College Trigonometry

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LSSU Math 131

1. Functions and Graphs
   - Equations and Inequalities
   - Inverse Functions

Subsection 1

Equations and Inequalities

Solving a Linear Equation

- Solve $5x - 11 = 3$;

$$\begin{aligned} & 5x - 11 = 3 \\ \Rightarrow \quad & 5x - 11 + 11 = 3 + 11 \\ \Rightarrow \quad & 5x = 14 \\ \Rightarrow \quad & \frac{1}{5} \cdot 5x = \frac{1}{5} \cdot 14 \\ \Rightarrow \quad & x = \frac{14}{5}; \end{aligned}$$

Solving a Literal Equation

- Solve $\frac{by}{c-y} = ax$ for y;

$$\begin{aligned} & \frac{by}{c-y} = ax \\ \Rightarrow \quad & by = ax(c-y) \\ \Rightarrow \quad & by = acx - axy \\ \Rightarrow \quad & axy + by = acx \\ \Rightarrow \quad & y(ax+b) = acx \\ \Rightarrow \quad & y = \frac{acx}{ax+b}; \end{aligned}$$

Quadratic Equations and the Quadratic Formula

- A **quadratic equation** is one of the form
$$ax^2 + bx + c = 0, \quad a \neq 0;$$
- The solution of $ax^2 + bx + c = 0$, with $a \neq 0$, is given by the **quadratic formula**
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a};$$

Example: Solve the equation $2x^2 - 4x + 1 = 0$ using the quadratic formula;
We have $a = 2, \quad b = -4, \quad c = 1$; Therefore,
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \frac{4 \pm \sqrt{8}}{4} = \frac{4 \pm 2\sqrt{2}}{4} = \frac{2(2 \pm \sqrt{2})}{4} = \frac{2 \pm \sqrt{2}}{2};$$

The Zero Factor Property (Zero Product Principle)

- If a and b are algebraic expressions, then

$$ab = 0 \quad \text{if and only if} \quad a = 0 \text{ or } b = 0;$$

- The Zero-Factor Property may be used to solve equations by factoring;

  Example: Solve $2x^2 + x - 6 = 0$;

$$\begin{aligned} & 2x^2 + x - 6 = 0 \\ \Rightarrow \quad & (x+2)(2x-3) = 0 \\ \Rightarrow \quad & x + 2 = 0 \text{ or } 2x - 3 = 0 \\ \Rightarrow \quad & x = -2 \text{ or } x = \frac{3}{2}; \end{aligned}$$

Applying the Zero-Factor Property

- Example: Solve $(2x-1)(x-3)=x^2+x-4$;

$$\begin{aligned}
&(2x-1)(x-3)=x^2+x-4\\
&\Rightarrow\quad 2x^2-6x-x+3=x^2+x-4\\
&\Rightarrow\quad 2x^2-7x+3=x^2+x-4\\
&\Rightarrow\quad x^2-8x+7=0\\
&\Rightarrow\quad (x-1)(x-7)=0\\
&\Rightarrow\quad x-1=0 \text{ or } x-7=0\\
&\Rightarrow\quad x=1 \text{ or } x=7;
\end{aligned}$$

Solving Linear Inequalities

- The following properties are used to solve inequalities:

$$\textbf{Addition Property:} \quad a < b \quad \Rightarrow \quad a + c < b + c;$$
$$\textbf{Multiplication Property:} \quad a < b \quad \Rightarrow \quad \begin{cases} ac < bc, & \text{if } c > 0; \\ ac > bc, & \text{if } c < 0; \end{cases}$$

- Example: Solve the inequality $2(x+3) < 4x + 10$ and write the solution set in interval notation;

$$\begin{aligned} & 2(x+3) < 4x + 10 \\ \Rightarrow \quad & 2x + 6 < 4x + 10 \\ \Rightarrow \quad & -2x < 4 \\ \Rightarrow \quad & x > -2; \end{aligned}$$

In interval notation, we get: x in $(-2, \infty)$;

Solving Polynomial Inequalities

The Sign Property for Polynomials

Nonzero polynomials have the property that for any value of x between two consecutive real zeros, either all values of the polynomial are positive or all values of the polynomial are negative.

- Example: Solve the inequality $x^2 + 3x - 4 < 0$;
  First find the critical points by solving the equation:

$$x^2 + 3x - 4 = 0 \Rightarrow (x+4)(x-1) = 0 \Rightarrow x = -4 \text{ or } x = 1;$$

Then, form the sign table for $f(x) = x^2 + 3x - 4$ as follows:

	$x < -4$	$-4 < x < 1$	$x > 1$
	$f(-5) = 6 > 0$	$f(0) = -4 < 0$	$f(2) = 6 > 0$
$f(x)$	$+$	$-$	$+$

Since we want $f(x) < 0$, we must pick the interval with the "-"; So the solution set is $-4 < x < 1$ or in interval notation: x in $(-4, 1)$;

One More Example

- Example: Solve the inequality $12x^2 + 8x \geq 15$;
  We must solve $12x^2 + 8x \geq 15 \Rightarrow 12x^2 + 8x - 15 \geq 0$; First find the critical points by solving the equation:

$$12x^2 + 8x - 15 = 0 \ \Rightarrow \ (2x+3)(6x-5) = 0$$
$$\Rightarrow \ 2x + 3 = 0 \text{ or } 6x - 5 = 0 \ \Rightarrow \ x = -\frac{3}{2} \text{ or } x = \frac{5}{6};$$

  Then, form the sign table for $f(x) = 12x^2 + 8x - 15$ as follows:

	$x < -\frac{3}{2}$	$-\frac{3}{2} < x < \frac{5}{6}$	$x > \frac{5}{6}$
	$f(-2) = 17 > 0$	$f(0) = -15 < 0$	$f(1) = 5 > 0$
$f(x)$	$+$	$-$	$+$

  Since we want $f(x) \geq 0$, we must pick the intervals with the "+" including endpoints! So the solution set is $x \leq -\frac{3}{2}$ or $x \geq \frac{5}{6}$ or in interval notation: x in $(-\infty, -\frac{3}{2}] \cup [\frac{5}{6}, \infty)$;

Method for Solving Polynomial Inequalities

Summary of the Method

1. Rewrite the inequality so that one side is a nonzero polynomial and the other side is zero;
2. Find the real zeros of the polynomial; These are the **critical values** of the inequality;
3. Use test points to determine the sign of the polynomial in each of the intervals formed by the critical values;
4. Include either the intervals signed "+" if the polynomial is to be > 0 or those signed "-" if the polynomial is to be < 0;
5. Include endpoints in the solution set if the polynomial can be ≥ 0 or ≤ 0;

Absolute Value Inequalities

Properties of Absolute Value Inequalities

Suppose E is a variable expression and k a nonnegative real number;

- $|E| \leq k$ if and only if $-k \leq E \leq k$;
- $|E| \geq k$ if and only if $E \leq -k$ or $E \geq k$;

- Example: Solve the inequality $|2 - 3x| < 7$;

$$\begin{aligned} & |2-3x| < 7 \\ & \Rightarrow \quad -7 < 2 - 3x < 7 \\ & \stackrel{-2}{\Rightarrow} \quad -9 < -3x < 5 \\ & \stackrel{\div(-3)}{\Rightarrow} \quad 3 > x > -\tfrac{5}{3}; \end{aligned}$$

Therefore, in interval notation, x in $(-\frac{5}{3}, 3)$;

An Additional Example

- Example: Solve the inequality $|4x-3| \geq 5$;

$$\begin{array}{ll} |4x-3| \geq 5 & \\ \Rightarrow & 4x-3 \leq -5 \text{ or } 4x-3 \geq 5 \\ \stackrel{+3}{\Rightarrow} & 4x \leq -2 \text{ or } 4x \geq 8 \\ \stackrel{\div 4}{\Rightarrow} & x \leq -\frac{1}{2} \text{ or } x \geq 2; \end{array}$$

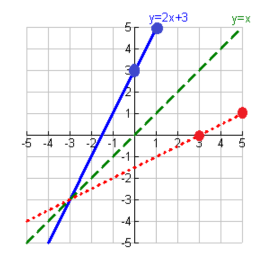
Therefore, in interval notation, x in $(-\infty, -\frac{1}{2}] \cup [2, \infty)$;

Subsection 2

Inverse Functions

Intuition Behind Inverse Functions

- Consider $f(x) = 2x + 3$; "Multiply by 2 and add 3"; For instance $f(0) = 3$ and $f(1) = 5$;
- Now consider $g(x) = \frac{x-3}{2}$; "Subtract 3 and divide by 2"; For instance $g(3) = 0$ and $g(5) = 1$;

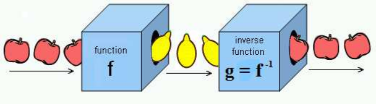


- We now compute

$$g(f(x)) = g(2x+3) = \frac{(2x+3)-3}{2} = \frac{2x}{2} = x;$$

$$f(g(x)) = f(\frac{x-3}{2}) = 2 \cdot \frac{x-3}{2} + 3 = x - 3 + 3 = x;$$

- It is clear that g undoes what f does and vice-versa!



Inverse Functions

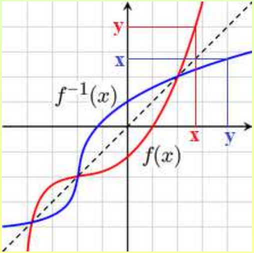
- The function g is called the **inverse function** of the function f and denoted $g = f^{-1}$ if

$$f(x) = y \quad \text{if and only if} \quad f^{-1}(y) = x;$$

- Note that this is the same as saying that

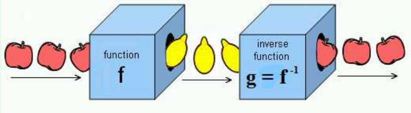
$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y;$$

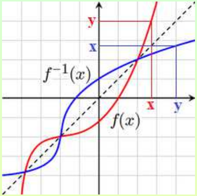
- Since $x \overset{f}{\mapsto} y$ is equivalent to $y \overset{f^{-1}}{\mapsto} x$ a point (x, y) is on the graph of f if and only if (y, x) is on the graph of f^{-1}; This implies symmetry of the graphs with respect to the line $y = x$!
- Note, also, that $\mathrm{Dom}(f^{-1}) = \mathrm{Ran}(f)$ and $\mathrm{Ran}(f^{-1}) = \mathrm{Dom}(f)$;



Summary of Properties of Inverse Functions

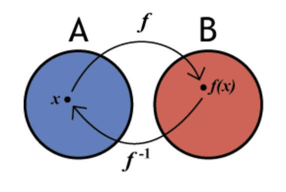
- Application of $f^{-1}(y)$ reverses the effect of $f(x)$ and vice-versa;



- 

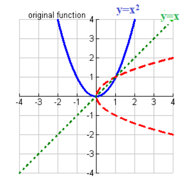
  Graphs of f and f^{-1} are symmetric with respect to $y = x$;

- Domain of f becomes range of f^{-1} and vice-versa;

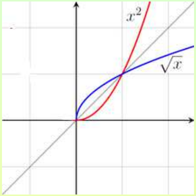


Do All Functions Have Inverses?

- If the graph of a function $y = f(x)$ does not pass the horizontal line test, then its symmetric graph with respect to $y = x$ does not pass the vertical line text! Thus, such a function cannot have an inverse!



- To fix this, we "cut" $f(x)$ by changing its domain to force its graph to pass the horizontal line test! Then, e.g., $f(x) = x^2$ with domain $[0, \infty)$ does have an inverse: The function $f^{-1}(x) = \sqrt{x}$ with domain $[0, \infty)$ also!



Condition for Existence of an Inverse Function

A function f has an inverse if and only if it is **one-to-one**, i.e., its graph passes the horizontal line test.

How to Find an Inverse Function

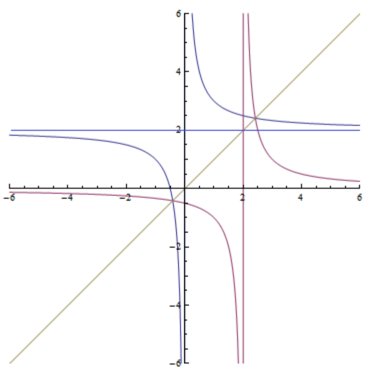
- To find a formula for the inverse $f^{-1}(x)$ of a given function $f(x)$:
  1. Substitute y in for $f(x)$;
  2. Interchange x and y;
  3. Solve, if possible for y in terms of x;
  4. Substitute $f^{-1}(x)$ in for y;
- Example: Find a formula for the inverse $f^{-1}(x)$ if $f(x) = 3x + 2$;

$$\begin{aligned} & f(x) = 3x + 2 \\ \Rightarrow \quad & y = 3x + 2 \\ \Rightarrow \quad & x = 3y + 2 \\ \Rightarrow \quad & 3y = x - 2 \quad \Rightarrow \quad y = \frac{x-2}{3} \\ \Rightarrow \quad & f^{-1}(x) = \frac{x-2}{3}; \end{aligned}$$

Finding an Inverse

- Example: Find a formula for the inverse $f^{-1}(x)$ if $f(x) = \frac{2x+1}{x}$, with $x \neq 0$;

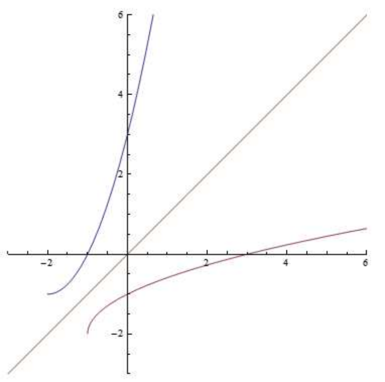
$$\begin{aligned} f(x) &= \frac{2x+1}{x} \\ \Rightarrow \quad y &= \frac{2x+1}{x} \\ \Rightarrow \quad x &= \frac{2y+1}{y} \\ \Rightarrow \quad yx &= 2y+1 \\ \Rightarrow \quad yx - 2y &= 1 \\ \Rightarrow \quad y(x-2) &= 1 \\ \Rightarrow \quad y &= \frac{1}{x-2} \\ \Rightarrow \quad f^{-1}(x) &= \frac{1}{x-2}, \ x \neq 2; \end{aligned}$$



Finding an Inverse of a Function with a Restricted Domain

- Example: Find a formula for the inverse $f^{-1}(x)$ if $f(x) = x^2 + 4x + 3$, with $x \geq -2$;

$$\begin{aligned} & f(x) = x^2 + 4x + 3 \\ \Rightarrow \quad & y = x^2 + 4x + 3 \\ \Rightarrow \quad & x = y^2 + 4y + 3 \\ \Rightarrow \quad & x = (y+2)^2 - 1 \\ \Rightarrow \quad & x + 1 = (y+2)^2 \\ \Rightarrow \quad & y + 2 = \pm\sqrt{x+1} \\ \Rightarrow \quad & y = \pm\sqrt{x+1} - 2 \\ \Rightarrow \quad & f^{-1}(x) = \sqrt{x+1} - 2, \\ & \qquad x \neq -1; \end{aligned}$$



An Application: Retail Pricing

- Suppose $S(x) = \frac{4}{3}x + 100$ determines the retail price of a gold bracelet when the whole sale price is x;
  - If the merchant paid \$672 wholesale, what is the retail price?

$$S(672) = \frac{4}{3} \cdot 672 + 100 = 896 + 100 = \$996;$$

  - Find S^{-1} and use it to determine the merchant's wholesale price for a bracelet that retails at \$1,596;

$$S(x) = \tfrac{4}{3}x + 100 \quad \Rightarrow \quad y = \tfrac{4}{3}x + 1003 \quad \Rightarrow \quad x = \tfrac{4}{3}y + 100$$
$$\Rightarrow \quad \tfrac{4}{3}y = x - 100 \quad \Rightarrow \quad y = \tfrac{3}{4}(x - 100) \quad \Rightarrow \quad y = \tfrac{3}{4}x - 75$$
$$\Rightarrow \quad S^{-1}(x) = \tfrac{3}{4}x - 75;$$

So $S^{-1}(1596) = \frac{3}{4} \cdot 1596 - 75 = 1197 - 75 = \$1,122;$