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Subsection 1

Initial Ordinals
Finite Ordinals and Cardinals

- We proved results involving the cardinality \( |X| \) of a set \( X \), but we have not defined \( |X| \) itself, except in the case when \( X \) is finite or countable.
- We now find “representatives” of cardinalities.
- Natural numbers play this role satisfactorily for finite sets.
- We showed that ordinal numbers have many properties of natural numbers, e.g., inductive proofs and recursive constructions.
- However, ordinal numbers do not represent cardinalities; instead, they represent types of well-orderings.
- Since any infinite set can be well-ordered in many different ways, there are many ordinal numbers of the same cardinality;
- E.g., \( \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega, \ldots, \omega \cdot \omega, \omega \cdot \omega + 1, \ldots \) are all countable ordinal numbers; i.e., \( |\omega| = |\omega + 1| = |\omega + \omega| = \cdots = \aleph_0 \).
- The good behavior of ordinal numbers of finite cardinalities is due to the fact that all linear orderings of a finite set are isomorphic, and they are well-orderings. Thus, for any finite \( X \), there exists unique ordinal \( n \) such that \( |n| = |X| \), called the \textbf{cardinal number} of \( X \).
Initial Ordinals

- To get representatives for cardinalities of infinite (well-orderable) sets, we take the least ordinal number of any given cardinality as the representative of that cardinality:

**Definition (Initial Ordinal)**

An ordinal number $\alpha$ is called an **initial ordinal** if it is not equipotent to any $\beta < \alpha$.

**Example:**
- Every natural number is an initial ordinal.
- $\omega$ is an initial ordinal, because $\omega$ is not equipotent to any natural number.
- $\omega + 1$ is not initial, because $|\omega| = |\omega + 1|$.
- Similarly, none of $\omega + 2, \omega + 3, \omega + \omega, \omega \cdot \omega, \omega^\omega, \ldots$ is initial.
Cardinal Numbers

Theorem
Each well-orderable set $X$ is equipotent to a unique initial ordinal number.

- By a preceding theorem, $X$ is equipotent to some ordinal $\alpha$. Let $\alpha_0$ be the least ordinal equipotent to $X$. Then $\alpha_0$ is an initial ordinal because $|\alpha_0| = |\beta|$, for some $\beta < \alpha_0$, would imply $|X| = |\beta|$, a contradiction.
  If $\alpha_0 \neq \alpha_1$ are initial ordinals, they cannot be equipotent, because $|\alpha_0| = |\alpha_1|$ and, say, $\alpha_0 < \alpha_1$, would violate the fact that $\alpha_1$ is initial. This proves the uniqueness.

Definition (Cardinal Number)
If $X$ is a well-orderable set, then the cardinal number of $X$, denoted $|X|$, is the unique initial ordinal equipotent to $X$. In particular, $|X| = \omega$ for any countable set $X$, and $|X| = n$ for any finite set of $n$ elements.
Hartogs Number of a Set

- Are there other initial ordinals besides the natural numbers and \( \omega \)?
- Let \( A \) be any set; \( A \) may not be well-orderable itself, but it certainly has some well-orderable subsets; for example, all finite subsets of \( A \) are well-orderable.

**Definition (Hartogs Number)**

For any \( A \), let \( h(A) \) be the least ordinal number which is not equipotent to any subset of \( A \). \( h(A) \) is called the *Hartogs number* of \( A \).

By definition, \( h(A) \) is the least ordinal \( \alpha \) such that \( |\alpha| \not\preceq |A| \).

**Lemma**

For any \( A \), \( h(A) \) is an initial ordinal number.

Assume that \( |\beta| = |h(A)| \) for some \( \beta < h(A) \). Then \( \beta \) is equipotent to a subset of \( A \), and \( \beta \) is equipotent to \( h(A) \). We conclude that \( h(A) \) is equipotent to a subset of \( A \), i.e., \( h(A) < h(A) \), a contradiction.
Existence of Hartogs Numbers

- How do we know that the Hartogs number of $A$ exists?
  If all infinite ordinals were countable, $h(\omega)$ would consist of all ordinals!

**Lemma**

The Hartogs number of $A$ exists for all $A$.

- By a preceding theorem, for every well-ordered set $(W, R)$ where $W \subseteq A$, there is a unique ordinal $\alpha$, such that $(\alpha, <)$ is isomorphic to $(W, R)$. By Replacement, there exists a set $H$ such that, for every well-ordering $R \in \mathcal{P}(A \times A)$, its isomorphic ordinal $\alpha$ is in $H$.

Claim: $H$ contains all ordinals equipotent to a subset of $A$.

If $f$ is a one-to-one function mapping $\alpha$ into $A$, we set $W = \text{ran} f$ and $R = \{(f(\beta), f(\gamma)) : \beta < \gamma < \alpha\}$. $R \subseteq A \times A$ is then a well-ordering isomorphic to $\alpha$ (by the isomorphism $f$). These considerations show that $h(A) = \{\alpha \in H : \alpha$ is an ordinal equipotent to a subset of $A\}$. Thus, by Axiom Schema of Comprehension, $h(A)$ exists.
The Hierarchy of Omegas

- We can now define a “scale” of larger and larger initial ordinal numbers by transfinite recursion:

**Definition (Omegas)**

\[
\begin{align*}
\omega_0 &= \omega; \\
\omega_{\alpha+1} &= h(\omega_\alpha), \text{ for all } \alpha; \\
\omega_\alpha &= \sup \{\omega_\beta : \beta < \alpha\}, \text{ if } \alpha \text{ is limit } \alpha \neq 0.
\end{align*}
\]

- We know that \(|\omega_{\alpha+1}| > |\omega_\alpha|\), for each \(\alpha\), and so \(|\omega_\alpha| < |\omega_\beta|\) whenever \(\alpha < \beta\).

**Theorem**

(a) \(\omega_\alpha\) is an infinite initial ordinal number for each \(\alpha\).

(b) If \(\Omega\) is an infinite initial ordinal number, then \(\Omega = \omega_\alpha\) for some \(\alpha\).
Proof of the Theorem

(a) The proof is by induction on $\alpha$. The only nontrivial case is when $\alpha$ is a limit ordinal. Suppose that $|\omega_\alpha| = |\gamma|$ for some $\gamma < \omega_\alpha$; then there is $\beta < \alpha$ such that $\gamma \leq \omega_\beta$ (by the definition of supremum). But this implies $|\omega_\alpha| = |\gamma| \leq |\omega_\beta| \leq |\omega_\alpha|$ and yields a contradiction.

(b) First, an easy induction shows that $\alpha \leq \omega_\alpha$ for all $\alpha$. Therefore, for every infinite initial ordinal $\Omega$, there is an ordinal $\alpha$ such that $\Omega < \omega_\alpha$, (for example, $\alpha = \Omega + 1$). Thus, it suffices to prove the following:

Claim: For every infinite initial ordinal $\Omega < \omega_\alpha$, there is some $\gamma < \alpha$ such that $\Omega = \omega_\gamma$.

By induction on $\alpha$.

- The claim is trivially true for $\alpha = 0$.
- If $\alpha = \beta + 1$, $\Omega < \omega_\alpha = h(\omega_\beta)$ implies that $|\Omega| \leq |\omega_\beta|$ so either $\Omega = \omega_\beta$ and we can let $\gamma = \beta$, or $\Omega < \omega_\beta$ and existence of $\gamma < \beta < \alpha$ follows from the inductive assumption.
- If $\alpha$ is a limit ordinal, $\Omega < \omega_\alpha = \sup \{\omega_\beta : \beta < \alpha\}$ implies that $\Omega < \omega_\beta$ for some $\beta < \alpha$. The inductive assumption again guarantees the existence of some $\gamma < \beta$, such that $\Omega = \omega_\gamma$. 
Conclusions

- Every well-orderable set is equipotent to a unique initial ordinal.
- Infinite initial ordinal numbers form a transfinite sequence $\omega_\alpha$ with $\alpha$ ranging over all ordinal numbers.
- Infinite initial ordinals are, by definition, the cardinalities of infinite well-orderable sets. It is customary to call these cardinal numbers **alephs**, i.e., we define $\aleph_\alpha = \omega_\alpha$, for each $\alpha$.

The cardinal number of a well-orderable set is thus either a natural number or an aleph.

Note that the ordering of cardinal numbers by size defined previously agrees with the ordering of natural numbers and alephs as ordinals by $<$ (i.e., $\in$):

If $|X| = \aleph_\alpha$ and $|Y| = \aleph_\beta$, then $|X| < |Y|$ if and only if $\aleph_\alpha < \aleph_\beta$ (i.e., $\omega_\alpha \in \omega_\beta$).

A similar equivalence holds if one or both of $|X|$ and $|Y|$ are natural numbers.
Cardinal and Ordinal Operations

- We have defined addition, multiplication, and exponentiation of cardinal numbers.
- These agree with the corresponding ordinal operations if the ordinals involved are natural numbers but they may differ for infinite ordinals.
- Example:
  - $\omega_0 + \omega_0 \neq \omega_0$ if $+$ stands for the ordinal addition; but $\omega_0 + \omega_0 = \omega_0$ if $+$ stands for the cardinal addition.
  - The addition of cardinal numbers is commutative, but the addition of ordinal numbers is not.
- For clarity, the $\omega$-symbolism is used when the ordinal operations are involved, and the $\aleph$-symbolism for the cardinal operations.
- Thus:
  - $\omega_0 + \omega_0$ and $2^{\omega_0}$ indicate ordinal addition and exponentiation:
    - $\omega_0 + \omega_0 = \sup \{\omega + n : n < \omega_0\} > \omega_0$;
    - $2^{\omega_0} = \sup \{2^n : n < \omega_0\} = \omega_0$.
  - $\aleph_0 + \aleph_0$ and $2^{\aleph_0}$ cardinal operations:
    - $\aleph_0 + \aleph_0 = \aleph_0$;
    - $2^{\aleph_0}$ is uncountable.
Subsection 2

Addition and Multiplication of Alephs
Let \( \kappa \) and \( \lambda \) be cardinal numbers. We have defined \( \kappa + \lambda \) as the cardinality of the set \( X \cup Y \), where \( |X| = \kappa \), \( |Y| = \lambda \), and \( X \) and \( Y \) are disjoint:

\[
|X| + |Y| = |X \cup Y|, \text{ if } X \cap Y = \emptyset.
\]

This definition does not depend on the choice of \( X \) and \( Y \).

The product \( \kappa \cdot \lambda \) has been defined as the cardinality of the cartesian product \( X \times Y \), where \( X \) and \( Y \) are any two sets of respective cardinalities \( \kappa \) and \( \lambda \):

\[
|X| \cdot |Y| = |X \times Y|.
\]

This definition is also independent of the choice of \( X \) and \( Y \).

Addition and multiplication satisfy:

\[
\begin{align*}
\kappa + \lambda &= \lambda + \kappa \\
\kappa + (\lambda + \mu) &= (\kappa + \lambda) + \mu \\
\kappa \cdot (\lambda + \mu) &= \kappa \cdot \lambda + \kappa \cdot \mu
\end{align*}
\]

\[
\begin{align*}
\kappa \cdot \lambda &= \lambda \cdot \kappa \\
\kappa \cdot (\lambda \cdot \mu) &= (\kappa \cdot \lambda) \cdot \mu \\
\kappa \cdot (\lambda + \mu) &= \kappa \cdot \lambda + \kappa \cdot \mu
\end{align*}
\]
Some Examples of Operations involving Alephs

- The arithmetic of infinite numbers differs substantially from the arithmetic of finite numbers.
- In fact, the rules for addition and multiplication of alephs are very simple.
- Example:
  - $\aleph_0 + n = \aleph_0$, for every natural number $n$.
  - $\aleph_0 + \aleph_0 = \aleph_0$, since the set of all natural numbers is the union of two disjoint countable sets: the set of even numbers and the set of odd numbers.
  - $\aleph_0 \cdot \aleph_0 = \aleph_0$ (The set of all pairs of natural numbers is countable.)
- We prove, next, a general theorem that determines completely the result of addition and multiplication of alephs.
We prove the theorem by transfinite induction. For every $\alpha$, we construct a certain well-ordering $\prec$ of the set $\omega_\alpha \times \omega_\alpha$ and show, using the induction hypothesis $\aleph_\beta \cdot \aleph_\beta \leq \aleph_\beta$ that the order-type of the well-ordered set $(\omega_\alpha \times \omega_\alpha, \prec)$ is at most $\omega_\alpha$. Then, it follows that $\aleph_\alpha \cdot \aleph_\alpha \leq \aleph_\alpha$ and since $\aleph_\alpha \cdot \aleph_\alpha \geq \aleph_\alpha$, we have $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$.

We construct the well-ordering $\prec$ of $\omega_\alpha \times \omega_\alpha$ uniformly for all $\omega_\alpha$, i.e., we define a property $\prec$ of pairs of ordinals and show that $\prec$ well-orders $\omega_\alpha \times \omega_\alpha$, for every $\omega_\alpha$.

$$(\alpha_1, \alpha_2) \prec (\beta_1, \beta_2)$$ if and only if either $\max\{\alpha_1, \alpha_2\} < \max\{\beta_1, \beta_2\}$ or $\max\{\alpha_1, \alpha_2\} = \max\{\beta_1, \beta_2\}$ and $\alpha_1 < \beta_1$ or $\max\{\alpha_1, \alpha_2\} = \max\{\beta_1, \beta_2\}, \alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$.

We show that $\prec$ is a well-ordering (of any set of pairs of ordinals).
≺ is an Ordering

≺ is transitive: Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ be such that $(\alpha_1, \alpha_2) \prec (\beta_1, \beta_2)$ and $(\beta_1, \beta_2) \prec (\gamma_1, \gamma_2)$. By definition
\[
\max \{ \alpha_1, \alpha_2 \} \leq \max \{ \beta_1, \beta_2 \} \leq \max \{ \gamma_1, \gamma_2 \},
\]
whence
\[
\max \{ \alpha_1, \alpha_2 \} \leq \max \{ \gamma_1, \gamma_2 \}.
\]

- If $\max \{ \alpha_1, \alpha_2 \} < \max \{ \gamma_1, \gamma_2 \}$, then $(\alpha_1, \alpha_2) \prec (\gamma_1, \gamma_2)$.
- If $\max \{ \alpha_1, \alpha_2 \} = \max \{ \beta_1, \beta_2 \} = \max \{ \gamma_1, \gamma_2 \}$, then we have $\alpha_1 \leq \beta_1 \leq \gamma_1$, and so $\alpha_1 \leq \gamma_1$.
  - If $\alpha_1 < \gamma_1$, then $(\alpha_1, \alpha_2) \prec (\gamma_1, \gamma_2)$;
  - Otherwise, we have $\alpha_1 = \beta_1 = \gamma_1$. In this last case, $\max \{ \alpha_1, \alpha_2 \} = \max \{ \beta_1, \beta_2 \} = \max \{ \gamma_1, \gamma_2 \}$, and $\alpha_1 = \beta_1 = \gamma_1$, so, necessarily, $\alpha_2 < \beta_2 < \gamma_2$, and it follows again that $(\alpha_1, \alpha_2) \prec (\gamma_1, \gamma_2)$.
\(<\) is Linear

We verify that for any \(\alpha_1, \alpha_2, \beta_1, \beta_2\), either

\[(\alpha_1, \alpha_2) < (\beta_1, \beta_2) \text{ or } (\beta_1, \beta_2) < (\alpha_1, \alpha_2) \text{ or } (\alpha_1, \alpha_2) = (\beta_1, \beta_2)\]

and that these three cases are mutually exclusive. This follows directly from the definition:

Given \((\alpha_1, \alpha_2)\) and \((\beta_1, \beta_2)\)

- we first compare \(\max \{\alpha_1, \alpha_2\}\) and \(\max \{\beta_1, \beta_2\}\),
- then \(\alpha_1\) and \(\beta_1\),
- and last the ordinals \(\alpha_2\) and \(\beta_2\).
≺ is a Well-Ordering

≺ is a well-ordering: Let $X$ be a nonempty set of pairs of ordinals. We find the $≺$-least element of $X$. Let $\delta$ be the least maximum of the pairs in $X$, i.e., let $\delta$ be least element in $\{\max \{\alpha, \beta\} : (\alpha, \beta) \in X\}$. Let $Y = \{(\alpha, \beta) \in X : \max \{\alpha, \beta\} = \delta\}$. The set $Y$ is a nonempty subset of $X$, and, for every $(\alpha, \beta) \in Y$, we have $\max \{\alpha, \beta\} = \delta$. Moreover, $\delta < \max \{\alpha', \beta'\}$, for any $(\alpha', \beta') \in X - Y$, and hence $(\alpha, \beta) \prec (\alpha', \beta')$ whenever $(\alpha, \beta) \in Y$ and $(\alpha', \beta') \in X - Y$.

Therefore, the least element of $Y$, if it exists, is also the least element of $X$. Now let $\alpha_0$ be the least ordinal in the set $\{\alpha : (\alpha, \beta) \in Y \text{ for some } \beta\}$ and let $Z = \{(\alpha, \beta) \in Y : \alpha = \alpha_0\}$. The set $Z$ is a nonempty subset of $Y$. Also $(\alpha, \beta) \prec (\alpha', \beta')$ whenever $(\alpha, \beta) \in Z$ and $(\alpha', \beta') \in Y - Z$.

Finally, let $\beta_0$ be the least ordinal in the set $\{\beta : (\alpha_0, \beta) \in Z\}$. Clearly, $(\alpha_0, \beta_0)$ is the least element of $Z$. It follows that $(\alpha_0, \beta_0)$ is the least element of $X$. 

Finishing the Proof

Having shown that $\prec$ is a well-ordering of $\omega_\alpha \times \omega_\alpha$ for every $\alpha$, we use this well-ordering to prove, by transfinite induction on $\alpha$, that $|\omega_\alpha \times \omega_\alpha| \leq \aleph_\alpha$, i.e., $\aleph_\alpha \cdot \aleph_\alpha \leq \aleph_\alpha$.

For $\alpha = 0$, we know that $\aleph_0 \cdot \aleph_0 = \aleph_0$.

So let $\alpha > 0$, and let us assume that $\aleph_\beta \cdot \aleph_\beta \leq \aleph_\beta$, for all $\beta < \alpha$. We prove that $|\omega_\alpha \times \omega_\alpha| \leq \aleph_\alpha$. If suffices to show that the order-type of the well-ordered set $(\omega_\alpha \times \omega_\alpha, \prec)$ is at most $\omega_\alpha$.

If the order-type of $(\omega_\alpha \times \omega_\alpha, \prec)$ were greater than $\omega_\alpha$, then there would exist $(\alpha_1, \alpha_2) \in \omega_\alpha \times \omega_\alpha$, such that the cardinality of the set $X = \{(\xi_1, \xi_2) \in \omega_\alpha \times \omega_\alpha : (\xi_1, \xi_2) \prec (\alpha_1, \alpha_2)\}$ is at least $\aleph_\alpha$. Thus, it suffices to prove that, for any $(\alpha_1, \alpha_2) \in \omega_\alpha \times \omega_\alpha$, we have $|X| < \aleph_\alpha$.

Let $\beta = \max \{\alpha_1, \alpha_2\} + 1$. Then $\beta \in \omega_\alpha$ and, for every $(\xi_1, \xi_2) \in X$, we have $\max \{\xi_1, \xi_2\} \leq \max \{\alpha_1, \alpha_2\} < \beta$, so $\xi_1 \in \beta$ and $\xi_2 \in \beta$, i.e., $X \subseteq \beta \times \beta$.

Let $\gamma < \alpha$ be such that $|\beta| \leq \aleph_\gamma$. Then $|X| \leq |\beta \times \beta| = |\beta| \cdot |\beta| \leq \aleph_\gamma \cdot \aleph_\gamma$ and $\aleph_\gamma \cdot \aleph_\gamma \leq \aleph_\gamma$, by the induction hypothesis. Thus, $|X| \leq \aleph_\gamma$, and, hence, $|X| < \aleph_\alpha$. 
Corollary

For every $\alpha$ and $\beta$ such that $\alpha \leq \beta$, we have $\aleph_\alpha \cdot \aleph_\beta = \aleph_\beta$. Also, $n \cdot \aleph_\alpha = \aleph_\alpha$, for every positive natural number $n$.

If $\alpha \leq \beta$, then

- $\aleph_\beta = 1 \cdot \aleph_\beta \leq \aleph_\alpha \cdot \aleph_\beta$
- $\aleph_\alpha \cdot \aleph_\beta \leq \aleph_\beta \cdot \aleph_\beta = \aleph_\beta$, by the theorem.

Thus by the Cantor-Bernstein Theorem $\aleph_\alpha \cdot \aleph_\beta = \aleph_\beta$. The equality $n \cdot \aleph_\alpha = \aleph_\alpha$ is proved similarly.

Corollary

For every $\alpha$ and $\beta$ such that $\alpha \leq \beta$, we have $\aleph_\alpha + \aleph_\beta = \aleph_\beta$. Also, $n + \aleph_\alpha = \aleph_\alpha$, for all natural numbers $n$.

If $\alpha \leq \beta$, then $\aleph_\beta \leq \aleph_\alpha + \aleph_\beta \leq \aleph_\beta + \aleph_\beta = 2 \cdot \aleph_\beta = \aleph_\beta$ and the assertion follows. The second part is proved similarly.