College Trigonometry

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LSSU Math 131

Subsection 1

Angles and Arcs

Terminology on Angles

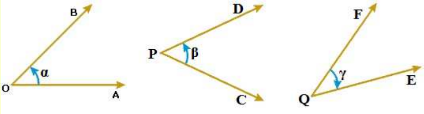
- The two parts into which a point P on a line separates the line are called **half-lines** or **rays**;
- The half-line formed by P that includes a point A on the line is denoted by $\overrightarrow{PA}$; P is the **endpoint** of $\overrightarrow{PA}$;

Definition of Angle

An **angle** is formed by rotating a given ray about its endpoint to some terminal position; The original ray is called the **initial side** of the angle and the second ray is the **terminal side**; The common endpoint is the **vertex** of the angle.

- Angles formed by a counterclockwise rotation are **positive angles** and those formed by a clockwise rotation are **negative angles**;
- Notation for angles:

  $\angle\alpha = \angle AOB$; $\angle\beta = \angle CPD$; $\angle\gamma = \angle FQE$;

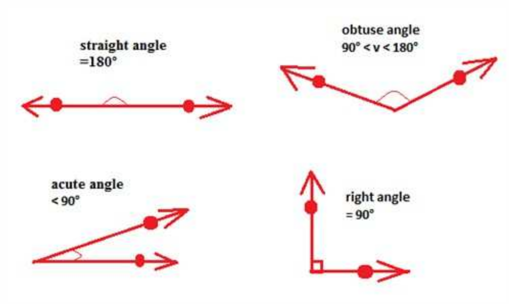


Degree Measure

Degree Measure

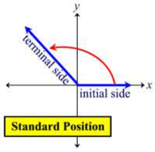
An angle formed by rotating the initial side counterclockwise exactly once until it coincides with itself is defined to have a measure of 360 degrees, written 360°; Therefore, **one degree** is the measure of an angle formed by rotating a ray $\frac{1}{360}$ of a complete revolution and it is written 1°;

- Angles are classified according to their degree measure:

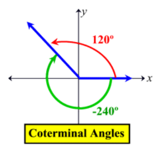


Standard Position, Complementary and Supplementary

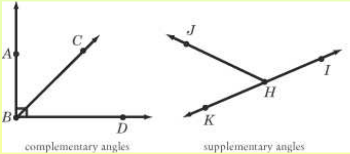
- An angle superimposed in a Cartesian coordinate system is in **standard position** if its vertex is at the origin and its initial side is on the positive x-axis:



- Two angles are **coterminal** if they share the same terminal side when placed in standard position;



- Two positive angles are **complementary** if the sum of their measures is 90° and they are **supplementary** if the sum of their measures is 180°;



Simple Examples

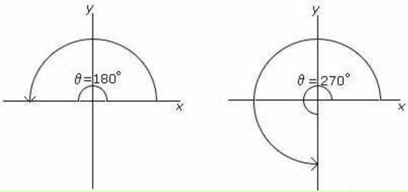
- Find, if possible the measure of the complement and the supplement of $\theta = 40^\circ$;

$$\begin{aligned} \text{Comp}(\theta) &= 90^\circ - 40^\circ = 50^\circ; \\ \text{Supp}(\theta) &= 180^\circ - 40^\circ = 140^\circ; \end{aligned}$$

- Find, if possible the measure of the complement and the supplement of $\theta = 125^\circ$;
  - θ does not have a complement since it is an angle with measure greater than 90°;
  - $\text{Supp}(\theta) = 180^\circ - 125^\circ = 55^\circ$;
- Are the two acute angles of any right triangle complementary angles? Yes! because their sum is 90°;

Quadrantal Angles

- An angle is a **quadrantal angle** if its terminal side in standard position lies on a coordinate axis;



For instance, the $90^\circ, 180^\circ$ and 270° angles are all quadrantal angles;

- Recall that two angles are coterminal if they share the same terminal side when placed in standard position;

Measures of Coterminal Angles

Given an angle $\angle\theta$ in standard position with measure x°, then the measures of the angles that are coterminal with $\angle\theta$ are given by $x^\circ + k \cdot 360^\circ$, where k is an integer.

An Example

- Assume that the following angles are in standard position; Determine the measure of the positive angle with measure less than 360° that is coterminal with the given angle and classify the angle by quadrant;
  - $\alpha = 550^\circ$;
    We have $\alpha = 550^\circ = 360^\circ + 190^\circ$; Therefore, α is coterminal with the 190° angle and the terminal side lies in Quadrant III;
  - $\beta = -255^\circ$;
    We have $\beta = -255^\circ = -360^\circ + 105^\circ$; Therefore, β is coterminal with the 105° angle and the terminal side lies in Quadrant II;
  - $\gamma = 1105^\circ$;
    We have $\gamma = 1105^\circ = 3 \cdot 360^\circ + 25^\circ$; Therefore, γ is coterminal with the 25° angle and the terminal side lies in Quadrant I;

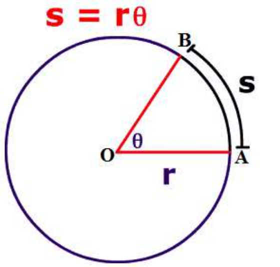
Decimal Degrees and DMS (Degree, Minute, Second)

- To represent a fractional part of a degree, there are two popular methods:
  - The **decimal degree method** uses a decimal number;
    For instance, 34.42° means 34° plus 42 hundredths of 1°;
  - The **DMS** (**D**egree, **M**inute, **S**econd) **method** subdivides a degree into 60 **minutes** ($1^\circ = 60'$) and each minute into 60 **seconds** ($1' = 60''$);
- Example: Write $126^\circ 12' 27''$ as a decimal degree;

$$126^\circ 12' 27'' = (126 + \frac{12}{60} + \frac{27}{3600})^\circ =$$
$$(126 + 0.2 + 0.0075)^\circ = 126.2075^\circ;$$

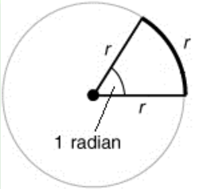
Central Angles and the Radian

- Consider a circle of radius r and two radii OA and OB;
- The angle θ formed by OA and OB is called a **central angle**;
- The portion of the arc between A and B is an **arc** of the circle and is denoted by $\overset{\frown}{AB}$;
- The arc $\overset{\frown}{AB}$ is said to **subtend** the angle θ;



Definition of a Radian

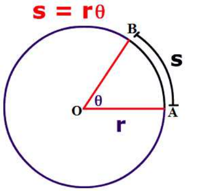
One **radian** is defined to be the measure of the central angle subtended by an arc of length r on a circle of radius r;



Radian Measure of an Angle

Definition of Radian Measure

Given an arc of length s on a circle of radius r, the measure of the central angle subtended by the arc is $\theta = \frac{s}{r}$ radians.



- Example: Suppose an arc has length 15 cm on a circle of radius 5 cm. What is the radian measure of the central angle subtended by the arc?
$$\theta = \frac{s}{r} = \frac{15}{5} = 3 \text{ radians;}$$
- Example: An arc of length 12 cm has radian measure $\frac{4}{3}$ radians; What is the radius of the corresponding circle?
$$\theta = \frac{s}{r} \quad \Rightarrow \quad r = \frac{s}{\theta} \quad \Rightarrow \quad r = \frac{12}{4/3} = 9 \text{ cm;}$$

Conversion Between Radians and Degrees

Radian-Degree Conversions

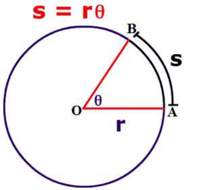
- To convert from radians to degrees, multiply by $\frac{180^\circ}{\pi \text{ rads}}$;
- To convert from degrees to radians, multiply by $\frac{\pi \text{ rads}}{180^\circ}$;

- Example: Convert from degrees to radians:
  - $60^\circ = 60^\circ \cdot \dfrac{\pi \text{ rads}}{180^\circ} = \dfrac{\pi}{3}$ rads;
  - $315^\circ = 315^\circ \cdot \dfrac{\pi \text{ rads}}{180^\circ} = \dfrac{7\pi}{4}$ rads;
  - $-150^\circ = -150^\circ \cdot \dfrac{\pi \text{ rads}}{180^\circ} = -\dfrac{5\pi}{6}$ rads;
- Example: Convert from radians to degrees:
  - $\dfrac{3\pi}{4}$ rads $= \dfrac{3\pi}{4}$ rads $\cdot \dfrac{180^\circ}{\pi \text{ rads}} = 135^\circ$;
  - 1 rad $= 1$ rad $\cdot \dfrac{180^\circ}{\pi \text{ rads}} = \dfrac{180}{\pi}^\circ$;
  - $-\dfrac{5\pi}{2}$ rads $= -\dfrac{5\pi}{2}$ rads $\cdot \dfrac{180^\circ}{\pi \text{ rads}} = -450^\circ$;

Arc and Arc Length

The length s of the arc subtending a central angle of nonnegative radian measure θ of a circle of radius r is given by

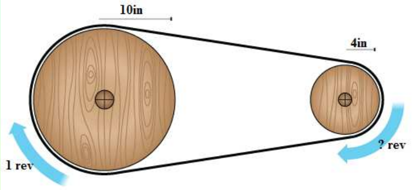
$$s = r\theta;$$



- Example: What is the length of the arc that subtends a central angle of 120° in a circle of radius 10 cm?
  First, convert degrees to radians:
  $$120^\circ = 120^\circ \cdot \frac{\pi \text{ rads}}{180^\circ} = \frac{2\pi}{3} \text{ rads};$$
  Then, use the formula:
  $$s = r\theta = 10 \text{ cm} \cdot \frac{2\pi}{3} \text{ rad} = \frac{20\pi}{3} \text{ cm};$$

A More Challenging Application

A pulley with a radius of 10 inches uses a belt to drive a pulley with a radius of 4 inches; Find the angle through which the smaller pulley turns as the 10-inch pulley makes one full revolution; State answer in both radians and degrees;



For the large pulley, through one revolution we obtain
$s_1 = r_1\theta_1 = 10 \text{ in} \cdot 2\pi \text{ rads} = 20\pi \text{ in}$;
During that revolution, since the two pulleys are connected through the belt, we get $s_2 = s_1$; Therefore, $s_1 = r_2\theta_2 \Rightarrow \theta_2 = \dfrac{s_1}{r_2} = \dfrac{20\pi}{4} = 5\pi$ rads;

In degrees $5\pi \text{ rads} = 5\pi \text{ rads} \cdot \dfrac{180^\circ}{\pi \text{ rads}} = 900^\circ$;

Linear and Angular Speed and Their Relation

Definition of Linear and Angular Speed

Suppose that a point moves on a circular path of radius r at a constant rate of θ radians per unit of time t; If s is the distance that the point travels, then $s = r\theta$; The **linear speed** of the point is $v = \frac{s}{t}$; The **angular speed** of the point is $\omega = \frac{\theta}{t}$;

- To reveal the relation between the linear and the angular speeds, note that
$$v = \frac{s}{t} = \frac{r\theta}{t} = r\frac{\theta}{t} = r\omega;$$
- Example: A hard disk rotates at 7200 revolutions per minute; What is its angular speed in radians per second?
$$7200 \text{ rev/min} = 7200 \ \frac{\text{rev}}{\text{min}} \cdot \frac{2\pi \text{ rad}}{1 \text{ rev}} \cdot \frac{1 \text{ min}}{60 \text{ sec}} = 240\pi \text{ rad/sec};$$

Another Example

A windmill has blades that are 12 feet in length; If it is rotating at 3 revolutions per second, what is the linear speed in feet per second of the tips of the blades;

The angular speed of the point is:

$$\omega = 3\frac{\text{rev}}{\text{sec}} \cdot 2\pi\frac{\text{rad}}{\text{rev}} = 6\pi\frac{\text{rad}}{\text{sec}}.$$

Thus, the linear speed is

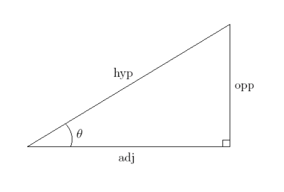
$$v = r\omega = 12 \text{ ft} \cdot 6\pi\frac{\text{rad}}{\text{sec}} = 72\pi \text{ ft/sec};$$

Subsection 2

Right Triangle Trigonometry

Definitions of Trigonometric Functions

- Consider an acute angle θ of a right triangle; We refer to the vertical side **opposite** and the vertical side **adjacent** to the angle θ;



Definition of Trigonometric Functions of θ

The values of the trigonometric functions of θ are defined as follows:

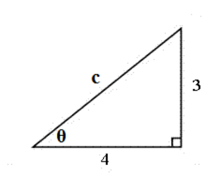
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

$$\sec\theta = \frac{\text{hyp}}{\text{adj}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

Computing Trig Functions

- Find the values of the trig functions of the angle θ of the triangle given in the figure



First, compute the length c of the hypothenuse using the Pythagorean Theorem:

$$c^2 = a^2 + b^2 = 3^2 + 4^2 = 25 \quad \Rightarrow \quad c = 5;$$

Now set up the trig functions of θ:

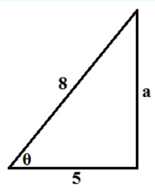
$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{3}{5}; \quad \cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{4}{5}; \quad \tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{3}{4};$$

$$\cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{4}{3}; \quad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{5}{4}; \quad \csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{5}{3};$$

A More Challenging Example

- Given that θ is an acute angle and $\cos\theta = \frac{5}{8}$, compute $\tan\theta$;

  Since $\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{5}{8}$, we get the following diagram:



  Now, compute the length a of the opposite side to θ using the Pythagorean Theorem:

$$a^2 = c^2 - b^2 = 8^2 - 5^2 = 64 - 25 = 39; \quad \Rightarrow \quad a = \sqrt{39};$$

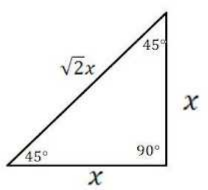
  Therefore, we obtain

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{39}}{5};$$

Trigonometric Numbers of $\theta = 45^\circ$

- We compute the trigonometric numbers of a 45° angle;

  Since a right triangle having a 45° angle is isosceles, we get the following diagram:



  Therefore, for the trigonometric numbers, we get:

$$\sin 45^\circ = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{2}x} = \frac{\sqrt{2}}{2}; \quad \cos 45^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{x}{\sqrt{2}x} = \frac{\sqrt{2}}{2};$$

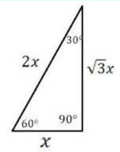
$$\tan 45^\circ = \frac{\text{opp}}{\text{adj}} = \frac{x}{x} = 1; \quad \cot 45^\circ = \frac{\text{adj}}{\text{opp}} = \frac{x}{x} = 1;$$

$$\sec 45^\circ = \frac{\text{hyp}}{\text{adj}} = \frac{\sqrt{2}x}{x} = \sqrt{2}; \quad \csc 45^\circ = \frac{\text{hyp}}{\text{opp}} = \frac{\sqrt{2}x}{x} = \sqrt{2};$$

Trigonometric Numbers of $\theta = 30^\circ$ and $\theta = 60^\circ$

- We compute the trigonometric numbers of a 30° and of a 60° angle;

  Since a right triangle having a 60° angle is "half" of an equilateral triangle, we get the following diagram:



  Therefore, for the trigonometric numbers, we get:

$$\sin 30^\circ = \cos 60^\circ = \frac{x}{2x} = \frac{1}{2}; \qquad \cos 30^\circ = \sin 60^\circ = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2};$$

$$\tan 30^\circ = \cot 60^\circ = \frac{x}{\sqrt{3}x} = \frac{\sqrt{3}}{3}; \qquad \cot 30^\circ = \tan 60^\circ = \frac{\sqrt{3}x}{x} = \sqrt{3};$$

$$\sec 30^\circ = \csc 60^\circ = \frac{2x}{\sqrt{3}x} = \frac{2\sqrt{3}}{3}; \qquad \csc 30^\circ = \sec 60^\circ = \frac{2x}{x} = 2;$$

Table of Trigonometric Numbers of $30^\circ, 45^\circ$ and 60°

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$30^\circ; \frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$	$\sqrt{3}$
$45^\circ; \frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	$\sqrt{2}$	$\sqrt{2}$	1
$60^\circ; \frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2	$\frac{\sqrt{3}}{3}$

Evaluating Expressions

- Find the exact value of the following expressions:
  - $\sin^2 45^\circ + \cos^2 60^\circ = \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$;
  - $2\csc\frac{\pi}{4} - \sec\frac{\pi}{3}\cos\frac{\pi}{6} = 2 \cdot \sqrt{2} - 2 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{2} - \sqrt{3}$;

Reciprocal Identities

- Recall that we have

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \quad \cos\theta = \frac{\text{adj}}{\text{hyp}} \quad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

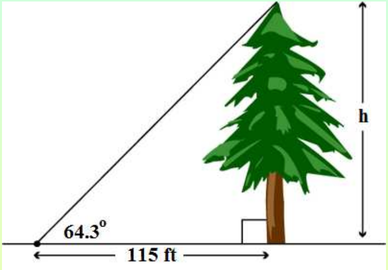
$$\csc\theta = \frac{\text{hyp}}{\text{opp}} \quad \sec\theta = \frac{\text{hyp}}{\text{adj}} \quad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

These imply the following important **reciprocal identities**:

$\sin\theta = \frac{1}{\csc\theta}$	$\cos\theta = \frac{1}{\sec\theta}$	$\tan\theta = \frac{1}{\cot\theta}$
$\csc\theta = \frac{1}{\sin\theta}$	$\sec\theta = \frac{1}{\cos\theta}$	$\cot\theta = \frac{1}{\tan\theta}$

Application: Angle of Elevation

- From a point 115 feet from the base of a tree, the angle of elevation to the top of the tree is $64.3°$; What is the height of the tree?

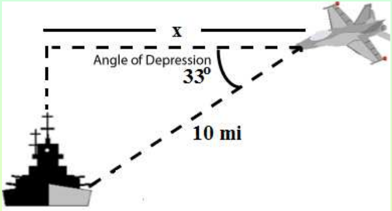


$$\tan 64.3° = \frac{\text{opp}}{\text{adj}} = \frac{h}{115}$$

$$\Rightarrow \quad h = 115 \cdot \tan 64.3° \approx 238.95 \text{ ft.}$$

Application: Angle of Depression

- Suppose the direct distance of a fighter jet from the landing deck of an aircraft carrier is 10 miles and the angle of depression is 33°; Find the horizontal ground distance from the jet to the carrier;

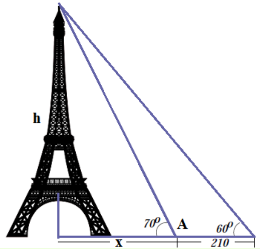


$$\cos 33^\circ = \frac{\text{adj}}{\text{hyp}} = \frac{x}{10}$$

$$\Rightarrow \quad x = 10 \cdot \cos 33^\circ \approx 8.387 \text{ miles};$$

Application: Angle of Elevation Revisited

- An observer notes that the angle of elevation from a point A to the top of the Eiffel tower is 70°; From another point 210 feet further from the base of the tower, the angle of elevation is 60°; Find the height of the Eiffel tower;



$$\tan 70^\circ = \frac{h}{x} \quad \Rightarrow \quad x = \frac{h}{\tan 70^\circ} = h \cot 70^\circ;$$

Moreover,

$$\begin{aligned} \tan 60^\circ &= \frac{h}{x+210} = \frac{h}{h\cot 70^\circ + 210} \\ \Rightarrow \quad h &= (\tan 60^\circ)(h\cot 70^\circ + 210) \\ \Rightarrow \quad h &= h\tan 60^\circ \cot 70^\circ + 210\tan 60^\circ \\ \Rightarrow \quad h - h\tan 60^\circ \cot 70^\circ &= 210\tan 60^\circ \\ \Rightarrow \quad h &= \frac{210\tan 60^\circ}{1-\tan 60^\circ \cot 70^\circ} \approx 984.16 \text{ feet;} \end{aligned}$$

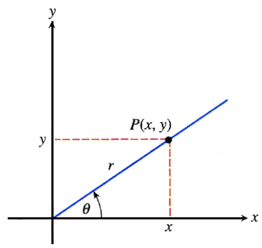
Subsection 3

Trigonometric Functions of Any Angle

Trigonometric Functions of Any Angle

Trigonometric Functions of Any Angle

Suppose $P(x, y)$ is a point different from the origin on the terminal side of an angle θ in standard position, such that $r = \sqrt{x^2 + y^2}$ is the distance from the origin to P;



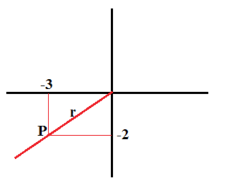
The six trigonometric functions of θ are defined as follows:

$$\sin\theta = \frac{y}{r}; \qquad \cos\theta = \frac{x}{r}; \qquad \tan\theta = \frac{y}{x}, x \neq 0;$$

$$\csc\theta = \frac{r}{y}, y \neq 0; \qquad \sec\theta = \frac{r}{x}, x \neq 0; \qquad \cot\theta = \frac{x}{y}, y \neq 0;$$

Evaluating Trigonometric Functions I

- Find the exact value of the six trigonometric functions of the angle θ in standard position whose terminal side contains the point $P(-3,-2)$;



We get $x = -3, y = -2$ and $r = \sqrt{(-3)^2 + (-2)^2} = \sqrt{13}$; Thus,

$$\sin\theta = \frac{y}{r} = \frac{-2}{\sqrt{13}} = -\frac{2\sqrt{13}}{13}; \quad \cos\theta = \frac{x}{r} = \frac{-3}{\sqrt{13}} = -\frac{3\sqrt{13}}{13};$$

$$\tan\theta = \frac{y}{x} = \frac{-2}{-3} = \frac{2}{3}; \quad \csc\theta = -\frac{\sqrt{13}}{2};$$

$$\sec\theta = -\frac{\sqrt{13}}{3}; \quad \cot\theta = \frac{3}{2};$$

Quadrantal Angles and Signs of Functions

Values of Trigonometric Functions of Quadrantal Angles:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
0°	0	1	0	☠	1	☠
90°	1	0	☠	1	☠	0
180°	0	-1	0	☠	-1	☠
270°	-1	0	☠	-1	☠	0

Signs of Trigonometric Functions:

Sign of	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
$\sin\theta$ and $\csc\theta$	$+$	$+$	$-$	$-$
$\cos\theta$ and $\sec\theta$	$+$	$-$	$-$	$+$
$\tan\theta$ and $\cot\theta$	$+$	$-$	$+$	$-$

Evaluating Trigonometric Functions II

- Given $\tan\theta = -\frac{7}{5}$ and $\sin\theta < 0$, find $\cos\theta$ and $\csc\theta$;

  Since $\tan\theta = -\frac{7}{5}$ and $\sin\theta < 0$, we get $\frac{y}{x} = -\frac{7}{5}$ and $y < 0$;
  Therefore $y = -7$ and $x = 5$;
  These imply that $r = \sqrt{5^2 + (-7)^2} = \sqrt{74}$;
  Therefore
  $$\cos\theta = \frac{x}{r} = \frac{5}{\sqrt{74}} = \frac{5\sqrt{74}}{74}$$
  and
  $$\csc\theta = \frac{r}{y} = -\frac{\sqrt{74}}{7};$$

The Reference Angle

- Given $\angle\theta$ in standard position, its **reference angle** θ' is the *acute angle* formed by the terminal side of $\angle\theta$ and the x-axis;
- Example: Find the measure of the reference angle θ' for each of the following:
  - $\theta = 120°$;
    Since $120° = 180° - 60°$, we have $\theta' = 60°$;
  - $\theta = 345°$;
    Since $345° = 360° - 15°$, we have $\theta' = 15°$;
  - $\theta = \frac{7\pi}{4}$;
    Since $\frac{7\pi}{4} = 2\pi - \frac{\pi}{4}$, we have $\theta' = \frac{\pi}{4}$;
  - $\theta = \frac{13\pi}{6}$;
    Since $\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$, we have $\theta' = \frac{\pi}{6}$;

Using the Reference Angle

Reference Angle Theorem

To evaluate $\sin\theta$, determine $\sin\theta'$; Then use either $\sin\theta'$ or $-\sin\theta'$, depending on which of the two has the correct sign.

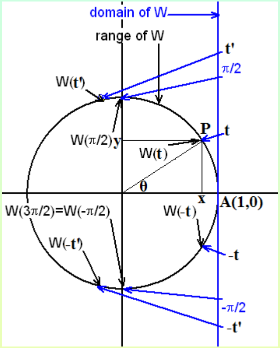
- Example: Determine the exact value of
  - $\sin 210°$;
    We have $\theta' = 30°$ and $210°$ is in Quadrant III; Thus,
    $\sin 210° = -\sin 30° = -\frac{1}{2}$;
  - $\cos 405°$;
    We have $\theta' = 45°$ and $405°$ is in Quadrant I; Thus,
    $\cos 405° = \cos 45° = \frac{\sqrt{2}}{2}$;
  - $\tan\frac{5\pi}{3}$;
    We have $\theta' = \frac{\pi}{3}$ and $\frac{5\pi}{3}$ is in Quadrant IV; Thus,
    $\tan\frac{5\pi}{3} = -\tan\frac{\pi}{3} = -\sqrt{3}$;

Subsection 4

Trigonometric Functions of Real Numbers

Wrapping Function

- Consider the **unit circle**, i.e., the circle of radius 1 centered at the origin;
- The **wrapping function** has domain all real numbers and maps a real number t to a point $W(t) = P(x, y)$ on the unit circle such that the length of the arc $\widehat{AP}$ is $|t|$, where $A(1, 0)$;
- Since $r = 1$, we have $s = 1 \cdot \theta$, i.e., the length s of the arc equals the measure θ of the central angle subtended by the arc!



- This allows one to associate an angle with any given real number t using the wrapping function and passing through the arc on the unit circle starting from A and having length $|t|$ (clockwise if $t < 0$ and counterclockwise if $t > 0$);

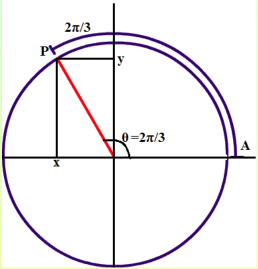
An Example of Evaluating the Wrapping Function

- Evaluate $W(\dfrac{2\pi}{3})$;

  We have $\cos\dfrac{2\pi}{3} = \dfrac{x}{r} \Rightarrow -\cos\dfrac{\pi}{3} = \dfrac{x}{1} \Rightarrow x = -\frac{1}{2}$;

  Similarly,

  $\sin\dfrac{2\pi}{3} = \dfrac{y}{r} \Rightarrow \sin\dfrac{\pi}{3} = \dfrac{y}{1} \Rightarrow y = \dfrac{\sqrt{3}}{2}$;



  Therefore, the point $W(\dfrac{2\pi}{3}) = (-\dfrac{1}{2}, \dfrac{\sqrt{3}}{2})$;

Trigonometric Functions of Any Real Number

- Use the wrapping function on the unit circle to map a real number t,
  - first to the arc $\overparen{\mathrm{AP}}$ of length $|t|$ (counterclockwise for $t > 0$ and clockwise for $t < 0$)
  - then to the central angle of measure t subtended by the arc $\overparen{\mathrm{AP}}$;
- We define the trigonometric functions of the real number t as the trigonometric functions of the angle corresponding to t, which (because $r = 1$) has measure t radians;

Trigonometric Functions of Real Numbers

Let t be a real number and $W(t) = P(x, y)$; Then, we define

$$\sin t = y, \qquad \cos t = x, \qquad \tan t = \frac{y}{x}, x \neq 0,$$

$$\csc t = \frac{1}{y}, y \neq 0, \qquad \sec t = \frac{1}{x}, x \neq 0, \qquad \cot t = \frac{x}{y}, y \neq 0;$$

Evaluating Trigonometric Functions

- Find the exact value of each function:
  - $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$;
  - $\sin\left(-\frac{7\pi}{6}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$;
  - $\tan\left(-\frac{5\pi}{4}\right) = -\tan\left(\frac{\pi}{4}\right) = -1$;
  - $\sec\left(\frac{5\pi}{3}\right) = \sec\left(\frac{\pi}{3}\right) = 2$;

Application: The Millenium Ferris Wheel in London

The Millenium Wheel has a diameter of 450 feet and completes one revolution every 30 minutes; Suppose that the height h in feet above the Thames River of a person riding on the Wheel can be estimated by

$$h(t) = 255 - 225\cos\left(\frac{\pi}{15}t\right),$$

where t in minutes is time since person started the ride;

- How high is the person at the start of the ride?

$$h(0) = 255 - 225\cos 0 = 255 - 225 \cdot 1 = 30 \text{ feet};$$

- How high is the person after 18 minutes?

$$h(18) = 255 - 225\cos\left(\frac{18\pi}{15}\right) = 255 - 225\cos\left(\frac{6\pi}{5}\right) \approx 437 \text{ feet};$$

Domains and Ranges of Trigonometric Functions

- The domain and ranges of the trigonometric functions:

Function	Domain	Range
$y = \sin t$	$\mathbb{R}$	$\{y : -1 \leq y \leq 1\}$
$y = \cos t$	$\mathbb{R}$	$\{y : -1 \leq y \leq 1\}$
$y = \tan t$	$\{t : t \neq \frac{(2k+1)\pi}{2}\}$	$\mathbb{R}$
$y = \csc t$	$\{t : t \neq k\pi\}$	$\{y : y \leq -1 \text{ or } y \geq 1\}$
$y = \sec t$	$\{t : t \neq \frac{(2k+1)\pi}{2}\}$	$\{y : y \leq -1 \text{ or } y \geq 1\}$
$y = \cot t$	$\{t : t \neq k\pi\}$	$\mathbb{R}$

Even and Odd Trigonometric Functions

- The four trigonometric functions
$$y = \sin t, \quad y = \csc t, \quad y = \tan t, \quad y = \cot t$$
are all odd functions;
- The two trigonometric functions
$$y = \cos t, \quad y = \sec t$$
are both even functions;
- These statements imply the following **even-odd identities**:
$$\begin{array}{lll} \sin(-t) = -\sin t & \cos(-t) = \cos t & \tan(-t) = -\tan t \\ \csc(-t) = -\csc t & \sec(-t) = \sec t & \cot(-t) = -\cot t \end{array}$$
- Example: Is $f(x) = x - \tan x$ even, odd or neither?
$$\begin{array}{l} f(-x) = (-x) - \tan(-x) = -x - (-\tan x) = \\ -x + \tan x = -(x - \tan x) = -f(x); \end{array}$$
Therefore, $f(x)$ is an odd function;

Periodicity

- A function f is **periodic** if there exists a positive constant p, such that
$$f(t+p)=f(t)$$
for all t in the domain of f; The smallest such positive p for which f is periodic is called the **period** of f;
- The functions
$$y = \sin t, \quad y = \cos t, \quad y = \csc t, \quad y = \sec t$$
are periodic with period 2π;
- The functions
$$y = \tan t, \quad y = \cot t$$
are periodic with period π;
- These statements imply the following **periodic identities**:

$$\begin{array}{lll} \sin(t+2k\pi) = \sin t & \cos(t+2k\pi) = \cos t & \tan(t+k\pi) = \tan t \\ \csc(t+2k\pi) = \csc t & \sec(t+2k\pi) = \sec t & \cot(t+k\pi) = \cot t \end{array}$$

Trigonometric Identities

Reciprocal Identities

$$\sin t = \frac{1}{\csc t}, \quad \cos t = \frac{1}{\sec t}, \quad \tan t = \frac{1}{\cot t};$$

Ratio Identities

$$\tan t = \frac{\sin t}{\cos t}; \qquad \cot t = \frac{\cos t}{\sin t};$$

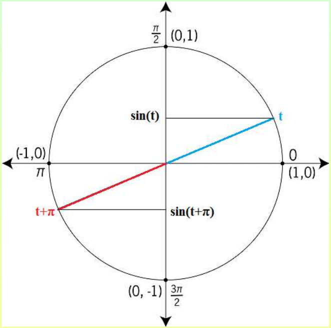
Pythagorean Identities

$$\cos^2 t + \sin^2 t = 1, \quad 1 + \tan^2 t = \sec^2 t, \quad 1 + \cot^2 t = \csc^2 t;$$

Example I

- Use the unit circle and the definitions of trigonometric functions to show that $\sin(t+\pi) = -\sin t$;

  If $W(t) = (x, y)$, then $W(t+\pi) = (-x, -y)$;



  Therefore $\sin(t+\pi) = -y = -\sin t$;

Example II

- Write the expression $\frac{1}{\sin^2 t} + \frac{1}{\cos^2 t}$ as a single term;

$$\begin{aligned}
\frac{1}{\sin^2 t} + \frac{1}{\cos^2 t} &= \frac{\cos^2 t}{\sin^2 t \cos^2 t} + \frac{\sin^2 t}{\sin^2 t \cos^2 t} \\
&= \frac{\cos^2 t + \sin^2 t}{\sin^2 t \cos^2 t} \\
&= \frac{1}{\sin^2 t \cos^2 t} \\
&= \frac{1}{\sin^2 t} \cdot \frac{1}{\cos^2 t} \\
&= \csc^2 t \sec^2 t;
\end{aligned}$$

Example III

- For $\frac{\pi}{2} < t < \pi$, write $\tan t$ in terms of only $\sin t$;

$$\begin{aligned} \cos^2 t + \sin^2 t = 1 \quad &\Rightarrow \quad \cos^2 t = 1 - \sin^2 t \\ &\Rightarrow \quad \cos t = \pm\sqrt{1 - \sin^2 t} \\ &\stackrel{\frac{\pi}{2}<t<\pi}{\Rightarrow} \quad \cos t = -\sqrt{1 - \sin^2 t}; \end{aligned}$$

Therefore, we get

$$\tan t = \frac{\sin t}{\cos t} = -\frac{\sin t}{\sqrt{1 - \sin^2 t}};$$

Subsection 5

Graphs of the Sine and Cosine Functions

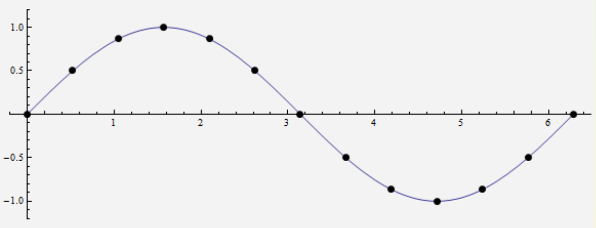
Graph of $y = \sin x$

- We create a small table of values:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$y = \sin x$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0

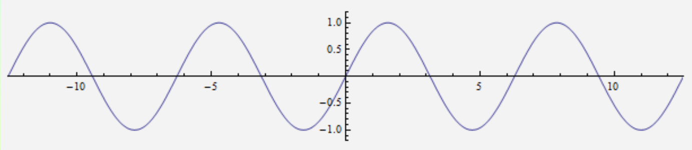
x	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$y = \sin x$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0

- We plot the points and connect:



Basic Properties of $y = \sin x$

- Extending the previous graph by periodicity, we get



- This graph has the following basic properties:
  - **Domain**: All real numbers;
  - **Range**: $\{y : -1 \leq y \leq 1\}$;
  - **Period**: 2π;
  - **Symmetry**: With respect to the origin (Odd);
  - x-**Intercepts**: $k\pi$;

Graph of $y = a\sin x$

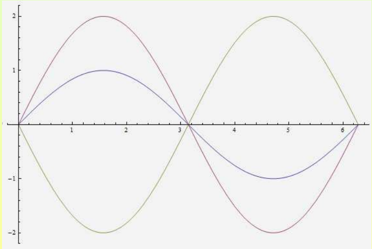
- The **amplitude** of a graph with maximum value $y = M$ and minimum value $y = m$ is defined by $A = \frac{1}{2}(M - m)$;
- Example: $y = \sin x$ has $M = 1$ and $m = -1$; Thus, it has amplitude $A = \frac{1}{2}(1 - (-1)) = 1$;

Amplitude of $y = a\sin x$

The amplitude of $y = a\sin x$ is $|a|$.

- Example: Graph $y = -2\sin x$;

  To graph this function, we start from $y = \sin x$, obtain $y = 2\sin x$ by a vertical stretch by a factor of 2 and then obtain $y = -2\sin x$ by flipping with respect to the x-axis;



Graph of $y = \sin bx$

Period of $y = \sin bx$

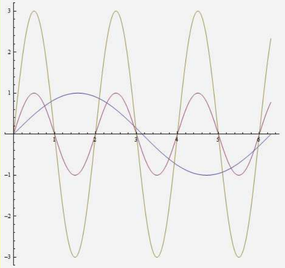
The period of $y = \sin bx$ is $\frac{2\pi}{|b|}$.

- Example: Find the amplitude and periods of the following functions:

Function	$y = a\sin bx$	$y = 3\sin(-2x)$	$y = -\sin\frac{x}{3}$	$y = -2\sin\frac{3x}{4}$
Amplitude	$\|a\|$	$\|3\| = 3$	$\|-1\| = 1$	$\|-2\| = 2$
Period	$\frac{2\pi}{\|b\|}$	$\frac{2\pi}{2} = \pi$	$\frac{2\pi}{1/3} = 6\pi$	$\frac{2\pi}{3/4} = \frac{8\pi}{3}$

- Example: Graph $y = 3\sin \pi x$;

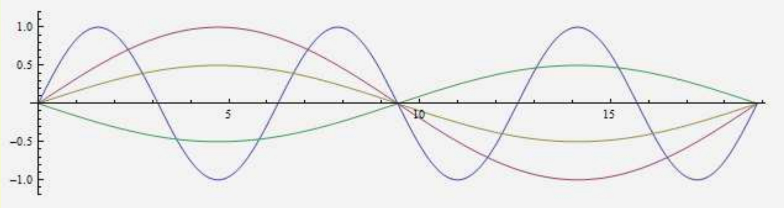
  To graph this function, we start from $y = \sin x$, obtain $y = \sin \pi x$ by a horizontal compression by a factor of π and then obtain $y = 3\sin \pi x$ by a vertical stretch by a factor of 3;



Graph of $y = a\sin bx$

- Example: Graph $y = -\frac{1}{2}\sin\frac{x}{3}$;

  To graph this function, we start from $y = \sin x$, obtain $y = \sin\frac{x}{3}$ by a horizontal stretching by a factor of 3, then obtain $y = \frac{1}{2}\sin\frac{x}{3}$ by a vertical compression by a factor of 2 and, finally, obtain $y = -\frac{1}{2}\sin\frac{x}{3}$ by a flipping with respect to the x-axis;



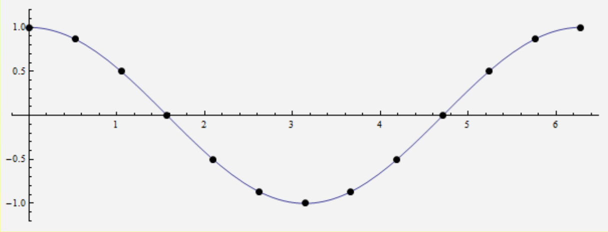
Graph of $y = \cos x$

- We create a small table of values:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$y = \cos x$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1

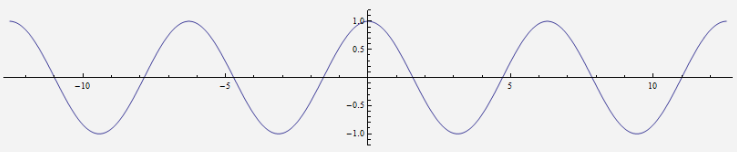
x	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$y = \cos x$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1

- We plot the points and connect:



Basic Properties of $y = \cos x$

- Extending the previous graph by periodicity, we get



- This graph has the following basic properties:
  - **Domain**: All real numbers;
  - **Range**: $\{y : -1 \leq y \leq 1\}$;
  - **Period**: 2π;
  - **Symmetry**: With respect to the y-axis (Even);
  - x-**Intercepts**: $(2k+1)\frac{\pi}{2}$;

Graph of $y = a\cos x$

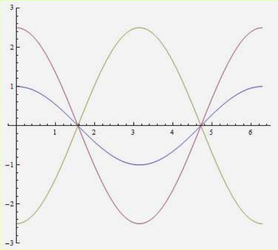
- Recall that the *amplitude* of a graph is defined by $A = \frac{1}{2}(M - m)$;
- Example: $y = \cos x$ has $M = 1$ and $m = -1$; Thus, it has amplitude $A = \frac{1}{2}(1 - (-1)) = 1$;

Amplitude of $y = a\cos x$

The amplitude of $y = a\cos x$ is $|a|$.

- Example: Graph $y = -\frac{5}{2}\sin x$;

  To graph this function, we start from $y = \cos x$, obtain $y = \frac{5}{2}\cos x$ by a vertical stretch by a factor of $\frac{5}{2}$ and then obtain $y = -\frac{5}{2}\cos x$ by flipping with respect to the x-axis;



Graph of $y = \cos bx$

Period of $y = \cos bx$

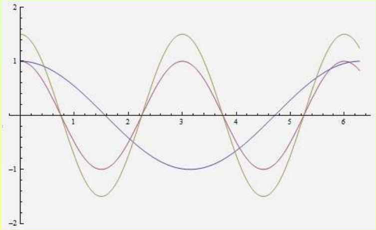
The period of $y = \cos bx$ is $\frac{2\pi}{|b|}$.

- Example: Find the amplitude and periods of the following functions:

Function	$y = a\cos bx$	$y = 2\cos 3x$	$y = -3\cos\frac{2x}{3}$
Amplitude	$\|a\|$	$\|2\| = 2$	$\|-3\| = 3$
Period	$\frac{2\pi}{\|b\|}$	$\frac{2\pi}{3}$	$\frac{2\pi}{2/3} = 3\pi$

- Example: Graph $y = \frac{3}{2}\cos\frac{2\pi}{3}x$;

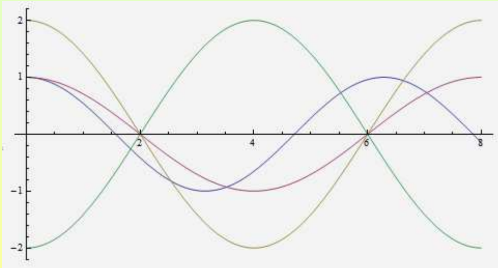
  To graph this function, we start from $y = \cos x$, obtain $y = \cos\frac{2\pi}{3}x$ by a horizontal compression by a factor of $\frac{2\pi}{3}$ and then obtain $y = \frac{3}{2}\cos\frac{2\pi}{3}x$ by a vertical stretch by a factor of $\frac{3}{2}$;



Graph of $y = a \cos bx$

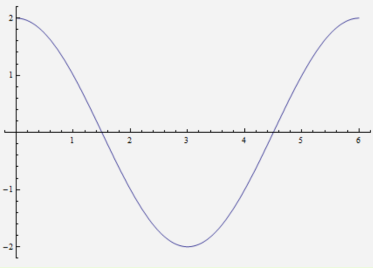
- Example: Graph $y = -2\cos\frac{\pi x}{4}$;

  To graph this function, we start from $y = \cos x$, obtain $y = \cos\frac{\pi x}{4}$ by a horizontal stretching by a factor of $\frac{4}{\pi}$, then obtain $y = 2\cos\frac{\pi x}{4}$ by a vertical stretching by a factor of 2 and, finally, obtain $y = -2\cos\frac{\pi x}{4}$ by a flipping with respect to the x-axis;



Finding an Equation for a Graph I

- The graph on the right shows a single cycle of a graph of a sine or cosine function; Find an equation for the graph;



The graph has

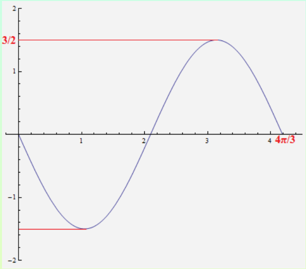
- Amplitude $|a| = 2$;
- Period $T = 6 \Rightarrow \dfrac{2\pi}{|b|} = 6 \Rightarrow |b| = \dfrac{2\pi}{6} = \dfrac{\pi}{3} \Rightarrow b = \pm\dfrac{\pi}{3}$;

Since at $x = 0$, it has value $y = +2$, we get an equation

$$y = a\cos bx = 2\cos\frac{\pi}{3}x;$$

Finding an Equation for a Graph II

- The graph on the right shows a single cycle of a graph of a sine or cosine function; Find an equation for the graph;



The graph has
  - Amplitude $|a| = \dfrac{3}{2}$;
  - Period $T = \dfrac{4\pi}{3} \Rightarrow \dfrac{2\pi}{|b|} = \dfrac{4\pi}{3} \Rightarrow |b| = \dfrac{2\pi}{4\pi/3} = \dfrac{3}{2} \Rightarrow b = \pm\dfrac{3}{2}$;

Since at $x = \frac{\pi}{3}$, it has value $y = -\frac{3}{2}$, we get an equation

$$y = a \sin bx = -\frac{3}{2} \sin \frac{3}{2} x;$$

Subsection 6

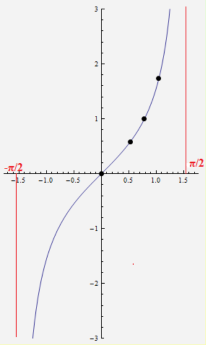
Graphs of the Other Trigonometric Functions

Graph of $y = \tan x$

- We create a small table of values:

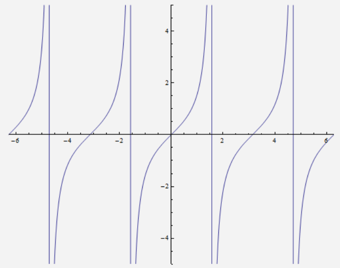
x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$y = \tan x$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	☠

- We plot the points, connect and use the odd property:



Basic Properties of $y = \tan x$

- Extending the previous graph by periodicity, we get

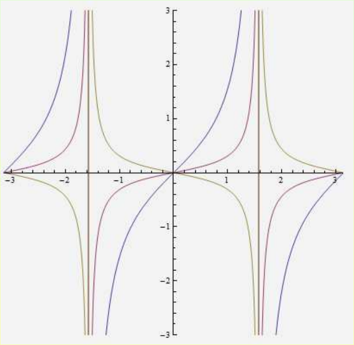


- This graph has the following basic properties:
  - **Domain**: $\mathbb{R} - \{(2k+1)\frac{\pi}{2} : k \in \mathbb{Z}\}$;
  - **Range**: All reals;
  - **Period**: π;
  - **Symmetry**: With respect to the origin (Odd);
  - x-**Intercepts**: $k\pi$;
  - **Vertical Asymptotes**: $x = (2k+1)\frac{\pi}{2}$;

Graph of $y = a\tan x$

- The graph of $y = a\tan x$ does not have an amplitude since it does not have a maximum or minimum value;
- It just represents either a vertical stretching or a vertical compression of the graph of $y = \pm\tan x$;
- Example: Graph $y = -\frac{1}{5}\tan x$;

  To graph this function, we start from $y = \tan x$, obtain $y = \frac{1}{5}\tan x$ by a vertical compression by a factor of 5 and then obtain $y = -\frac{1}{5}\tan x$ by flipping with respect to the x-axis;



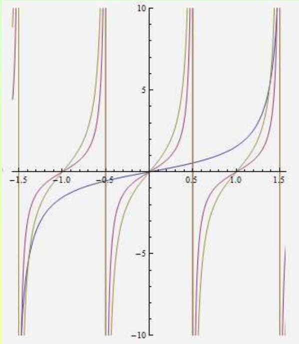
Graph of $y = \tan bx$

Period of $y = \tan bx$

The period of $y = \tan bx$ is $\frac{\pi}{|b|}$.

- Example: Graph $y = 2\tan \pi x$;

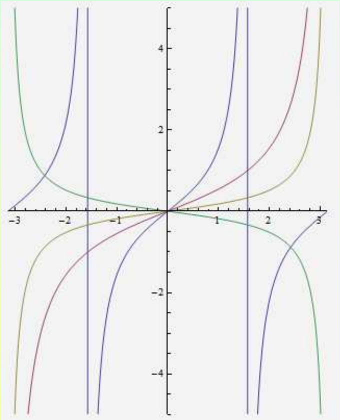
  To graph this function, we start from $y = \tan x$, obtain $y = \tan \pi x$ by a horizontal compression by a factor of π and then obtain $y = 2\tan \pi x$ by a vertical stretch by a factor of 2;



Graph of $y = a \tan bx$

- Example: Graph $y = -\frac{1}{3}\tan\frac{x}{2}$;

  To graph this function, we start from $y = \tan x$, obtain $y = \tan\dfrac{x}{2}$ by a horizontal stretching by a factor of 2, then obtain $y = \dfrac{1}{3}\tan\dfrac{x}{2}$ by a vertical compression by a factor of 3 and, finally, obtain $y = -\dfrac{1}{3}\tan\dfrac{x}{2}$ by a flipping with respect to the x-axis;

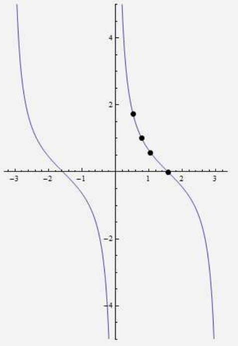


Graph of $y = \cot x$

- We create a small table of values:

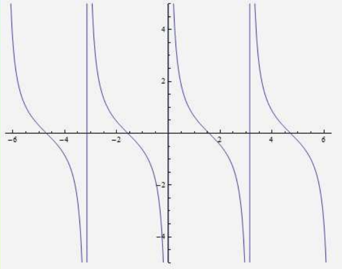
x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$y = \cot x$	☠	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

- We plot the points, connect and use the odd property:



Basic Properties of $y = \cot x$

- Extending the previous graph by periodicity, we get



- This graph has the following basic properties:
  - **Domain**: $\mathbb{R} - \{k\pi : k \in \mathbb{Z}\}$;
  - **Range**: All reals;
  - **Period**: π;
  - **Symmetry**: With respect to the origin (Odd);
  - x-**Intercepts**: $(2k+1)\frac{\pi}{2}$;
  - **Vertical Asymptotes**: $x = k\pi$;

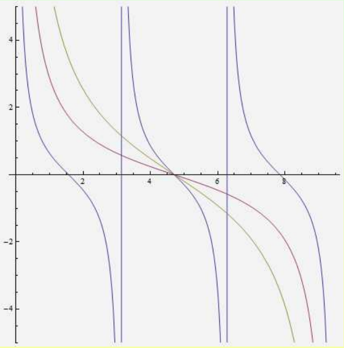
Graph of $y = a \cot bx$

Period of $y = a \cot bx$

The period of $y = a \cot bx$ is $\frac{\pi}{|b|}$.

- Example: Graph $y = 2 \cot \frac{x}{3}$;

  To graph this function, we start from $y = \cot x$, obtain $y = \cot \frac{x}{3}$ by a horizontal stretching by a factor of 3 and then obtain $y = 2 \cot \frac{x}{3}$ by a vertical stretch by a factor of 2;

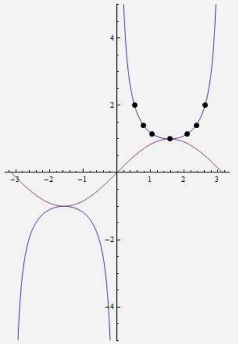


Graph of $y = \csc x$

- We create a small table of values:

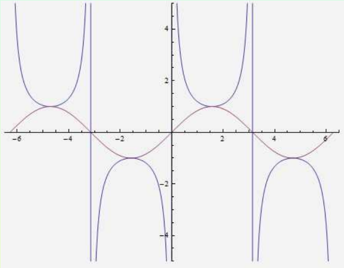
x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$y = \csc x$	☠	2	$\sqrt{2}$	$\frac{2\sqrt{3}}{3}$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	☠

- We plot the points, connect and use the odd property:



Basic Properties of $y = \csc x$

- Extending the previous graph by periodicity, we get



- This graph has the following basic properties:
  - **Domain**: $\mathbb{R} - \{k\pi : k \in \mathbb{Z}\}$;
  - **Range**: $\{y : y \leq -1 \text{ or } y \geq 1\}$;
  - **Period**: 2π;
  - **Symmetry**: With respect to the origin (Odd);
  - x-**Intercepts**: None;
  - **Vertical Asymptotes**: $x = k\pi$;

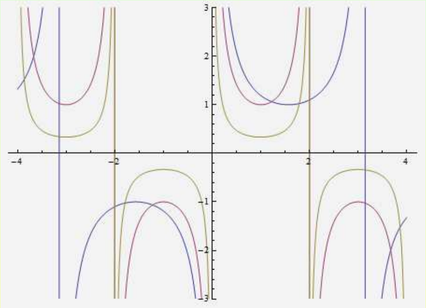
Graph of $y = a \csc bx$

Period of $y = a \csc bx$

The period of $y = a \csc bx$ is $\frac{2\pi}{|b|}$.

- Example: Graph $y = \frac{1}{3} \csc \frac{\pi x}{2}$;

  To graph this function, we start from $y = \csc x$, obtain $y = \csc \frac{\pi x}{2}$ by a horizontal compression by a factor of $\frac{\pi}{2}$ and then obtain $y = \frac{1}{3} \csc \frac{\pi x}{2}$ by a vertical compression by a factor of 3;

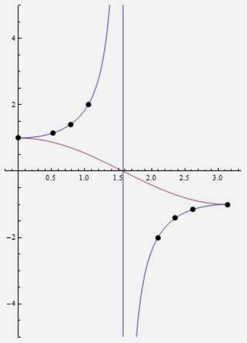


Graph of $y = \sec x$

- We create a small table of values:

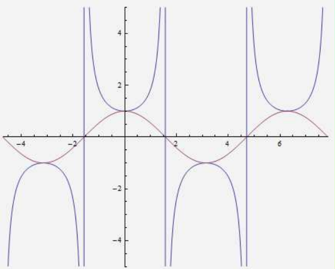
x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$y = \sec x$	1	$\frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	☠	-2	$-\sqrt{2}$	$-\frac{2\sqrt{3}}{3}$	-1

- We plot the points and connect:



Basic Properties of $y = \sec x$

- Extending the previous graph by symmetry and periodicity, we get



- This graph has the following basic properties:
  - **Domain**: $\mathbb{R} - \{(2k+1)\frac{\pi}{2} : k \in \mathbb{Z}\}$;
  - **Range**: $\{y : y \leq -1 \text{ or } y \geq 1\}$;
  - **Period**: 2π;
  - **Symmetry**: With respect to the x-axis (Even);
  - x-**Intercepts**: None;
  - **Vertical Asymptotes**: $x = (2k+1)\frac{\pi}{2}$;

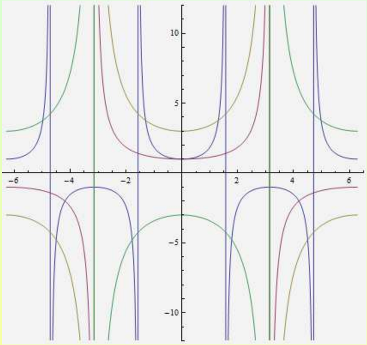
Graph of $y = a \sec bx$

Period of $y = a \sec bx$

The period of $y = a \sec bx$ is $\frac{2\pi}{|b|}$.

- Example: Graph $y = -3 \sec \frac{x}{2}$;

  To graph this function, we start from $y = \sec x$, obtain $y = \sec \frac{x}{2}$ by a horizontal stretching by a factor of 2, then obtain $y = 3 \sec \frac{x}{2}$ by a vertical stretching by a factor of 3 and, finally, $y = -3 \sec \frac{x}{2}$ by flipping with respect to the x-axis;



Subsection 7

Graphing Techniques

Amplitude, Period and Phase Shift of Sinusoidal Graphs

Graphs of $y = a\sin(bx+c)$ and $y = a\cos(bx+c)$

The graphs of $y = a\sin(bx+c)$ and $y = a\cos(bx+c)$ have

$$\text{Amplitude}: |a|, \quad \text{Period}: \frac{2\pi}{|b|}, \quad \text{Phase Shift}: -\frac{c}{b};$$

- The graph $y = a\sin(bx+c)$ shifts the graph of $y = a\sin bx$ horizontally $-\frac{c}{b}$ units;
- The graph $y = a\cos(bx+c)$ shifts the graph of $y = a\cos bx$ horizontally $-\frac{c}{b}$ units;

- Example: What is the phase shift of $y = 3\sin(\frac{1}{2}x - \frac{\pi}{6})$?

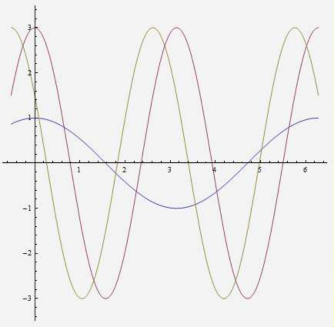
$$\phi = -\frac{c}{b} = -\frac{-\pi/6}{1/2} = \frac{\pi}{3};$$

Using Amplitudes, Periods and Phase Shifts to Graph

- Find the amplitude, period and phase shift of $y = 3\cos\left(2x + \frac{\pi}{3}\right)$ and use them to sketch the graph;

$$\text{Amplitude:}|a| = 3; \qquad \text{Period:} T = \frac{2\pi}{|b|} = \frac{2\pi}{2} = \pi;$$

$$\text{Phase Shift:}\phi = -\frac{c}{b} = -\frac{\pi/3}{2} = -\frac{\pi}{6};$$



Period and Phase Shift of Tangent and Cotangent

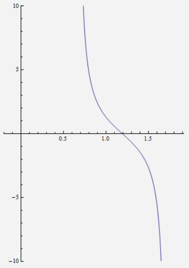
Graphs of $y = a\tan(bx+c)$ and $y = a\cot(bx+c)$

The graphs of $y = a\tan(bx+c)$ and $y = a\cot(bx+c)$ have

$$\text{Period}: \frac{\pi}{|b|}, \quad \text{Phase Shift}: -\frac{c}{b};$$

- The graph $y = a\tan(bx+c)$ shifts the graph of $y = a\tan bx$ horizontally $-\frac{c}{b}$ units;
- The graph $y = a\cot(bx+c)$ shifts the graph of $y = a\cot bx$ horizontally $-\frac{c}{b}$ units;

- Example: graph one period of $y = 2\cot(3x-2)$; Since the period is $T = \frac{\pi}{|b|} = \frac{\pi}{3}$ and the phase shift is $\phi = -\frac{c}{b} = -\frac{-2}{3} = \frac{2}{3}$, we start the graph at $x = \frac{2}{3}$ and end it at $x = \frac{2}{3} + \frac{\pi}{3} = \frac{\pi+2}{3}$;

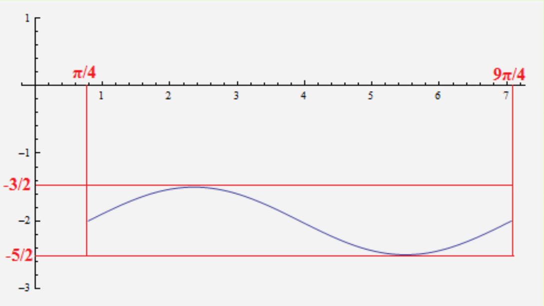


Using Amplitudes, Periods and Shifts to Graph I

- Graph $y = \frac{1}{2}\sin\left(x - \frac{\pi}{4}\right) - 2$;

  Amplitude: $|a| = \frac{1}{2}$; Period: $T = \dfrac{2\pi}{|b|} = \dfrac{2\pi}{1} = 2\pi$;

  Phase Shift: $\phi = -\dfrac{c}{b} = -\dfrac{-\pi/4}{1} = \dfrac{\pi}{4}$; Vertical Shift: $y_0 = -2$;



Using Amplitudes, Periods and Shifts to Graph II

- Find the amplitude, period and phase shift of $y = -2\cos(\pi x + \frac{\pi}{2}) + 1$ and used them to sketch the graph;

  Amplitude: $|a| = 2$; Period: $T = \frac{2\pi}{|b|} = \frac{2\pi}{\pi} = 2$;

  Phase Shift: $\phi = -\frac{c}{b} = -\frac{\pi/2}{\pi} = -\frac{1}{2}$; Vertical Shift: $y_0 = 1$;