Introduction to Set Theory

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1. Arithmetic of Cardinal Numbers
   - Infinite Sums of Cardinal Numbers
   - Infinite Products of Cardinal Numbers
   - König's Theorem

Subsection 1

Infinite Sums of Cardinal Numbers

Infinite Sum of Cardinal Numbers

- We define sums and products of infinitely many cardinal numbers.
- E.g., it is natural to expect that $\underbrace{1+1+\cdots}_{\aleph_0 \text{ times}} = \aleph_0$ or, more generally, $\underbrace{\kappa+\kappa+\cdots}_{\lambda \text{ times}} = \kappa\cdot\lambda$.
- The sum of two cardinals κ_1 and κ_2 is the cardinality of $A_1 \cup A_2$, where A_1 and A_2 are disjoint sets such that $|A_1| = \kappa_1$ and $|A_2| = \kappa_2$.

Definition (Sum of Infinitely Many Cardinals)

Let $\langle A_i : i \in I\rangle$ be a system of mutually disjoint sets and $|A_i| = \kappa_i$, for all $i \in I$. We define the **sum** of $\langle \kappa_i : i \in I\rangle$ by $\sum_{i\in I}\kappa_i = |\bigcup_{i\in I} A_i|$.

- The definition of $\sum_{i\in I}\kappa_i$ uses particular sets A_i, $i \in I$. Unlike in the finite case one needs the Axiom of Choice in order to prove the independence of the sum from the choice of the particular sets A_i.
- Without the Axiom of Choice, there may exist $\langle A_n : n \in \mathbb{N}\rangle$, $\langle A_n' : n \in \mathbb{N}\rangle$, of mutually disjoint sets, with $|A_n| = 2$ and $|A_n'| = 2$, but $\bigcup_{n=0}^{\infty} A_n$ not equipotent to $\bigcup_{n=0}^{\infty} A_n'$.

Independence of the Choice of Sets

- The Axiom of Choice is assumed from now on.

Lemma

If $\langle A_i : i \in I\rangle$ and $\langle A'_i : i \in I\rangle$ are systems of mutually disjoint sets, such that $|A_i| = |A'_i|$, for all $i \in I$, then $|\bigcup_{i\in I} A_i| = |\bigcup_{i\in I} A'_i|$.

- For each $i \in I$, choose a one-to-one mapping f_i of A_i onto A'_i. Then $f = \bigcup_{i\in I} f_i$ is a one-to-one mapping of $\bigcup_{i\in I} A_i$ onto $\bigcup_{i\in I} A'_i$.
- This lemma makes the definition of $\sum_{i\in I} \kappa_i$ legitimate.
- Since infinite unions of sets satisfy the associative law, it follows that the infinite sums of cardinals are also associative.
- The operation $\sum$ has other reasonable properties:
  - If $\kappa_i \leq \lambda_i$, for all $i \in I$, then $\sum_{i\in I} \kappa_i \leq \sum_{i\in I} \lambda_i$.
  - However, if $\kappa_i < \lambda_i$, for all $i \in I$, it does not necessarily follow that $\sum_{i\in I} \kappa_i < \sum_{i\in I} \lambda_i$.

Theorem for Infinite Sum

- If $\kappa_i = \kappa$, for all $i \in \lambda$, then $\sum_{i\in\lambda} \kappa_i = \underbrace{\kappa + \kappa + \cdots}_{\lambda \text{ times}} = \kappa \cdot \lambda$.
- It is not very difficult to evaluate infinite sums, e.g., consider $\sum_{n\in\mathbb{N}} n = 1 + 2 + 3 + \cdots$. This sum is equal to $\aleph_0$.

Theorem

Let λ be an infinite cardinal, $\kappa_\alpha, \alpha < \lambda$, be nonzero cardinal numbers, and $\kappa = \sup\{\kappa_\alpha : \alpha < \lambda\}$. Then $\sum_{\alpha<\lambda} \kappa_\alpha = \lambda \cdot \kappa = \lambda \cdot \sup\{\kappa_\alpha : \alpha < \lambda\}$.

- On the one hand, $\kappa_\alpha \leq \kappa$, for each $\alpha < \lambda$. So $\sum_{\alpha<\lambda} \kappa_\alpha \leq \sum_{\alpha<\lambda} \kappa = \kappa \cdot \lambda$.

  On the other hand, $\lambda = \sum_{\alpha<\lambda} 1 \leq \sum_{\alpha<\lambda} \kappa_\alpha$. Also $\kappa \leq \sum_{\alpha<\lambda} \kappa_\alpha$: The sum $\sum_{\alpha<\lambda} \kappa_\alpha$ is an upper bound of the κ_α's and κ is the least upper bound. Now since both κ and λ are $\leq \sum_{\alpha<\lambda} \kappa_\alpha$, it follows that $\kappa \cdot \lambda$, which is the greater of the two, is also $\leq \sum_{\alpha<\lambda} \kappa_\alpha$. The conclusion is now a consequence of the Cantor-Bernstein Theorem.

Subsection 2

Infinite Products of Cardinal Numbers

Product of Infinitely Many Cardinal Numbers

Corollary

If κ_i, $i \in I$, are cardinal numbers, and if $|I| \leq \sup\{\kappa_i : i \in I\}$, then

$$\sum_{i \in I} \kappa_i = \sup_{i \in I} \kappa_i.$$

Note the assumption is satisfied if all the κ_i's are mutually distinct.

- The product of two cardinals κ_1 and κ_2 is the cardinality of the cartesian product $A_1 \times A_2$, where A_1 and A_2 are arbitrary sets such that $|A_1| = \kappa_1$ and $|A_2| = \kappa_2$.

Definition (Product of Infinitely Many Cardinals)

Let $\langle A_i : i \in I \rangle$ be a family of sets, such that $|A_i| = \kappa_i$, for all $i \in I$. We define the **product** of $\langle \kappa_i : i \in I \rangle$ by $\prod_{i \in I} \kappa_i = |\prod_{i \in I} A_i|$.

- We use the same symbol for the product of cardinals (the left-hand side) as for the cartesian product of the indexed family $\langle A_i : i \in I \rangle$.

Independence from Choice of Sets

- The definition of $\prod_{i\in I}\kappa_i$ does not depend on the particular sets A_i:

Lemma

If $\langle A_i : i\in I\rangle$ and $\langle A'_i : i\in I\rangle$ are such that $|A_i| = |A'_i|$, for all $i\in I$, then $|\prod_{i\in I}A_i| = |\prod_{i\in I}A'_i|$.

- For each $i\in I$, choose a one-to-one mapping f_i of A_i onto A'_i. Let f be the function on $\prod_{i\in I}A_i$ defined as follows: If $x = \langle x_i : i\in I\rangle \in \prod_{i\in I}A_i$, let $f(x) = \langle f_i(x_i) : i\in I\rangle$. Then f is a one-to-one mapping of $\prod_{i\in I}A_i$ onto $\prod_{i\in I}A'_i$
- The infinite products share many properties of finite products:
  - If at least one κ_i is 0, then $\prod_{i\in I}\kappa_i = 0$.
  - They satisfy the associative law.
  - If $\kappa_i \leq \lambda_i$, for all $i\in I$, then $\prod_{i\in I}\kappa_i \leq \prod_{i\in I}\lambda_i$.
  - If all the factors $\kappa_i = \kappa$, then we have $\prod_{i\in\lambda}\kappa_i = \underbrace{\kappa\cdot\kappa\cdot\cdots}_{\lambda\text{ times}} = \kappa^\lambda$.

Properties of Exponentiation

- If all the factors $\kappa_i = \kappa$, then we have $\prod_{i \in \lambda} \kappa_i = \underbrace{\kappa \cdot \kappa \cdot \cdots}_{\lambda \text{ times}} = \kappa^\lambda$.
- The following rules, involving exponentiation, also generalize from the finite to the infinite case:

$$\left(\prod_{i \in I} \kappa_i\right)^\lambda = \prod_{i \in I} (\kappa_i^\lambda)$$

$$\prod_{i \in I} (\kappa^{\lambda_i}) = \kappa^{\sum_{i \in I} \lambda_i}.$$

Evaluation of Infinite Products

- Infinite products are more difficult to evaluate than infinite sums.
- In some special cases, for instance when evaluating the product $\prod_{\alpha<\lambda} \kappa_\alpha$, of an increasing sequence $\langle \kappa_\alpha : \alpha < \lambda \rangle$ of cardinals, some simple rules can be proved.
- Consider the following very special case:

$$\prod_{n=1}^{\infty} n = 1 \cdot 2 \cdot 3 \cdot \cdots .$$

First, note that $\prod_{n=1}^{\infty} n \leq \prod_{i=1}^{\infty} \aleph_0 = \aleph_0^{\aleph_0} = 2^{\aleph_0}$.

Conversely, we have $2^{\aleph_0} \leq \prod_{n=1}^{\infty} 2 \leq \prod_{n=2}^{\infty} n = \prod_{n=1}^{\infty} n$.

Therefore, $1 \cdot 2 \cdot 3 \cdot \cdots = 2^{\aleph_0}$.

Subsection 3

König's Theorem

König's Theorem

König's Theorem

If κ_i and λ_i, $i \in I$, are cardinal numbers, and if $\kappa_i < \lambda_i$, for all $i \in I$, then $\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$.

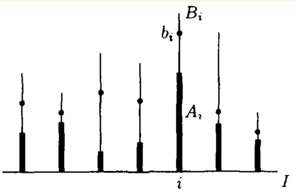
- First, let us show $\sum_{i \in I} \kappa_i \leq \prod_{i \in I} \lambda_i$. Let $\langle A_i : i \in I \rangle$ and $\langle B_i : i \in I \rangle$ be such that $|A_i| = \kappa_i$, and $|B_i| = \lambda_i$, for all $i \in I$ and the A_i's are mutually disjoint. We may further assume that $A_i \subset B_i$, for all $i \in I$. We find a one-to-one mapping f of $\bigcup_{i \in I} A_i$ into $\prod_{i \in I} B_i$.
  We choose $d_i \in B_i - A_i$, for each $i \in I$, and define a function f as follows: For each $x \in \bigcup_{i \in I} A_i$, let i_x be the unique $i \in I$, such that $x \in A_i$. Let $f(x) = \langle a_i : i \in I \rangle$, where $a_i = \begin{cases} x, & \text{if } i = i_x \\ d_i, & \text{if } i \neq i_x \end{cases}$. If $x \neq y$, let $f(x) = a$ and $f(y) = b$ and let us show that $a \neq b$.
  - If $i_x = i_y = i$, then $a_i = x$ while $b_i = y$.
  - If $i_x \neq i_y = i$, then $a_i = d_i \notin A_i$ while $b_i = y \in A_i$.

  In either case $f(x) \neq f(y)$.

Proof of König's Theorem

- We show that $\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$. Let $B_i, i \in I$, be such that $|B_i| = \lambda_i$, for all $i \in I$. If the product $\prod_{i \in I} \lambda_i$ were equal to the sum $\sum_{i \in I} \kappa_i$, we could find mutually disjoint subsets X_i, of the cartesian product $\prod_{i \in I} B_i$, such thai $|X_i| = \kappa_i$, for all i, and $\bigcup_{i \in I} X_i = \prod_{i \in I} B_i$. We show that this is impossible.

  For each $i \in I$, let $A_i = \{a_i : a \in X_i\}$:

  For every $i \in I$, we have $A_i \subset B_i$, since $|A_i| \leq |X_i| = \kappa_i < \lambda_i = |B_i|$. Hence there exists $b_i \in B_i$, such that $b_i \notin A_i$. Let $b = \langle b_i : i \in I \rangle$.

  Now we can easily show that b is not a member of any X_i, $i \in I$: For any $i \in I$, $b_i \notin A_i$, and so $b \notin X_i$. Hence $\bigcup_{i \in I} X_i$ is not the whole set $\prod_{i \in I} B_i$, a contradiction.

Cantor's Theorem as a Special Case of König's Theorem

- König's Theorem and its proof are generalizations of Cantor's Theorem which states that $2^\kappa > \kappa$, for all κ.
- If we express κ as the infinite sum

$$\kappa = 1 + 1 + 1 + \cdots \quad (\kappa \text{ times})$$

and 2^κ as the infinite product

$$2^\kappa = 2 \cdot 2 \cdot 2 \cdot \cdots \quad (\kappa \text{ times}),$$

we can apply König's Theorem (since $1 < 2$) and obtain

$$\kappa = \sum_{i \in \kappa} 1 < \prod_{i \in \kappa} 2 = 2^\kappa.$$