Introduction to Set Theory

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- Initial Ordinals
- Addition and Multiplication of Alephs

Subsection 1

Initial Ordinals

Finite Ordinals and Cardinals

- We proved results involving the cardinality $|X|$ of a set X, but we have not defined $|X|$ itself, except in the case when X is finite or countable.
- We now find "representatives" of cardinalities.
- Natural numbers play this role satisfactorily for finite sets.
- We showed that ordinal numbers have many properties of natural numbers, e.g, inductive proofs and recursive constructions.
- However, ordinal numbers do not represent cardinalities; instead, they represent types of well-orderings.
- Since any infinite set can be well-ordered in many different ways, there are many ordinal numbers of the same cardinality;
- E.g., $\omega, \omega+1, \omega+2, \ldots, \omega+\omega, \ldots, \omega\cdot\omega, \omega\cdot\omega+1, \ldots$ are all countable ordinal numbers; i.e., $|\omega| = |\omega+1| = |\omega+\omega| = \cdots = \aleph_0$.
- The good behavior of ordinal numbers of finite cardinalities is due to the fact that all linear orderings of a finite set are isomorphic, and they are well-orderings. Thus, for any finite X, there exists unique ordinal n such that $|n| = |X|$, called the **cardinal number** of X.

Initial Ordinals

- To get representatives for cardinalities of infinite (well-orderable) sets, we take the least ordinal number of any given cardinality as the representative of that cardinality:

Definition (Initial Ordinal)

An ordinal number α is called an **initial ordinal** if it is not equipotent to any $\beta < \alpha$.

- Example:
  - Every natural number is an initial ordinal.
  - ω is an initial ordinal, because ω is not equipotent to any natural number.
  - $\omega + 1$ is not initial, because $|\omega| = |\omega + 1|$.
  - Similarly, none of $\omega + 2, \omega + 3, \omega + \omega, \omega \cdot \omega, \omega^\omega, \ldots$ is initial.

Cardinal Numbers

Theorem

Each well-orderable set X is equipotent to a unique initial ordinal number.

- By a preceding theorem, X is equipotent to some ordinal α. Let α_0 be the least ordinal equipotent to X. Then α_0 is an initial ordinal because $|\alpha_0| = |\beta|$, for some $\beta < \alpha_0$, would imply $|X| = |\beta|$, a contradiction.
  If $\alpha_0 \neq \alpha_1$ are initial ordinals, they cannot be equipotent, because $|\alpha_0| = |\alpha_1|$ and, say, $\alpha_0 < \alpha_1$, would violate the fact that α_1 is initial. This proves the uniqueness.

Definition (Cardinal Number)

If X is a well-orderable set, then the cardinal number of X, denoted $|X|$, is the unique initial ordinal equipotent to X. In particular, $|X| = \omega$ for any countable set X, and $|X| = n$ for any finite set of n elements.

Hartogs Number of a Set

- Are there other initial ordinals besides the natural numbers and ω?
- Let A be any set; A may not be well-orderable itself, but it certainly has some well-orderable subsets; for example, all finite subsets of A are well-orderable.

Definition (Hartogs Number)

For any A, let $h(A)$ be the least ordinal number which is not equipotent to any subset of A. $h(A)$ is called the **Hartogs number** of A.

- By definition, $h(A)$ is the least ordinal α such that $|\alpha| \nleq |A|$.

Lemma

For any A, $h(A)$ is an initial ordinal number.

- Assume that $|\beta| = |h(A)|$ for some $\beta < h(A)$. Then β is equipotent to a subset of A, and β is equipotent to $h(A)$. We conclude that $h(A)$ is equipotent to a subset of A, i.e., $h(A) < h(A)$, a contradiction.

Existence of Hartogs Numbers

- How do we know that the Hartogs number of A exists?

  If all infinite ordinals were countable, $h(\omega)$ would consist of all ordinals!

Lemma

The Hartogs number of A exists for all A.

- By a preceding theorem, for every well-ordered set (W, R) where $W \subseteq A$, there is a unique ordinal α, such that $(\alpha, <)$ is isomorphic to (W, R). By Replacement, there exists a set H such that, for every well-ordering $R \in \mathcal{P}(A \times A)$, its isomorphic ordinal α is in H.

  Claim: H contains all ordinals equipotent to a subset of A.

  If f is a one-to-one function mapping α into A, we set $W = \operatorname{ran} f$ and $R = \{(f(\beta), f(\gamma)) : \beta < \gamma < \alpha\}$. $R \subseteq A \times A$ is then a well-ordering isomorphic to α (by the isomorphism f). These considerations show that $h(A) = \{\alpha \in H : \alpha \text{ is an ordinal equipotent to a subset of } A\}$. Thus, by Axiom Schema of Comprehension, $h(A)$ exists.

The Hierarchy of Omegas

- We can now define a "scale" of larger and larger initial ordinal numbers by transfinite recursion:

Definition (Omegas)

$$\begin{aligned} \omega_0 &= \omega; \\ \omega_{\alpha+1} &= h(\omega_\alpha), \text{ for all } \alpha; \\ \omega_\alpha &= \sup\{\omega_\beta : \beta < \alpha\}, \text{ if } \alpha \text{ is limit } \alpha \neq 0. \end{aligned}$$

- We know that $|\omega_{\alpha+1}| > |\omega_\alpha|$, for each α, and so $|\omega_\alpha| < |\omega_\beta|$ whenever $\alpha < \beta$.

Theorem

(a) ω_α is an infinite initial ordinal number for each α.

(b) If Ω is an infinite initial ordinal number, then $\Omega = \omega_\alpha$ for some α.

Proof of the Theorem

(a) The proof is by induction on α. The only nontrivial case is when α is a limit ordinal. Suppose that $|\omega_\alpha| = |\gamma|$ for some $\gamma < \omega_\alpha$; then there is $\beta < \alpha$ such that $\gamma \leq \omega_\beta$ (by the definition of supremum). But this implies $|\omega_\alpha| = |\gamma| \leq |\omega_\beta| \leq |\omega_\alpha|$ and yields a contradiction.

(b) First, an easy induction shows that $\alpha \leq \omega_\alpha$ for all α. Therefore, for every infinite initial ordinal Ω, there is an ordinal α such that $\Omega < \omega_\alpha$, (for example, $\alpha = \Omega + 1$). Thus, it suffices to prove the following:
Claim: For every infinite initial ordinal $\Omega < \omega_\alpha$, there is some $\gamma < \alpha$ such that $\Omega = \omega_\gamma$.
By induction on α.

- The claim is trivially true for $\alpha = 0$.
- If $\alpha = \beta + 1$, $\Omega < \omega_\alpha = h(\omega_\beta)$ implies that $|\Omega| \leq |\omega_\beta|$ so either $\Omega = \omega_\beta$ and we can let $\gamma = \beta$, or $\Omega < \omega_\beta$ and existence of $\gamma < \beta < \alpha$ follows from the inductive assumption.
- If α is a limit ordinal, $\Omega < \omega_\alpha = \sup\{\omega_\beta : \beta < \alpha\}$ implies that $\Omega < \omega_\beta$ for some $\beta < \alpha$. The inductive assumption again guarantees the existence of some $\gamma < \beta$, such that $\Omega = \omega_\gamma$.

Conclusions

- Every well-orderable set is equipotent to a unique initial ordinal.
- Infinite initial ordinal numbers form a transfinite sequence ω_α with α ranging over all ordinal numbers.
- Infinite initial ordinals are, by definition, the cardinalities of infinite well-orderable sets. It is customary to call these cardinal numbers **alephs**, i.e., we define
$$\aleph_\alpha = \omega_\alpha, \text{ for each } \alpha.$$
- The cardinal number of a well-orderable set is thus either a natural number or an aleph.
- Note that the ordering of cardinal numbers by size defined previously agrees with the ordering of natural numbers and alephs as ordinals by $<$ (i.e., $\in$):
  If $|X| = \aleph_\alpha$ and $|Y| = \aleph_\beta$, then $|X| < |Y|$ if and only if $\aleph_\alpha < \aleph_\beta$ (i.e., $\omega_\alpha \in \omega_\beta$).
- A similar equivalence holds if one or both of $|X|$ and $|Y|$ are natural numbers.

Cardinal and Ordinal Operations

- We have defined addition, multiplication, and exponentiation of cardinal numbers.
- These agree with the corresponding ordinal operations if the ordinals involved are natural numbers but they may differ for infinite ordinals.
- Example:
  - $\omega_0 + \omega_0 \neq \omega_0$ if $+$ stands for the ordinal addition; but $\omega_0 + \omega_0 = \omega_0$ if $+$ stands for the cardinal addition.
  - The addition of cardinal numbers is commutative, but the addition of ordinal numbers is not.
- For clarity, the ω-symbolism is used when the ordinal operations are involved, and the $\aleph$-symbolism for the cardinal operations.
- Thus:
  - $\omega_0 + \omega_0$ and 2^{ω_0} indicate ordinal addition and exponentiation:
    - $\omega_0 + \omega_0 = \sup\{\omega + n : n < \omega_0\} > \omega_0$;
    - $2^{\omega_0} = \sup\{2^n : n < \omega_0\} = \omega_0$.
  - $\aleph_0 + \aleph_0$ and $2^{\aleph_0}$ cardinal operations:
    - $\aleph_0 + \aleph_0 = \aleph_0$;
    - $2^{\aleph_0}$ is uncountable.

Subsection 2

Addition and Multiplication of Alephs

Revisiting Cardinal Addition and Multiplication

- Let κ and λ be cardinal numbers. We have defined $\kappa + \lambda$ as the cardinality of the set $X \cup Y$, where $|X| = \kappa$, $|Y| = \lambda$, and X and Y are disjoint:
$$|X| + |Y| = |X \cup Y|, \text{ if } X \cap Y = \emptyset.$$
This definition does not depend on the choice of X and Y.
- The product $\kappa \cdot \lambda$ has been defined as the cardinality of the cartesian product $X \times Y$, where X and Y are any two sets of respective cardinalities κ and λ:
$$|X| \cdot |Y| = |X \times Y|.$$
This definition is also independent of the choice of X and Y.
- Addition and multiplication satisfy:
$$\begin{array}{cc} \kappa + \lambda = \lambda + \kappa & \kappa \cdot \lambda = \lambda \cdot \kappa \\ \kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu & \kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu \end{array}$$
$$\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$$

Some Examples of Operations involving Alephs

- The arithmetic of infinite numbers differs substantially from the arithmetic of finite numbers.
- In fact, the rules for addition and multiplication of alephs are very simple.
- Example:
  - $\aleph_0 + n = \aleph_0$, for every natural number n.
  - $\aleph_0 + \aleph_0 = \aleph_0$, since the set of all natural numbers is the union of two disjoint countable sets: the set of even numbers and the set of odd numbers.
  - $\aleph_0 \cdot \aleph_0 = \aleph_0$ (The set of all pairs of natural numbers is countable.)
- We prove, next, a general theorem that determines completely the result of addition and multiplication of alephs.

$\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$

Theorem

$\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$, for every α.

- We prove the theorem by transfinite induction. For every α, we construct a certain well-ordering $\prec$ of the set $\omega_\alpha \times \omega_\alpha$ and show, using the induction hypothesis $\aleph_\beta \cdot \aleph_\beta \leq \aleph_\beta$ that the order-type of the well-ordered set $(\omega_\alpha \times \omega_\alpha, \prec)$ is at most ω_α. Then, it follows that $\aleph_\alpha \cdot \aleph_\alpha \leq \aleph_\alpha$ and since $\aleph_\alpha \cdot \aleph_\alpha \geq \aleph_\alpha$, we have $\aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha$.
  We construct the well-ordering $\prec$ of $\omega_\alpha \times \omega_\alpha$ uniformly for all ω_α, i.e., we define a property $\prec$ of pairs of ordinals and show that $\prec$ well-orders $\omega_\alpha \times \omega_\alpha$, for every ω_α.

$(\alpha_1, \alpha_2) \prec (\beta_1, \beta_2)$ if and only if either $\max\{\alpha_1, \alpha_2\} < \max\{\beta_1, \beta_2\}$
or $\max\{\alpha_1, \alpha_2\} = \max\{\beta_1, \beta_2\}$ and $\alpha_1 < \beta_1$
or $\max\{\alpha_1, \alpha_2\} = \max\{\beta_1, \beta_2\}$, $\alpha_1 = \beta_1$ and $\alpha_2 < \beta_2$.

We show that $\prec$ is a well-ordering (of any set of pairs of ordinals).

$\prec$ is an Ordering

- $\prec$ is transitive: Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ be such that $(\alpha_1, \alpha_2) \prec (\beta_1, \beta_2)$ and $(\beta_1, \beta_2) \prec (\gamma_1, \gamma_2)$. By definition $\max\{\alpha_1, \alpha_2\} \leq \max\{\beta_1, \beta_2\} \leq \max\{\gamma_1, \gamma_2\}$, whence $\max\{\alpha_1, \alpha_2\} \leq \max\{\gamma_1, \gamma_2\}$.
    - If $\max\{\alpha_1, \alpha_2\} < \max\{\gamma_1, \gamma_2\}$, then $(\alpha_1, \alpha_2) \prec (\gamma_1, \gamma_2)$.
    - If $\max\{\alpha_1, \alpha_2\} = \max\{\beta_1, \beta_2\} = \max\{\gamma_1, \gamma_2\}$, then we have $\alpha_1 \leq \beta_1 \leq \gamma_1$, and so $\alpha_1 \leq \gamma_1$.
        - If $\alpha_1 < \gamma_1$, then $(\alpha_1, \alpha_2) \prec (\gamma_1, \gamma_2)$;
        - Otherwise, we have $\alpha_1 = \beta_1 = \gamma_1$. In this last case, $\max\{\alpha_1, \alpha_2\} = \max\{\beta_1, \beta_2\} = \max\{\gamma_1, \gamma_2\}$, and $\alpha_1 = \beta_1 = \gamma_1$, so, necessarily, $\alpha_2 < \beta_2 < \gamma_2$, and it follows again that $(\alpha_1, \alpha_2) \prec (\gamma_1, \gamma_2)$.

$\prec$ is Linear

- We verify that for any $\alpha_1, \alpha_2, \beta_1, \beta_2$, either

$$(\alpha_1, \alpha_2) \prec (\beta_1, \beta_2) \text{ or } (\beta_1, \beta_2) \prec (\alpha_1, \alpha_2) \text{ or } (\alpha_1, \alpha_2) = (\beta_1, \beta_2)$$

  and that these three cases are mutually exclusive. This follows directly from the definition:

  Given (α_1, α_2) and (β_1, β_2)
  - we first compare $\max\{\alpha_1, \alpha_2\}$ and $\max\{\beta_1, \beta_2\}$,
  - then α_1 and β_1
  - and last the ordinals α_2 and β_2.

$\prec$ is a Well-Ordering

- $\prec$ is a well-ordering: Let X be a nonempty set of pairs of ordinals. We find the $\prec$-least element of X. Let δ be the least maximum of the pairs in X, i.e., let δ be least element in $\{\max\{\alpha,\beta\}:(\alpha,\beta)\in X\}$. Let $Y=\{(\alpha,\beta)\in X:\max\{\alpha,\beta\}=\delta\}$. The set Y is a nonempty subset of X, and, for every $(\alpha,\beta)\in Y$, we have $\max\{\alpha,\beta\}=\delta$. Moreover, $\delta<\max\{\alpha',\beta'\}$, for any $(\alpha',\beta')\in X-Y$, and hence $(\alpha,\beta)\prec(\alpha',\beta')$ whenever $(\alpha,\beta)\in Y$ and $(\alpha',\beta')\in X-Y$. Therefore, the least element of Y, if it exists, is also the least element of X. Now let α_0 be the least ordinal in the set $\{\alpha:(\alpha,\beta)\in Y \text{ for some } \beta\}$ and let $Z=\{(\alpha,\beta)\in Y:\alpha=\alpha_0\}$. The set Z is a nonempty subset of Y. Also $(\alpha,\beta)\prec(\alpha',\beta')$ whenever $(\alpha,\beta)\in Z$ and $(\alpha',\beta')\in Y-Z$.

  Finally, let β_0 be the least ordinal in the set $\{\beta:(\alpha_0,\beta)\in Z\}$. Clearly, (α_0,β_0) is the least element of Z. It follows that (α_0,β_0) is the least element of X.

Finishing the Proof

- Having shown that $\prec$ is a well-ordering of $\omega_\alpha \times \omega_\alpha$ for every α, we use this well-ordering to prove, by transfinite induction on α, that $|\omega_\alpha \times \omega_\alpha| \leq \aleph_\alpha$, i.e., $\aleph_\alpha \cdot \aleph_\alpha \leq \aleph_\alpha$.
  - For $\alpha = 0$, we know that $\aleph_0 \cdot \aleph_0 = \aleph_0$.
  - So let $\alpha > 0$, and let us assume that $\aleph_\beta \cdot \aleph_\beta \leq \aleph_\beta$, for all $\beta < \alpha$. We prove that $|\omega_\alpha \times \omega_\alpha| \leq \aleph_\alpha$. If suffices to show that the order-type of the well-ordered set $(\omega_\alpha \times \omega_\alpha, \prec)$ is at most ω_α.
    If the order-type of $(\omega_\alpha \times \omega_\alpha, \prec)$ were greater than ω_α, then there would exist $(\alpha_1, \alpha_2) \in \omega_\alpha \times \omega_\alpha$, such that the cardinality of the set $X = \{(\xi_1, \xi_2) \in \omega_\alpha \times \omega_\alpha : (\xi_1, \xi_2) \prec (\alpha_1, \alpha_2)\}$ is at least $\aleph_\alpha$. Thus, it suffices to prove that, for any $(\alpha_1, \alpha_2) \in \omega_\alpha \times \omega_\alpha$, we have $|X| < \aleph_\alpha$.
    Let $\beta = \max\{\alpha_1, \alpha_2\} + 1$. Then $\beta \in \omega_\alpha$ and, for every $(\xi_1, \xi_2) \in X$, we have $\max\{\xi_1, \xi_2\} \leq \max\{\alpha_1, \alpha_2\} < \beta$, so $\xi_1 \in \beta$ and $\xi_2 \in \beta$, i.e., $X \subseteq \beta \times \beta$.
    Let $\gamma < \alpha$ be such that $|\beta| \leq \aleph_\gamma$. Then $|X| \leq |\beta \times \beta| = |\beta| \cdot |\beta| \leq \aleph_\gamma \cdot \aleph_\gamma$ and $\aleph_\gamma \cdot \aleph_\gamma \leq \aleph_\gamma$, by the induction hypothesis. Thus, $|X| \leq \aleph_\gamma$, and, hence, $|X| < \aleph_\alpha$.

Rules of Cardinal Arithmetic

Corollary

For every α and β such that $\alpha \leq \beta$, we have $\aleph_\alpha \cdot \aleph_\beta = \aleph_\beta$.
Also, $n \cdot \aleph_\alpha = \aleph_\alpha$, for every positive natural number n.

- If $\alpha \leq \beta$, then
  - $\aleph_\beta = 1 \cdot \aleph_\beta \leq \aleph_\alpha \cdot \aleph_\beta$
  - $\aleph_\alpha \cdot \aleph_\beta \leq \aleph_\beta \cdot \aleph_\beta = \aleph_\beta$, by the theorem.

  Thus by the Cantor-Bernstein Theorem $\aleph_\alpha \cdot \aleph_\beta = \aleph_\beta$. The equality $n \cdot \aleph_\alpha = \aleph_\alpha$ is proved similarly.

Corollary

For every α and β such that $\alpha \leq \beta$, we have $\aleph_\alpha + \aleph_\beta = \aleph_\beta$.
Also, $n + \aleph_\alpha = \aleph_\alpha$, for all natural numbers n.

- If $\alpha \leq \beta$, then $\aleph_\beta \leq \aleph_\alpha + \aleph_\beta \leq \aleph_\beta + \aleph_\beta = 2 \cdot \aleph_\beta = \aleph_\beta$ and the assertion follows. The second part is proved similarly.