College Trigonometry

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LSSU Math 131

Subsection 1

Verification of Trigonometric Identities

Review of Fundamental Trigonometric Identities

- **Reciprocal Identities**:
$$\sin x = \frac{1}{\csc x} \qquad \cos x = \frac{1}{\sec x} \qquad \tan x = \frac{1}{\cot x}$$
- **Ratio Identities**:
$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$
- **Pythagorean Identities**:
$$\sin^2 x + \cos^2 x = 1 \quad \tan^2 x + 1 = \sec^2 x \quad 1 + \cot^2 x = \csc^2 x$$
- **Odd-Even Identities**:
$$\sin(-x) = -\sin x \quad \tan(-x) = -\tan x \quad \sec(-x) = \sec x$$
$$\cos(-x) = \cos x \quad \cot(-x) = -\cot x \quad \csc(-x) = -\csc x$$

General Guidelines for Verifying Trigonometric Identities

- If one side is more complex, try to simplify it to match the simpler side;
- Perform indicated operations (e.g., adding fractions or expanding powers) and be aware of potential factorizations;
- Use previously established identities that allow rewriting expressions in equivalent forms;
- Rewrite one side so that it involves only sines and cosines;
- Rewrite one side in terms of single trigonometric function;
- Multiplying both numerator and denominator of a fraction by the same expression may be useful;
- Keeping the final goal in mind is paramount: Does it involve products, quotients, sums, radicals, powers? Does the form provide insight on the most likely way to reach the goal?
- Proving an identity is, sometimes, partly art and partly science!

Changing to Sines and Cosines

- Verify the Identity $\sin x \cot x \sec x = 1$;

  The strategy:

  - Start from the complicated side;
  - Change to sines and cosines;
  - Simplify to get to the simpler side!

$$\begin{aligned} \sin x \cot x \sec x &= \sin x \cdot \frac{\cos x}{\sin x} \cdot \frac{1}{\cos x} \\ &= \frac{\sin x \cos x}{\sin x \cos x} \\ &= 1; \end{aligned}$$

Using a Pythagorean Identity

- Verify the Identity $1 - 2\sin^2 x = 2\cos^2 x - 1$;

  The strategy:
  - Start from the left side;
  - Use $\sin^2 x = 1 - \cos^2 x$;
  - Simplify to get to the right side!

$$\begin{aligned} 1 - 2\sin^2 x &= 1 - 2(1 - \cos^2 x) \\ &= 1 - 2 + 2\cos^2 x \\ &= 2\cos^2 x - 1; \end{aligned}$$

Factoring to Get to the Simpler Side

- Verify the Identity $\csc^2 x - \cos^2 x \csc^2 x = 1$;

  The strategy:

  - Start from the most complicated (left) side;
  - Factor out $\csc^2 x$;
  - Use an identity;
  - Simplify to get to the right side!

$$\begin{aligned} \csc^2 x - \cos^2 x \csc^2 x &= \csc^2 x(1 - \cos^2 x) \\ &= \csc^2 x \sin^2 x \\ &= \frac{1}{\sin^2 x} \sin^2 x \\ &= 1; \end{aligned}$$

Multiplying by a Conjugate

- Verify the Identity $\dfrac{\sin x}{1+\cos x} = \dfrac{1-\cos x}{\sin x}$;

  The strategy:
  - Start from the left side;
  - Multiply both numerator and denominator by the conjugate $1-\cos x$ of $1+\cos x$;
  - Use an identity;
  - Simplify to get to the right side!

$$\begin{aligned}\frac{\sin x}{1+\cos x} &= \frac{\sin x(1-\cos x)}{(1+\cos x)(1-\cos x)} \\ &= \frac{\sin x(1-\cos x)}{1-\cos^2 x} \\ &= \frac{\sin x(1-\cos x)}{\sin^2 x} \\ &= \frac{1-\cos x}{\sin x};\end{aligned}$$

Changing to Sines and Cosines

- Verify the Identity $\frac{\sin x + \tan x}{1 + \cos x} = \tan x$;

  The strategy:
  - Start from the left side;
  - Use ratio identity for $\tan x$;
  - Perform algebraic operations to simplify;
  - Factor and simplify again;

$$\begin{aligned}
\frac{\sin x + \tan x}{1 + \cos x} &= \frac{\sin x + \frac{\sin x}{\cos x}}{1 + \cos x} = \frac{\frac{\sin x \cos x}{\cos x} + \frac{\sin x}{\cos x}}{1 + \cos x} \\
&= \frac{\frac{\sin x \cos x + \sin x}{\cos x}}{1 + \cos x} = \frac{\sin x(\cos x + 1)}{\cos x(1 + \cos x)} \\
&= \frac{\sin x}{\cos x} = \tan x;
\end{aligned}$$

Working With Both Sides

- Verify the Identity $\frac{1+\cos x}{1-\cos x} = (\csc x + \cot x)^2$;

  The strategy:
  - Work with both sides;
  - On the left multiply by the conjugate;
  - On the right rewrite in terms of sine and cosine;
  - Try to show both sides equal to the same expression;

$$\begin{aligned}\frac{1+\cos x}{1-\cos x} &= \frac{(1+\cos x)(1+\cos x)}{(1-\cos x)(1+\cos x)} \\ &= \frac{1+2\cos x+\cos^2 x}{1-\cos^2 x} \\ &= \frac{1+2\cos x+\cos^2 x}{\sin^2 x};\end{aligned}$$

$$\begin{aligned}(\csc x+\cot x)^2 &= \left(\frac{1}{\sin x}+\frac{\cos x}{\sin x}\right)^2 \\ &= \left(\frac{1+\cos x}{\sin x}\right)^2 \\ &= \frac{(1+\cos x)^2}{\sin^2 x} \\ &= \frac{1+2\cos x+\cos^2 x}{\sin^2 x};\end{aligned}$$

Subsection 2

Sum, Difference and Cofunction Identities

Identities Involving $(\alpha \pm \beta)$

Sum and Difference Identities

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$
$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \qquad \tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

- Example: Find the exact value of $\cos(60^\circ - 45^\circ)$;

$$\begin{aligned}\cos(60^\circ-45^\circ) &= \cos 60^\circ \cos 45^\circ + \sin 60^\circ \sin 45^\circ \\ &= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4} = \frac{\sqrt{2}+\sqrt{6}}{4};\end{aligned}$$

Cofunction Identities

- Any pair of trigonometric functions f and g are called **cofunctions** if
$$f(x) = g(90^\circ - x) \quad \text{and} \quad g(x) = f(90^\circ - x);$$
- They are called cofunctions because they have the same value on *complementary angles*;

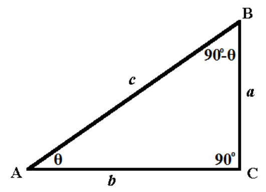
Cofunction Identities

$$\begin{array}{ll} \sin(90^\circ - \theta) = \cos\theta & \cos(90^\circ - \theta) = \sin\theta \\ \tan(90^\circ - \theta) = \cot\theta & \cot(90^\circ - \theta) = \tan\theta \\ \sec(90^\circ - \theta) = \csc\theta & \csc(90^\circ - \theta) = \sec\theta \end{array}$$

- Example: Use a cofunction identity to write an equivalent expression for $\sin 20^\circ$;
$$\sin 20^\circ = \cos(90^\circ - 20^\circ) = \cos 70^\circ;$$

Why Are Cofunction Identities True?

- Suppose we want to compute $\cos(90^\circ - \theta)$;



We have

$$\cos(90^\circ - \theta) = \frac{a}{c} = \sin\theta;$$

Alternatively, using the difference formulas:

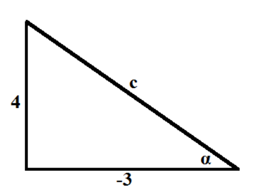
$$\begin{aligned}\cos(90^\circ - \theta) &= \cos 90^\circ \cos\theta + \sin 90^\circ \sin\theta \\ &= 0 \cdot \cos\theta + 1 \cdot \sin\theta = \sin\theta;\end{aligned}$$

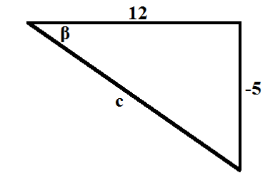
Example I

- Write each of the given expressions in terms of a single trigonometric function:
  - $\sin 5x \cos 3x - \cos 5x \sin 3x = \sin(5x - 3x) = \sin 2x$;
  - $\dfrac{\tan 4\alpha + \tan \alpha}{1 - \tan 4\alpha \tan \alpha} = \tan(4\alpha + \alpha) = \tan 5\alpha$;

Example II

- Given that $\tan\alpha = -\frac{4}{3}$ and α in Quadrant II and $\tan\beta = -\frac{5}{12}$ and β in Quadrant IV, find $\sin(\alpha+\beta)$;





$c = \sqrt{4^2+(-3)^2} = \sqrt{25} = 5$;
$\sin\alpha = \frac{4}{5}$; $\cos\alpha = -\frac{3}{5}$;

$c = \sqrt{(-5)^2+12^2} = \sqrt{169} = 13$;
$\sin\beta = -\frac{5}{13}$; $\cos\beta = \frac{12}{13}$;

Now we get

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha \\ &= \tfrac{4}{5}\cdot\tfrac{12}{13} + (-\tfrac{5}{13})(-\tfrac{3}{5}) = \tfrac{48}{65} + \tfrac{15}{65} = \tfrac{63}{65};\end{aligned}$$

Examples III

- Verify the identity $\cos(\pi - \theta) = -\cos\theta$;

$$\begin{aligned} \cos(\pi - \theta) &= \cos\pi\cos\theta + \sin\pi\sin\theta \\ &= -1\cdot\cos\theta + 0\cdot\sin\theta = -\cos\theta; \end{aligned}$$

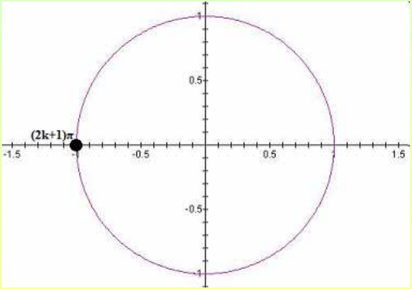
- Verify the identity $\frac{\cos 4\theta}{\sin\theta} - \frac{\sin 4\theta}{\cos\theta} = \frac{\cos 5\theta}{\sin\theta\cos\theta}$;

$$\begin{aligned} \frac{\cos 4\theta}{\sin\theta} - \frac{\sin 4\theta}{\cos\theta} &= \frac{\cos 4\theta\cos\theta}{\sin\theta\cos\theta} - \frac{\sin 4\theta\sin\theta}{\sin\theta\cos\theta} \\ &= \frac{\cos 4\theta\cos\theta - \sin 4\theta\sin\theta}{\sin\theta\cos\theta} \\ &= \frac{\cos(4\theta + \theta)}{\sin\theta\cos\theta} = \frac{\cos 5\theta}{\sin\theta\cos\theta}; \end{aligned}$$

Reduction Formulas

- Rewrite the following expression as a function involving only $\sin\theta$:

$$\begin{aligned}\sin[\theta+(2k+1)\pi] &= \sin\theta\cos(2k+1)\pi+\sin(2k+1)\pi\cos\theta\\ &= \sin\theta\cdot(-1)+0\cdot\cos\theta\\ &= -\sin\theta;\end{aligned}$$



Subsection 3

Double- and Half-Angle Identities

Double-Angle Identities

Double-Angle Identities

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$
$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$$

- Example: Write $4\sin 5\theta\cos 5\theta$ in terms of a single trigonometric function;

$$4\sin 5\theta\cos 5\theta = 2(2\sin 5\theta\cos 5\theta) = 2\sin 2(5\theta) = 2\sin 10\theta;$$

Example I

- If $\sin\alpha = \frac{4}{5}$ and $0^\circ < \alpha < 90^\circ$, find the exact value of $\sin 2\alpha$;

$$\begin{array}{ll} \sin^2\alpha + \cos^2\alpha = 1 & \\ \Rightarrow \quad \cos^2\alpha = 1 - \sin^2\alpha = 1 - \frac{16}{25} = \frac{9}{25} & \\ \stackrel{0^\circ < \alpha < 90^\circ}{\Rightarrow} \quad \cos\alpha = +\sqrt{\frac{9}{25}} = \frac{3}{5}; & \end{array}$$

Therefore,

$$\begin{array}{rcl} \sin 2\alpha & = & 2\sin\alpha\cos\alpha \\ & = & 2 \cdot \frac{4}{5} \cdot \frac{3}{5} = \frac{24}{25}; \end{array}$$

Example II

- Verify the identity $\csc 2\alpha = \frac{1}{2}(\tan\alpha + \cot\alpha)$;

$$\begin{aligned}
\frac{1}{2}(\tan\alpha + \cot\alpha) &= \frac{1}{2}\left(\frac{\sin\alpha}{\cos\alpha} + \frac{\cos\alpha}{\sin\alpha}\right) \\
&= \frac{1}{2}\left(\frac{\sin^2\alpha}{\sin\alpha\cos\alpha} + \frac{\cos^2\alpha}{\sin\alpha\cos\alpha}\right) \\
&= \frac{1}{2}\left(\frac{\sin^2\alpha + \cos^2\alpha}{\sin\alpha\cos\alpha}\right) \\
&= \frac{1}{2\sin\alpha\cos\alpha} \\
&= \frac{1}{\sin 2\alpha} = \csc 2\alpha;
\end{aligned}$$

Power-Reducing Identities

- Recall the double-angle formulas:

$$\cos 2\alpha = 2\cos^2\alpha - 1 \qquad \cos 2\alpha = 1 - 2\sin^2\alpha$$

- Solve the second for $\sin^2\alpha$ and the first for $\cos^2\alpha$ to get:

Power-Reducing Identities

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2} \quad \cos^2\alpha = \frac{1+\cos 2\alpha}{2} \quad \tan^2\alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha}$$

- Example: Write $\sin^4\alpha$ in terms of the first power of one or more cosine functions;

$$\begin{aligned}\sin^4\alpha &= (\sin^2\alpha)^2 = \left(\frac{1-\cos 2\alpha}{2}\right)^2 \\ &= \frac{1}{4}(1 - 2\cos 2\alpha + \cos^2 2\alpha) = \frac{1}{4}\left(1 - 2\cos 2\alpha + \frac{1+\cos 4\alpha}{2}\right) \\ &= \frac{1}{4}\left(\frac{2 - 4\cos 2\alpha + 1 + \cos 4\alpha}{2}\right) = \frac{1}{8}(3 - 4\cos 2\alpha + \cos 4\alpha);\end{aligned}$$

Half-Angle Identities

- Recall the power-reducing identities:

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2} \quad \cos^2\alpha = \frac{1+\cos 2\alpha}{2} \quad \tan^2\alpha = \frac{1-\cos 2\alpha}{1+\cos 2\alpha}$$

- Replace α by $\frac{\alpha}{2}$ and solve for the quantities on the left to get:

Half-Angle Identities

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1-\cos\alpha}{\sin\alpha}$$

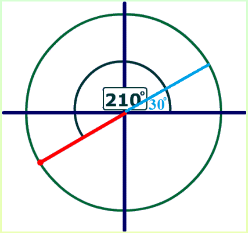
The choice of $+$ or $-$ depends on the quadrant of $\frac{\alpha}{2}$.

Example I

- Find the exact value of $\cos 105^\circ$;

  First, use reference angles to compute

$$\cos 210^\circ = -\cos 30^\circ = -\frac{\sqrt{3}}{2};$$



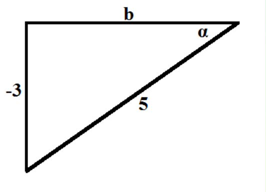
Now, use the half-angle formula:

$$\cos 105^\circ = \cos \frac{210^\circ}{2} = -\sqrt{\frac{1+\cos 210^\circ}{2}}$$
$$= -\sqrt{\frac{1+(-\frac{\sqrt{3}}{2})}{2}} = -\sqrt{\frac{2-\sqrt{3}}{4}} = -\frac{\sqrt{2-\sqrt{3}}}{2};$$

Example II

- If $\sin\alpha = -\frac{3}{5}$ and $180^\circ < \alpha < 270^\circ$, find the exact values of $\cos\frac{\alpha}{2}$ and $\tan\frac{\alpha}{2}$;

  Since α is in quadrant III and $\sin\alpha = -\frac{3}{5}$, we get the figure:



  Thus, $b = -\sqrt{5^2-(-3)^2} = -\sqrt{16} = -4$; This shows that $\cos\alpha = -\frac{4}{5}$;
  Now, note that $180^\circ < \alpha < 270^\circ \Rightarrow 90^\circ < \frac{\alpha}{2} < 135^\circ$; Therefore $\frac{\alpha}{2}$ is in quadrant II and therefore has negative cosine and negative tangent:

$$\cos\frac{\alpha}{2} = -\sqrt{\frac{1+\cos\alpha}{2}} = -\sqrt{\frac{1-\frac{4}{5}}{2}} = -\sqrt{\frac{1}{10}} = -\frac{\sqrt{10}}{10};$$

$$\tan\frac{\alpha}{2} = -\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = -\sqrt{\frac{1-(-\frac{4}{5})}{1+(-\frac{4}{5})}} = -\sqrt{\frac{\frac{9}{5}}{\frac{1}{5}}} = -3;$$

Example III

- Verify the identity $2\csc x\cos^2\frac{x}{2} = \frac{\sin x}{1-\cos x}$;

$$\begin{aligned} 2\csc x\cos^2\frac{x}{2} &= 2\cdot\frac{1}{\sin x}\cdot\frac{1+\cos x}{2} \\ &= \frac{1+\cos x}{\sin x} \\ &= \frac{(1+\cos x)(1-\cos x)}{\sin x(1-\cos x)} \\ &= \frac{1-\cos^2 x}{\sin x(1-\cos x)} \\ &= \frac{\sin^2 x}{\sin x(1-\cos x)} \\ &= \frac{\sin x}{1-\cos x}; \end{aligned}$$

Subsection 4

Identities Involving Sum of Trigonometric Functions

Product-to-Sum Identities

Product-to-Sum Identities

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$
$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$$
$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$
$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

- Let us see why one of them holds:

$$\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] =$$
$$\frac{1}{2}[(\sin\alpha\cos\beta+\sin\beta\cos\alpha)+(\sin\alpha\cos\beta-\sin\beta\cos\alpha)] =$$
$$\frac{1}{2}[2\sin\alpha\cos\beta] = \sin\alpha\cos\beta;$$

Examples

- Verify the identity

$$\cos 2x \sin 5x = \frac{1}{2}(\sin 7x + \sin 3x);$$

$$\begin{aligned} \cos 2x \sin 5x &= \tfrac{1}{2}[\sin(2x+5x) - \sin(2x-5x)] \\ &= \tfrac{1}{2}[\sin 7x - \sin(-3x)] \\ &= \tfrac{1}{2}[\sin 7x + \sin 3x]; \end{aligned}$$

- Verify the identity

$$\sin 5x \cos 3x = \sin 4x \cos 4x + \sin x \cos x;$$

$$\begin{aligned} \sin 5x \cos 3x &= \tfrac{1}{2}[\sin(5x+3x) + \sin(5x-3x)] \\ &= \tfrac{1}{2}[\sin 8x + \sin 2x] \\ &= \tfrac{1}{2}[2\sin 4x \cos 4x + 2\sin x \cos x] \\ &= \sin 4x \cos 4x + \sin x \cos x; \end{aligned}$$

Sum-to-Product Identities

Sum-to-Product Identities

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}$$
$$\cos x - \cos y = 2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$$

- To see why, e.g., the first one holds, recall that $\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] = \sin\alpha\cos\beta$;
  Now substitute $\alpha = \dfrac{x+y}{2}$ and $\beta = \dfrac{x-y}{2}$:

$$\frac{1}{2}[\sin(\frac{x+y}{2}+\frac{x-y}{2})+\sin(\frac{x+y}{2}-\frac{x-y}{2})] = \sin\frac{x+y}{2}\cos\frac{x-y}{2}$$
$$\Rightarrow \quad \frac{1}{2}[\sin x + \sin y] = \sin\frac{x+y}{2}\cos\frac{x-y}{2};$$

Examples

- Write $\sin 4\theta - \sin\theta$ as a product of two functions;

$$\sin 4\theta - \sin\theta = 2\cos\frac{4\theta+\theta}{2}\sin\frac{4\theta-\theta}{2} = 2\cos\frac{5\theta}{2}\sin\frac{3\theta}{2};$$

- Verify the identity $\dfrac{\sin 6x + \sin 2x}{\sin 6x - \sin 2x} = \tan 4x \cot 2x$;

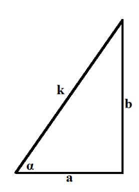
$$\frac{\sin 6x + \sin 2x}{\sin 6x - \sin 2x} = \frac{2\sin\frac{6x+2x}{2}\cos\frac{6x-2x}{2}}{2\cos\frac{6x+2x}{2}\sin\frac{6x-2x}{2}} =$$

$$\frac{\sin 4x\cos 2x}{\cos 4x\sin 2x} = \tan 4x\cot 2x;$$

Functions of Form $f(x) = a\sin x + b\cos x$

- The goal is to rewrite $f(x) = a\sin x + b\cos x$ in terms of a single trigonometric function;
  Set $k = \sqrt{a^2+b^2}$ and consider the right triangle:

$$f(x) = a\sin x + b\cos x = \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x\right] = k[\cos\alpha\sin x + \sin\alpha\cos x] = k\sin(x+\alpha);$$



Functions of the Form $a\sin x + b\cos x$

$$a\sin x + b\cos x = k\sin(x+\alpha),$$

where $k = \sqrt{a^2+b^2}$ and $\sin\alpha = \dfrac{b}{\sqrt{a^2+b^2}}$, $\cos\alpha = \dfrac{a}{\sqrt{a^2+b^2}}$.

Examples

- Rewrite $\sin x + \cos x$ in the form $k \sin(x + \alpha)$;
  We calculate: $k = \sqrt{1^2 + 1^2} = \sqrt{2}$; Therefore,

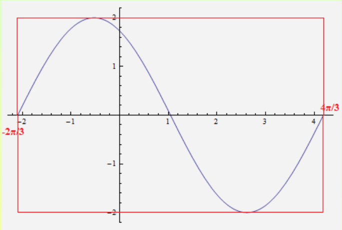
$$\sin x + \cos x = \sqrt{2}[\sin x(\frac{\sqrt{2}}{2}) + (\frac{\sqrt{2}}{2}) \cos x] =$$
$$\sqrt{2}[\sin x \cos \frac{\pi}{4} + \sin \frac{\pi}{4} \cos x] = \sqrt{2} \sin(x + \frac{\pi}{4});$$

- Graph the function $f(x) = -\sin x + \sqrt{3} \cos x$;

  Calculate $k = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$; Therefore,

$$f(x) = -\sin x + \sqrt{3} \cos x =$$
$$2(-\frac{1}{2} \sin x + \frac{\sqrt{3}}{2} \cos x) =$$
$$2(\sin x \cos \frac{2\pi}{3} + \sin \frac{2\pi}{3} \cos x) =$$
$$2 \sin(x + \frac{2\pi}{3});$$

  This function has amplitude 2, period 2π and phase shift $-\frac{2\pi}{3}$:



Subsection 5

Inverse Trigonometric Functions

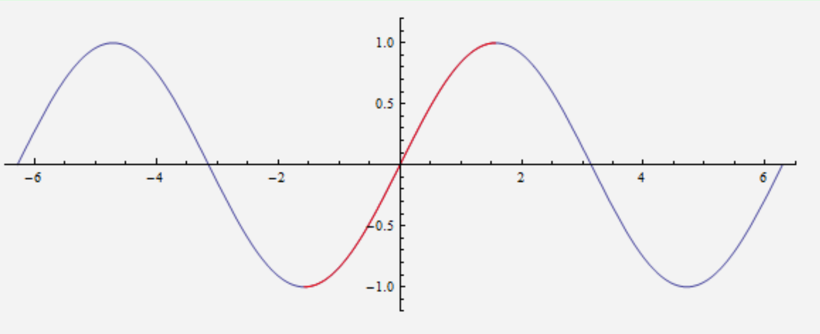
Restricting the Domain to Enforce the 1-1 Property

- Recall that a function $g = f^{-1}$ is the inverse function of a function f if it satisfies

$$y = f^{-1}(x) \quad \text{if and only if} \quad x = f(y);$$

- This condition, interpreted geometrically, means that the graph of f^{-1} is symmetric to the graph of f with respect to the line $y = x$;
- Therefore, since $y = f^{-1}(x)$ must pass the vertical line test in order to be the graph of a function, its symmetric $y = f(x)$ must pass the horizontal line test in order to have an inverse;
- Since all trigonometric functions fail to pass the horizontal line test, they do not have inverses;
- To fix this shortcoming, we restrict the domain so that the respective graphs are forced to pass the horizontal line test!

Forcing $\sin x$ to Pass the Horizontal Line Test

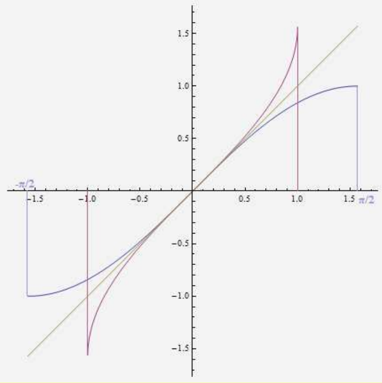


Definition of $\sin^{-1} x$

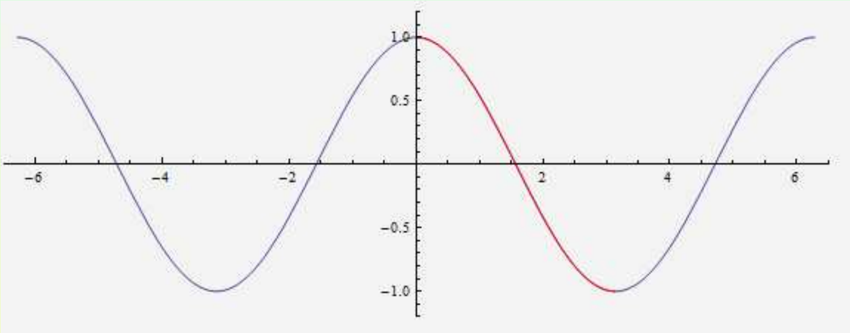
Definition of $\sin^{-1} x$

$$y = \sin^{-1} x \quad \text{if and only if} \quad x = \sin y$$

where $-1 \leq x \leq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$;



Forcing $\cos x$ to Pass the Horizontal Line Test

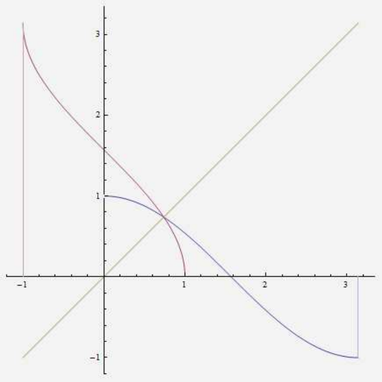


Definition of $\cos^{-1} x$

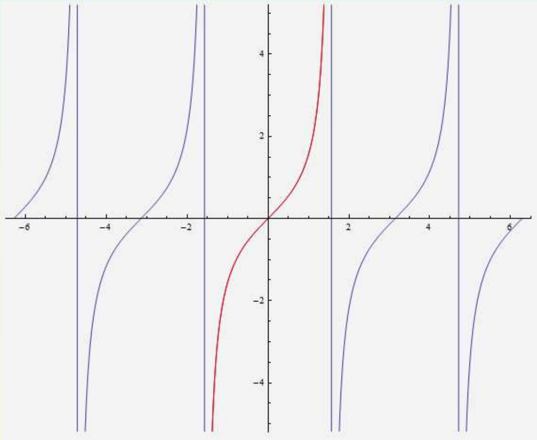
Definition of $\cos^{-1} x$

$$y = \cos^{-1} x \quad \text{if and only if} \quad x = \cos y$$

where $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$;



Forcing tan x to Pass the Horizontal Line Test

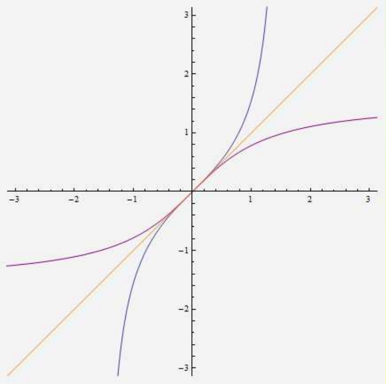


Definition of $\tan^{-1} x$

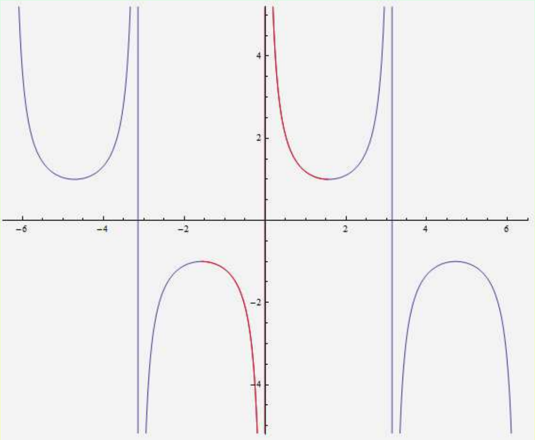
Definition of $\tan^{-1} x$

$$y = \tan^{-1} x \quad \text{if and only if} \quad x = \tan y$$

where $-\infty \leq x \leq \infty$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$;



Forcing $\csc x$ to Pass the Horizontal Line Test

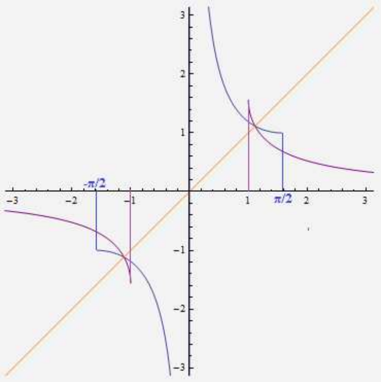


Definition of $\csc^{-1} x$

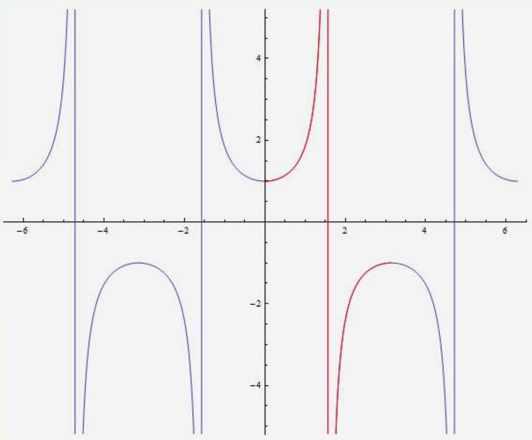
Definition of $\csc^{-1} x$

$$y = \csc^{-1} x \quad \text{if and only if} \quad x = \sec y$$

where $x \leq -1$ or $x \geq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$;



Forcing $\sec x$ to Pass the Horizontal Line Test

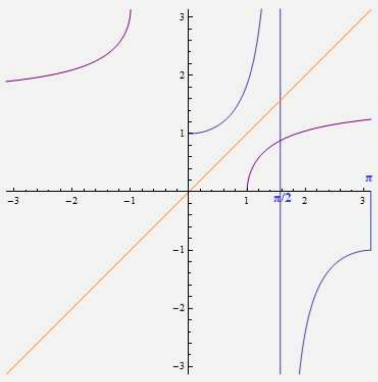


Definition of $\sec^{-1} x$

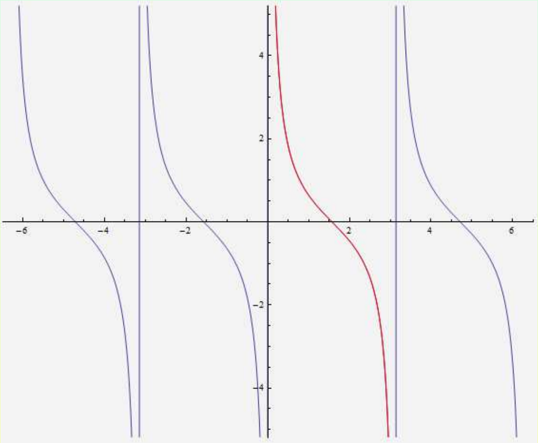
Definition of $\sec^{-1} x$

$$y = \sec^{-1} x \quad \text{if and only if} \quad x = \sec y$$

where $x \leq -1$ or $x \geq 1$ and $0 \leq y \leq \pi, y \neq \frac{\pi}{2}$;



Forcing $\cot x$ to Pass the Horizontal Line Test

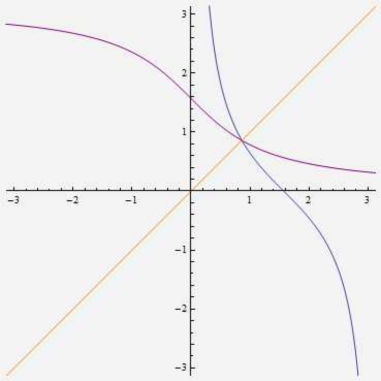


Definition of $\cot^{-1} x$

Definition of $\cot^{-1} x$

$$y = \cot^{-1} x \quad \text{if and only if} \quad x = \cot y$$

where $-\infty \leq x \leq \infty$ and $0 < y < \pi$;



Examples

- Find the exact value of $y = \tan^{-1} \frac{\sqrt{3}}{3}$;
  Follow closely the definition:
  $y = \tan^{-1} \frac{\sqrt{3}}{3}$ if and only if $\tan y = \frac{\sqrt{3}}{3}$ and $-\frac{\pi}{2} < y < \frac{\pi}{2}$;
  Therefore, $y = \frac{\pi}{6}$;
- Find the exact value of $y = \cos^{-1} (-\frac{\sqrt{2}}{2})$;
  Same method:
  $y = \cos^{-1} (-\frac{\sqrt{2}}{2})$ if and only if $\cos y = -\frac{\sqrt{2}}{2}$ and $0 \leq y \leq \pi$;
  Therefore, $y = \frac{3\pi}{4}$;

Identities for $\sec^{-1} x, \csc^{-1} x$ and $\cot^{-1} x$

Identities for Inverse Secant, Cosecant and Cotangent

If $x \leq -1$ or $x \geq 1$, then

$$\csc^{-1} x = \sin^{-1} \frac{1}{x} \quad \text{and} \quad \sec^{-1} x = \cos^{-1} \frac{1}{x};$$

For any real x,

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x;$$

- To see why the first holds, note that

$$y = \csc^{-1} x \quad \Rightarrow \quad \csc y = x \quad \Rightarrow \quad \frac{1}{\sin y} = x$$
$$\Rightarrow \quad \sin y = \frac{1}{x} \quad \Rightarrow \quad y = \sin^{-1} \frac{1}{x};$$

Composition of Functions and Their Inverses

Composition of Trigonometric Functions and Their Inverses

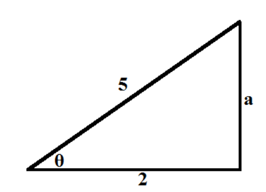
- If $-1 \leq x \leq 1$, then $\sin(\sin^{-1} x) = x$ and $\cos(\cos^{-1} x) = x$;
- If x is any real, $\tan(\tan^{-1} x) = x$;
- If $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, then $\sin^{-1}(\sin x) = x$;
- If $0 \leq x \leq \pi$, then $\cos^{-1}(\cos x) = x$;
- If $-\frac{\pi}{2} < x < \frac{\pi}{2}$, then $\tan^{-1}(\tan x) = x$;

- Find the exact value of each of the following expressions:
  - $\sin(\sin^{-1} 0.357) = 0.357$;
  - $\cos^{-1}(\cos 3) = 3$;
  - $\tan(\tan^{-1}(-11.27)) = -11.27$;
  - $\sin(\sin^{-1} \pi) =$ undefined;
  - $\cos(\cos^{-1} 0.277) = 0.277$;
  - $\tan^{-1}(\tan \frac{4\pi}{3}) = \tan^{-1} \sqrt{3} = \frac{\pi}{3}$;

More Examples I

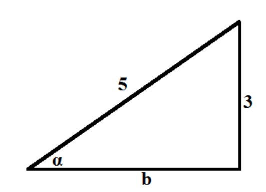
- Find the exact value of $\sin\left(\cos^{-1}\frac{2}{5}\right)$;

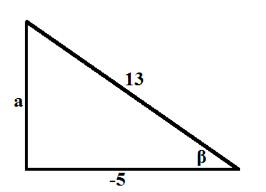
  We set $\theta = \cos^{-1}\frac{2}{5}$; This implies that $\cos\theta = \frac{2}{5}$ and that $0 \leq \theta \leq \pi$; Thus, we get the following: and we compute $a = \sqrt{21}$; Therefore, $\sin\left(\cos^{-1}\frac{2}{5}\right) = \sin\theta = \frac{\sqrt{21}}{5}$;



More Examples II

- Find the exact value of $\sin[\sin^{-1}\frac{3}{5} + \cos^1(-\frac{5}{13})]$;
  We set $\alpha = \sin^{-1}\frac{3}{5}$ and $\beta = \cos^1(-\frac{5}{13})$; Then, we have $\sin\alpha = \frac{3}{5}$ and $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$, and $\cos\beta = -\frac{5}{13}$ and $0 \leq \beta \leq \pi$; Thus, we get the following:





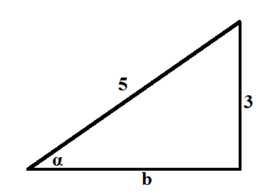
These imply that $\cos\alpha = \frac{4}{5}$ and that $\sin\beta = \frac{12}{13}$; Therefore, we work as follows:

$$\sin[\sin^{-1}\frac{3}{5} + \cos^1(-\frac{5}{13})] = \sin(\alpha + \beta) =$$
$$\sin\alpha\cos\beta + \sin\beta\cos\alpha = \frac{3}{5}(-\frac{5}{13}) + \frac{12}{13} \cdot \frac{4}{5} = \frac{33}{65};$$

Solving a Trigonometric Equation

- Solve $\sin^{-1}\frac{3}{5}+\cos^{-1}x=\pi$;

  We set $\theta=\sin^{-1}\frac{3}{5}$; This implies that $\sin\theta=\frac{3}{5}$ and that $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$; Thus, we get the triangle and we compute $b=4$; Therefore, $\cos(\sin^{-1}\frac{3}{5})=\cos\theta=\frac{4}{5}$;

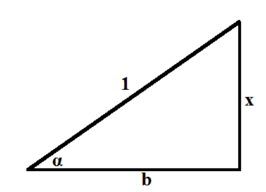


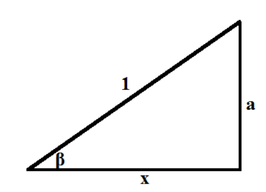
  Now work as follows:

$$\begin{aligned}\sin^{-1}\frac{3}{5}+\cos^{-1}x=\pi \quad &\Rightarrow \quad \cos^{-1}x=\pi-\sin^{-1}\frac{3}{5}\\ \Rightarrow \quad & x=\cos(\pi-\sin^{-1}\frac{3}{5})\\ \Rightarrow \quad & x=\cos\pi\cos(\sin^{-1}\frac{3}{5})+\sin\pi\sin(\sin^{-1}\frac{3}{5})\\ \Rightarrow \quad & x=-1\cdot\frac{4}{5}+0\cdot\frac{3}{5}=-\frac{4}{5};\end{aligned}$$

Verifying a Trigonometric Identity

- Verify the identity $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$;
  We set $\alpha = \sin^{-1} x$ and $\beta = \cos^{-1} x$; Then, we have $\sin\alpha = x$ and $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$, and $\cos\beta = x$ and $0 \leq \beta \leq \pi$; Thus, we get the following:





These imply that $\cos\alpha = \sqrt{1-x^2}$ and that $\sin\beta = \sqrt{1-x^2}$;
Therefore, we work as follows:

$$\sin^{-1} x + \cos^{-1} x = \alpha + \beta = \cos^{-1}(\cos(\alpha+\beta)) =$$
$$\cos^{-1}(\cos\alpha\cos\beta - \sin\alpha\sin\beta) =$$
$$\cos^{-1}(\sqrt{1-x^2}\cdot x - x\cdot\sqrt{1-x^2}) = \cos^{-1} 0 = \frac{\pi}{2};$$

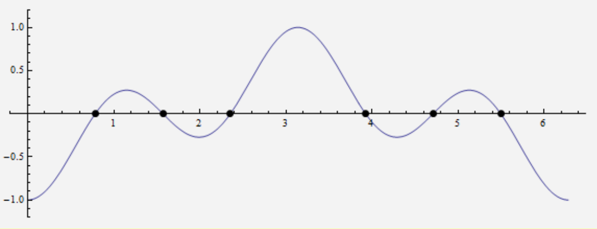
Subsection 6

Trigonometric Equations

Example I

- Solve $2\sin^2 x \cos x - \cos x = 0$, where $0 \le x < 2\pi$;

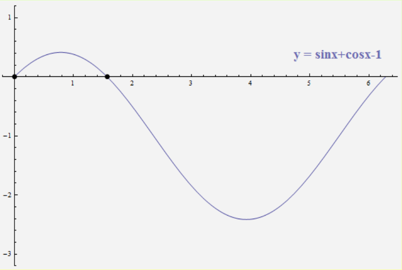
$$\begin{aligned} 2\sin^2 x\cos x - \cos x = 0 \quad &\Rightarrow \quad \cos x(2\sin^2 x - 1) = 0 \\ \Rightarrow \quad & \cos x = 0 \text{ or } \sin^2 x = \tfrac{1}{2} \\ \Rightarrow \quad & \cos x = 0 \text{ or } \sin x = \tfrac{\sqrt{2}}{2} \text{ or } \sin x = -\tfrac{\sqrt{2}}{2} \\ \Rightarrow \quad & x = \tfrac{\pi}{2}, \tfrac{3\pi}{2} \text{ or } x = \tfrac{\pi}{4}, \tfrac{3\pi}{4} \text{ or } x = \tfrac{5\pi}{4}, \tfrac{7\pi}{4}; \end{aligned}$$



Example II

- Solve $\sin x + \cos x = 1$, where $0 \leq x < 2\pi$;

$$\begin{aligned} \sin x + \cos x = 1 \quad &\Rightarrow \quad (\sin x + \cos x)^2 = 1 \\ &\Rightarrow \quad \sin^2 x + 2 \sin x \cos x + \cos^2 x = 1 \\ &\Rightarrow \quad 1 + 2 \sin x \cos x = 1 \\ &\Rightarrow \quad 2 \sin x \cos x = 0 \quad \Rightarrow \quad \sin x = 0 \text{ or } \cos x = 0 \\ &\Rightarrow \quad x = 0, \cancel{\pi} \text{ or } x = \frac{\pi}{2}, \cancel{\frac{3\pi}{2}}; \end{aligned}$$



Example III

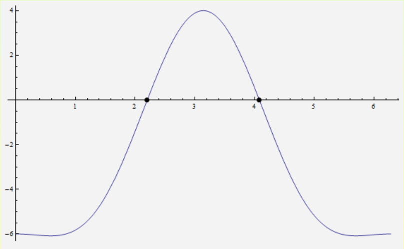
- Solve $3\cos^2 x - 5\cos x - 4 = 0$, where $0 \leq x < 2\pi$;

  We use the quadratic formula to solve for $\cos x$:

$$\cos x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{5 \pm \sqrt{25 - 4 \cdot 3 \cdot (-4)}}{6} = \frac{5 \pm \sqrt{73}}{6};$$

  $\cos x = \frac{5+\sqrt{73}}{6} > 1$ does not have a solution!

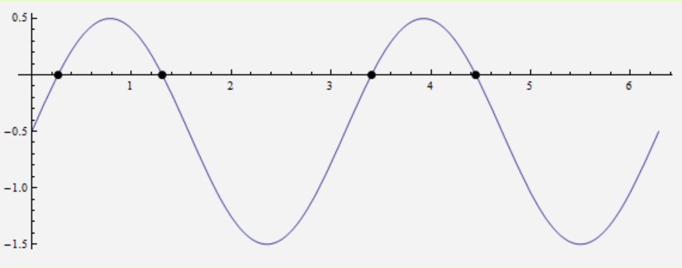
$$\cos x = \frac{5-\sqrt{73}}{6} \Rightarrow x = \cos^{-1}\left(\frac{5-\sqrt{73}}{6}\right) \text{ or } x = 2\pi - \cos^{-1}\left(\frac{5-\sqrt{73}}{6}\right);$$



Example IV

- Solve $\sin 2x = \frac{1}{2}$, where $0 \leq x < 2\pi$;
  Note that $0 \leq x < 2\pi \Rightarrow 0 \leq 2x < 4\pi$;

$$\begin{aligned} & \sin 2x = \tfrac{1}{2} \\ \Rightarrow \quad & 2x = \tfrac{\pi}{6}, \tfrac{5\pi}{6}, \tfrac{13\pi}{6}, \tfrac{17\pi}{6} \\ \Rightarrow \quad & x = \tfrac{\pi}{12}, \tfrac{5\pi}{12}, \tfrac{13\pi}{12}, \tfrac{17\pi}{12}; \end{aligned}$$



Example V

- Solve $\sin^2 2x - \frac{\sqrt{3}}{2}\sin 2x + \sin 2x - \frac{\sqrt{3}}{2} = 0$, where $0^\circ \leq x < 360^\circ$;

$$\begin{aligned} & \sin^2 2x - \frac{\sqrt{3}}{2}\sin 2x + \sin 2x - \frac{\sqrt{3}}{2} = 0 \\ \Rightarrow\quad & \sin 2x(\sin 2x - \frac{\sqrt{3}}{2}) + (\sin 2x - \frac{\sqrt{3}}{2}) = 0 \\ \Rightarrow\quad & (\sin 2x - \frac{\sqrt{3}}{2})(\sin 2x + 1) = 0 \\ \Rightarrow\quad & \sin 2x = \frac{\sqrt{3}}{2} \text{ or } \sin 2x = -1 \\ \Rightarrow\quad & 2x = 60^\circ, 120^\circ, 420^\circ, 480^\circ \text{ or } 2x = 270^\circ, 630^\circ \\ \Rightarrow\quad & x = 30^\circ, 60^\circ, 210^\circ, 240^\circ \text{ or } x = 135^\circ, 315^\circ; \end{aligned}$$