$. This means that $a \in A \cap B$ or $a \in A \cap C$. Therefore, $a \in (A \cap B) \cup (A \cap C)$.
  (b) Every element of $(A \cap B) \cup (A \cap C)$ belongs to $A \cap (B \cup C)$.
  Let $a \in (A \cap B) \cup (A \cap C)$. Then $a \in A \cap B$ or $a \in A \cap C$. In the first case, $a \in A$ and $a \in B$, so that $a \in A$ and $a \in B \cup C$ and, hence, $a \in A \cap (B \cup C)$. In the second $a \in A$ and $a \in C$, so again $a \in A$ and $a \in B \cup C$, so that $a \in A \cap (B \cup C)$.

Intersection of a System of Sets and Disjoint Sets

- The union of a system of sets S was defined in the preceding section.
- The **intersection** $\bigcap S$ of a nonempty system of sets S is defined by
$$x \in \bigcap S \quad \text{if and only if} \quad x \in A, \text{ for all } A \in S.$$
- Then intersection of two sets is a special case: $A \cap B = \bigcap\{A, B\}$.
- Notice that we do not define $\bigcap \emptyset$; the reason is that every x belongs to all $A \in \emptyset$ (since there is no such A), so $\bigcap \emptyset$ would have to be a set of all sets.
- We postpone further study of unions and intersections until later.
- We say that sets A and B are **disjoint** if $A \cap B = \emptyset$.
- More generally, S is a **system of mutually disjoint sets** if $A \cap B = \emptyset$, for all $A, B \in S$ such that $A \neq B$.