Introduction to Set Theory

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

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Subsection 1

Cardinal Arithmetic

Addition of Cardinal Numbers

- To define the **sum** $\kappa + \lambda$ of two cardinals, we use the analogy with finite sets: If a set A has a elements, a set B has b elements, and if A and B are disjoint, then $A \cup B$ has $a + b$ elements.

Definition (Addition of Cardinal Numbers)

$\kappa + \lambda = |A \cup B|$, where $|A| = \kappa$, $|B| = \lambda$ and $A \cap B = \emptyset$.

- In order to make this definition legitimate, we have to show that $\kappa + \lambda$ does not depend on the choice of the sets A and B.

Lemma (Addition of Cardinals is Well-defined)

If A, B, A', B' are such that $|A| = |A'|$, $|B| = |B'|$ and $A \cap B = \emptyset = A' \cap B'$, then $|A \cup B| = |A' \cup B'|$.

- Let f and g be, respectively, a 1-1 mapping of A onto A' and of B onto B'. Then $f \cup g$ is a 1-1 mapping of $A \cup B$ onto $A' \cup B'$.

Some Properties of Addition of Cardinal Numbers

- Addition of cardinals coincides with the ordinary addition of numbers in case of finite cardinals.
- Moreover, many of the usual laws of addition remain valid.
  - (a) Addition of cardinal numbers is commutative: $\kappa + \lambda = \lambda + \kappa$;
  - (b) Addition of cardinal numbers is associative: $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$;
- These laws follow directly from the definition.
  - (c) $\kappa \leq \kappa + \lambda$;
  - (d) If $\kappa_1 \leq \kappa_2$ and $\lambda_1 \leq \lambda_2$, then $\kappa_1 + \lambda_1 \leq \kappa_2 + \lambda_2$.
- However, not all laws of addition of numbers hold for addition of cardinals. In particular, strict inequalities in formulas are rare in case of infinite cardinals and those that hold are quite difficult to establish.
- Example: Take the simple fact that if $n \neq 0$, then $n + n > n$. If κ is infinite, then this is no longer true: We have seen that $\aleph_0 + \aleph_0 = \aleph_0$. The Axiom of Choice implies that $\kappa + \kappa = \kappa$, for every infinite κ.

Multiplication of Cardinal Numbers

- If A and B are sets of a and b elements, respectively, then the product $A \times B$ has $a \cdot b$ elements.

Definition (Multiplication of Cardinal Numbers)

$\kappa \cdot \lambda = |A \times B|$, where $|A| = \kappa$ and $|B| = \lambda$.

- This definition does not depend on the specific sets:

Lemma (Multiplication is Well-defined)

If A, B, A', B' satisfy $|A| = |A'|$, $|B| = |B'|$, then $|A \times B| = |A' \times B'|$.

- Let $f : A \to A'$, $g : B \to B'$ be mappings. We define $h : A \times B \to A' \times B'$ as follows:

$$h(a, b) = (f(a), g(b)).$$

If f and g are one-to-one and onto, so is h.

Properties of Multiplication of Cardinal Numbers

- Multiplication has some expected properties, in particular, it is commutative, associative and the distributive law holds:
  (e) $\kappa \cdot \lambda = \lambda \cdot \kappa$;
  (f) $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$;
  (g) $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$.
  The last property is a consequence of the equality $A \times (B \cup C) = (A \times B) \cup (A \times C)$, that holds for any sets A, B and C.
  (h) $\kappa \leq \kappa \cdot \lambda$, if $\lambda > 0$;
  (i) If $\kappa_1 \leq \kappa_2$ and $\lambda_1 \leq \lambda_2$, then $\kappa_1 \cdot \lambda_1 \leq \kappa_2 \cdot \lambda_2$.
  (j) $\kappa + \kappa = 2 \cdot \kappa$.
  If $|A| = \kappa$, then $2 \cdot \kappa$ is the cardinal of $\{0,1\} \times A$. We note that $\{0,1\} \times A = (\{0\} \times A) \cup (\{1\} \times A)$, that $|\{0\} \times A| = |\{1\} \times A| = \kappa$ and that the two summands are disjoint. Hence $2 \cdot \kappa = \kappa + \kappa$.
  (k) $\kappa + \kappa \leq \kappa \cdot \kappa$ whenever $\kappa \geq 2$.
- As in the case of addition, multiplication of infinite cardinals has some properties different from those valid for finite numbers.
  - Example: $\aleph_0 \cdot \aleph_0 = \aleph_0$. Moreover, the Axiom of Choice implies that $\kappa \cdot \kappa = \kappa$, for all infinite cardinals.

Exponentiation of Cardinal Numbers

- If A and B are finite sets, with a and b elements, respectively, then a^b is the number of all functions from B to A.

Definition (Exponentiation of Cardinal Numbers)

$\kappa^\lambda = |A^B|$, where $|A| = \kappa$ and $|B| = \lambda$.

- The definition of κ^λ does not depend on the choice of A and B.

Lemma (Exponentiation is Well-defined)

If $|A| = |A'|$ and $|B| = |B'|$, then $|A^B| = |A'^{B'}|$.

- Let $f : A \to A'$ and $g : B \to B'$ be one-to-one and onto. Let $F : A^B \to A'^{B'}$ be defined as follows:

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "k"'] & B' \arrow[d, "h"] \\ A \arrow[r, "f"'] & A' \end{tikzcd}$$

If $k \in A^B$, let $F(k) = h \in A'^{B'}$, where $h(g(b)) = f(k(b))$, for all $b \in B$, i.e., $h = f \circ k \circ g^{-1}$. Then F is one-to-one and maps A^B onto $A'^{B'}$.

Properties of Exponentiation

(l) $\kappa \leq \kappa^\lambda$, if $\lambda > 0$;

(m) $\lambda \leq \kappa^\lambda$, if $\kappa > 1$;

(n) If $\kappa_1 \leq \kappa_2$ and $\lambda_1 \leq \lambda_2$, then $\kappa_1^{\lambda_1} \leq \kappa_2^{\lambda_2}$;

(o) $\kappa \cdot \kappa = \kappa^2$;

To see this, it suffices to have a one-to-one correspondence between $A \times A$, the set of all pairs (a, b) with $a, b \in A$, and the set of all functions from $\{0, 1\}$ into A. Such a correspondence has already been established.

- Some additional properties are given in the following theorem.

Additional Properties of Exponentiation

Theorem

(a) $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$ (b) $(\kappa^\lambda)^\mu = \kappa^{\lambda\cdot\mu}$ (c) $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$.

- Let $\kappa = |K|$, $\lambda = |L|$ and $\mu = |M|$.
  (a) Assume that L and M are disjoint. Construct a one-to-one mapping F of $K^L \times K^M$ onto $K^{L\cup M}$. If $(f,g) \in K^L \times K^M$, let $F(f,g) = f \cup g$. We note that $f \cup g$ is a function, in fact a member of $K^{L\cup M}$. Every $h \in K^{L\cup M}$ is equal to $F(f,g)$ for some $(f,g) \in K^L \times K^M$ (namely, $f = h \upharpoonright L, g = h \upharpoonright M$). It is easily seen that F is one-to-one.
  (b) We look for a one-to-one map F of $K^{L\times M}$ onto $(K^L)^M$. A typical element of $K^{L\times M}$ is a function $f : L \times M \to K$. We let F assign to f the function $g : M \to K^L$ defined, for all $m \in M$, by $g(m) = h \in K^L$, where $h(\ell) = f(\ell, m)$, $\ell \in L$. F is one-to-one and onto.
  (c) We need a one-to-one mapping F of $K^M \times L^M$ onto $(K \times L)^M$. For each $(f_1, f_2) \in K^M \times L^M$, let $F(f_1, f_2) = g : M \to K \times L$, where $g(m) = (f_1(m), f_2(m))$, for all $m \in M$. It is also easy to check that F is one-to-one and onto.

Cantor's Theorem

Cantor's Theorem

$|X| < |\mathcal{P}(X)|$, for every set X.

- The proof is a straightforward generalization of the proof of the corresponding theorem for $\mathbb{N}$. Its heart is an abstract form of the diagonalization argument.
  - The function $f : X \to \mathcal{P}(X)$ defined by $f(x) = \{x\}$ is clearly one-to-one. So $|X| \leq |\mathcal{P}(X)|$.
  - It remains to show that there is no mapping of X onto $\mathcal{P}(X)$. So let f be a mapping of X into $\mathcal{P}(X)$. Consider the set
  $$S = \{x \in X : x \notin f(x)\}.$$
  Claim: S is not in the range of f.
  Suppose that $S = f(z)$, for some $z \in X$. By definition of S, $z \in S$ if and only if $z \notin f(z)$. So we have $z \in S$ if and only if $z \notin S$, a contradiction.
  So f is not onto $\mathcal{P}(X)$. This completes the proof that $|X| < |\mathcal{P}(X)|$.

Additional Generalizations

Theorem

$|\mathcal{P}(X)| = 2^{|X|}$, for every set X.

- Replace $\mathbb{N}$ by X in the proof of $|\mathcal{P}(\mathbb{N})| = 2^{|\mathbb{N}|}$.
- Cantor's Theorem can now be restated as

$$\kappa < 2^{\kappa}, \quad \text{for every cardinal number } \kappa.$$

- Finally, for any set of cardinal numbers, there exists a cardinal number greater than all of them:

Corollary

For any system of sets S, there is a set Y such that $|Y| > |X|$, for all $X \in S$.

- Let $Y = \mathcal{P}(\bigcup S)$. By Cantor's Theorem, $|Y| > |\bigcup S|$. Clearly, $|\bigcup S| \geq |X|$, for all $X \in S$ (if $X \in S$, then $X \subseteq \bigcup S$). Thus, $|Y| > |X|$.

Subsection 2

The Cardinality of the Continuum

$\aleph_0$ and $2^{\aleph_0}$

- We summarize the properties of the cardinal number $\aleph_0$, the cardinality of countable sets, using the concepts of cardinal arithmetic:
  (a) $\kappa < \aleph_0$ if and only if $\kappa \in \mathbb{N}$.
  (b) $n + \aleph_0 = \aleph_0 + \aleph_0 = \aleph_0$ $(n \in \mathbb{N})$.
  (c) $n \cdot \aleph_0 = \aleph_0 \cdot \aleph_0 = \aleph_0$ $(n \in \mathbb{N}, n > 0)$.
  (d) $\aleph_0^n = \aleph_0$ $(n \in \mathbb{N}, n > 0)$.
- We now study the second most important infinite cardinal number, the cardinality of the continuum, $2^{\aleph_0}$.
- Recall that $2^{\aleph_0}$ is the cardinality of $\mathbb{R}$:

Theorem

$|\mathbb{R}| = 2^{\aleph_0}$.

- Has already been shown.

Properties of the Cardinality of the Continuum

Theorem

(a) $n + 2^{\aleph_0} = \aleph_0 + 2^{\aleph_0} = 2^{\aleph_0} + 2^{\aleph_0} = 2^{\aleph_0}$ $(n \in \mathbb{N})$.

(b) $n \cdot 2^{\aleph_0} = \aleph_0 \cdot 2^{\aleph_0} = 2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0}$ $(n \in \mathbb{N}, n > 0)$.

(c) $(2^{\aleph_0})^n = (2^{\aleph_0})^{\aleph_0} = n^{\aleph_0} = \aleph_0^{\aleph_0} = 2^{\aleph_0}$ $(n \in \mathbb{N}, n > 1)$.

(a) This follows from the obvious sequence of inequalities:

$$2^{\aleph_0} \leq n + 2^{\aleph_0} \leq \aleph_0 + 2^{\aleph_0} \leq 2^{\aleph_0} + 2^{\aleph_0} = 2 \cdot 2^{\aleph_0} = 2^{1+\aleph_0} = 2^{\aleph_0}$$

using the Cantor-Bernstein Theorem.

(b) Similarly, we have

$$2^{\aleph_0} \leq n \cdot 2^{\aleph_0} \leq \aleph_0 \cdot 2^{\aleph_0} \leq 2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0+\aleph_0} = 2^{\aleph_0}.$$

(c) We have:

$$2^{\aleph_0} \leq (2^{\aleph_0})^n \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0^2} = 2^{\aleph_0},$$
$$2^{\aleph_0} \leq n^{\aleph_0} \leq \aleph_0^{\aleph_0} \leq (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0^2} = 2^{\aleph_0}.$$

Unexpected Consequences

- The preceding theorem, has some rather unexpected consequences:
  - For example, $2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0}$ means that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$. The set $\mathbb{R} \times \mathbb{R}$ of all pairs of real numbers is in a one-to-one correspondence with the set of all points in the plane (via a cartesian coordinate system). Thus we see that there exists a one-to-one mapping of a straight line $\mathbb{R}$ onto a plane $\mathbb{R} \times \mathbb{R}$.
  - Similarly, $\mathbb{R}$ can be mapped in a 1-1 way onto a three-dimensional space $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, etc.
- These results (due to Cantor) astonished his contemporaries, since they seem rather counterintuitive.
- The next theorem shows that several important sets have the cardinality of the continuum.

Sets with the Cardinality of the Continuum

Theorem (Sets with the Cardinality of the Continuum)

(a) The set of all points in the n-dimensional space $\mathbb{R}^n$ has cardinality $2^{\aleph_0}$.

(b) The set of all complex numbers has cardinality $2^{\aleph_0}$.

(c) The set of all infinite sequences of natural numbers has cardinality $2^{\aleph_0}$.

(d) The set of all infinite sequences of real numbers has cardinality $2^{\aleph_0}$.

(a) $|\mathbb{R}^n| = (2^{\aleph_0})^n$ by definition of cardinal exponentiation; $(2^{\aleph_0})^n = 2^{\aleph_0}$ by the theorem.

(b) Complex numbers are represented by pairs of reals, so the cardinality of the set of all complex numbers is $|\mathbb{R} \times \mathbb{R}| = (2^{\aleph_0})^2 = 2^{\aleph_0}$.

(c) The set of all infinite sequences of natural numbers is $\mathbb{N}^{\mathbb{N}}$ and $|\mathbb{N}^{\mathbb{N}}| = \aleph_0^{\aleph_0} = \aleph_0$.

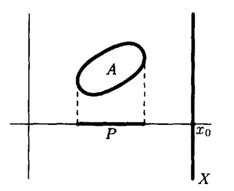
(d) $|\mathbb{R}^{\mathbb{N}}| = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$.

Complement of a Countable Set in a Continuum

Theorem (Complement of a Countable Set in a Continuum)

If A is a countable subset of B and $|B| = 2^{\aleph_0}$, then $|B - A| = 2^{\aleph_0}$.

- We can assume without loss of generality that $B = \mathbb{R} \times \mathbb{R}$. Let $P = \text{dom}A$, i.e., $P = \{x \in \mathbb{R} : (x, y) \in A, \text{ for some } y\}$.



  Since $|A| = \aleph_0$, we have $|P| \leq \aleph_0$. Thus, there exists $x_0 \in \mathbb{R}$, such that $x_0 \notin P$. Consequently, the set $X = \{x_0\} \times \mathbb{R}$ is disjoint from A, so $X \subseteq (\mathbb{R} \times \mathbb{R}) - A$. Clearly, $|X| = |\mathbb{R}| = 2^{\aleph_0}$ and we have $|(\mathbb{R} \times \mathbb{R}) - A| \geq 2^{\aleph_0}$.
- In general, using the Axiom of Choice, it can be shown that if $|A| < |B|$, then $|B - A| = |B|$.

More Sets with the Cardinality of the Continuum

Theorem (More Sets with the Cardinality of the Continuum)

(a) The set of all irrational numbers has cardinality $2^{\aleph_0}$.
(b) The set of all infinite sets of natural numbers has cardinality $2^{\aleph_0}$.
(c) The set of all one-to-one mappings of $\mathbb{N}$ onto $\mathbb{N}$ has cardinality $2^{\aleph_0}$.

(a) The set of all rationals $\mathbb{Q}$ is countable, hence the set $\mathbb{R} - \mathbb{Q}$ of all irrational numbers has cardinality $2^{\aleph_0}$.
(b) The set of all subsets of $\mathbb{N}$, $\mathcal{P}(\mathbb{N})$, has cardinality $2^{\aleph_0}$, and the set of all finite subsets of $\mathbb{N}$ is countable. Hence, the set of all infinite subsets of $\mathbb{N}$ has the cardinality of the continuum.

Proof of Part (c)

(c) Let P be the set of all one-to-one mappings of $\mathbb{N}$ onto $\mathbb{N}$.
Since $P \subseteq \mathbb{N}^{\mathbb{N}}$, $|P| \leq 2^{\aleph_0}$.
Let E and O, respectively, be the sets of all even and odd natural numbers. If $X \subseteq E$ is infinite, define a mapping $f_X : \mathbb{N} \to \mathbb{N}$ by

$$f_X(n) = \begin{cases} \text{the } k\text{th element of } X, & \text{if } n = 2k \\ \text{the } k\text{th element of } \mathbb{N} - X, & \text{if } n = 2k+1 \end{cases}$$

Notice that $\mathbb{N} - X \supseteq O$ is infinite, so f_X is a one-to-one mapping of $\mathbb{N}$ onto $\mathbb{N}$. Moreover, it is easy to show that $X_1 \neq X_2$ implies $f_{X_1} \neq f_{X_2}$. We thus have a one-to-one correspondence between infinite subsets of E and certain elements of P. Since there are $2^{\aleph_0}$ infinite subsets of E, we get $|P| \geq 2^{\aleph_0}$.

Continuous Functions and Open Sets

Theorem (Continuous Functions and Open Sets)

(a) The set of all continuous functions on $\mathbb{R}$ to $\mathbb{R}$ has cardinality $2^{\aleph_0}$.

(b) The set of all open sets of reals has cardinality $2^{\aleph_0}$.

(a) Every continuous function on $\mathbb{R}$ is determined by its values on a dense set, in particular on the rational arguments: If f and g are two continuous functions on $\mathbb{R}$, and if $f(q) = g(q)$, for all $q \in \mathbb{Q}$, then $f = g$. Let C be the set of all continuous real-valued functions on $\mathbb{R}$. Let F be a mapping of C into $\mathbb{R}^{\mathbb{Q}}$ defined by $F(f) = f \upharpoonright \mathbb{Q}$. By the fact above, F is one-to-one, so $|C| \leq |\mathbb{R}^{\mathbb{Q}}| = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$. On the other hand, clearly $|C| \geq 2^{\aleph_0}$ (consider the constant functions).

(b) Every open set is a union of a system of open intervals with rational endpoints. There are $\aleph_0$ open intervals with rational endpoints, and, hence, $2^{\aleph_0}$ such systems. So there are at most $2^{\aleph_0}$ open sets. Since for $a, b \in \mathbb{R}, a \neq b$, $(a, \infty) \neq (b, \infty)$, there are at least $2^{\aleph_0}$ open sets.

The Continuum Hypothesis

- We know that $2^{\aleph_0}$ is greater than $\aleph_0$, but how much greater?
- Cantor conjectured that $2^{\aleph_0}$ is the next cardinal number after $\aleph_0$:

The Continuum Hypothesis

There is no uncountable cardinal number κ such that $\kappa < 2^{\aleph_0}$.

- The Continuum Hypothesis asserts that every set of real numbers is either finite or countable, or else it is equipotent to the set of all real numbers, with no cardinalities in between.
- In 1900, David Hilbert included the Continuum Problem in his famous list of open problems in mathematics (as Problem 1).
- In 1939, Kurt Gödel showed that the Continuum Hypothesis is consistent with the axioms of set theory. That is, using the axioms of Zermelo-Fraenkel set theory (including the Axiom of Choice), one cannot refute the Continuum Hypothesis.
- In 1963, Paul Cohen proved that the Continuum Hypothesis is independent of (cannot be proved from) the axioms.

A Set with Cardinality Greater than the Continuum

Lemma

The set of all real-valued functions on real numbers has cardinality $2^{2^{\aleph_0}} > 2^{\aleph_0}$.

- The cardinal number of $\mathbb{R}^{\mathbb{R}}$ is

$$(2^{\aleph_0})^{2^{\aleph_0}} = 2^{\aleph_0 \cdot 2^{\aleph_0}} = 2^{2^{\aleph_0}}.$$