College Trigonometry

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 131

1. Functions and Graphs
   - Complex Numbers
   - Trigonometric Form of Complex Numbers
   - De Moivre's Theorem

Subsection 1

Complex Numbers

Complex Numbers

- The **imaginary unit** i is the number such that $i^2 = -1$;
- If a is a positive real number, then $\sqrt{-a} = \sqrt{(-1) \cdot a} = \sqrt{-1}\sqrt{a} = i\sqrt{a}$. The number $i\sqrt{a}$ is called an **imaginary number**;
- Example:
  - $\sqrt{-36} = i\sqrt{36} = 6i$;
  - $\sqrt{-18} = i\sqrt{18} = 3i\sqrt{2}$;
  - $\sqrt{-1} = i\sqrt{1} = i$;
- A **complex number** is a number of the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$; The number a is the **real part** of $a + bi$ and the number b is the **imaginary part**;
- Example:
  - $-3 + 5i$ has real part -3 and imaginary part 5;
  - $2 - 6i$ has real part 2 and imaginary part -6;
  - $7i$ has real part 0 and imaginary part 7;

Standard Form

- A complex number written in the form $a+bi$ is said to be in the **standard form**;
- Example: Write $7+\sqrt{-45}$ in the standard form;

$$7+\sqrt{-45}=7+i\sqrt{45}=7+3i\sqrt{5};$$

- Example: Write $6-\sqrt{-1}$ in the standard form;

$$6-\sqrt{-1}=6-i\sqrt{1}=6-i;$$

Addition and Subtraction of Complex Numbers

Addition and Subtraction of Complex Numbers

If $a + bi$ and $c + di$ are complex numbers, then

$$\textbf{Addition}: \quad (a + bi) + (c + di) = (a + c) + (b + d)i;$$
$$\textbf{Subtraction}: \quad (a + bi) - (c + di) = (a - c) + (b - d)i;$$

- Example: Simplify the expressions:
  - $(7 - 2i) + (-2 + 4i) = (7 - 2) + (-2 + 4)i = 5 + 2i;$
  - $(-9 + 4i) - (2 - 6i) = (-9 - 2) + (4 - (-6))i = -11 + 10i;$

Multiplication of Complex Numbers

Multiplication of Complex Numbers

If $a + bi$ and $c + di$ are complex numbers, then

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i;$$

- Example: Simplify the expressions:
  - $(3 - 4i)(2 + 5i) = 6 + 15i - 8i - 20i^2 = 6 + 15i - 8i + 20 = 26 + 7i;$
  - $(2 + \sqrt{-3})(4 - 5\sqrt{-3}) = (2 + i\sqrt{3})(4 - 5i\sqrt{3}) = 8 - 10i\sqrt{3} + 4i\sqrt{3} + 5 \cdot 3 = 23 - 6i\sqrt{3};$

Division of Complex Numbers

- The complex number $a - bi$ is called the **complex conjugate** of the complex number $a + bi$;
- The product of a complex number and its conjugate is a real number:

$$(a + bi)(a - bi) = a^2 + b^2;$$

Division of Complex Numbers

If $a + bi$ and $c + di \neq 0$ are complex numbers, then

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2} i;$$

- Example: Simplify the expression:

$$\frac{16 - 11i}{5 + 2i} = \frac{(16 - 11i)(5 - 2i)}{(5 + 2i)(5 - 2i)} = \frac{80 - 32i - 55i - 22}{25 + 4} =$$
$$\frac{58 - 87i}{29} = \frac{58}{29} - \frac{87}{29} i = 2 - 3i;$$

Another Example of Division

- Simplify the expression:

$$\begin{aligned}\frac{8-i}{2+3i} &= \frac{(8-i)(2-3i)}{(2+3i)(2-3i)} \\ &= \frac{16-24i-2i-3}{4+9} \\ &= \frac{13-26i}{13} \\ &= \frac{13}{13}-\frac{26}{13}i \\ &= 1-2i;\end{aligned}$$

Powers of i

- Let us compute a few of the first powers of i:

$$\begin{array}{llll} i^1 = i & i^2 = -1 & i^3 = -i & i^4 = 1 \\ i^5 = i & i^6 = -1 & i^7 = -i & i^8 = 1 \\ i^9 = i & i^{10} = -1 & i^{11} = -i & i^{12} = 1 \\ \ldots & & & \end{array}$$

Powers of i

If n is a positive integer, then $i^n = i^r$, where r is the remainder of the division of n by 4.

- Example: Evaluate i^{153};

$$i^{153} = i^{4 \cdot 38+1} = i^{4 \cdot 38} \cdot i = (i^4)^{38} \cdot i = 1^{38} \cdot i = i;$$

- Example: Evaluate i^{59};

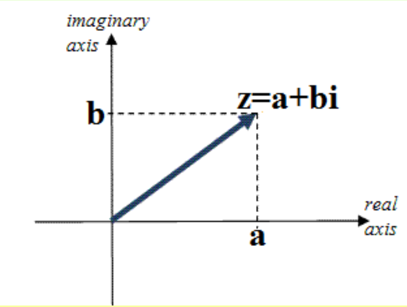
$$i^{59} = i^{4 \cdot 14+3} = i^3 = -i;$$

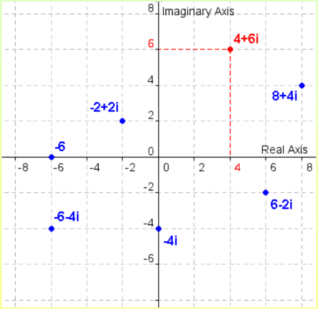
Subsection 2

Trigonometric Form of Complex Numbers

Complex Plane

- Complex numbers can be graphed in a coordinate plane called the **complex plane**;
- The horizontal axis is the **real axis** and the vertical axis the **imaginary axis**;
- A complex number in the **standard** or **rectangular form** $z = a + bi$ is associated with the point $P(a, b)$;





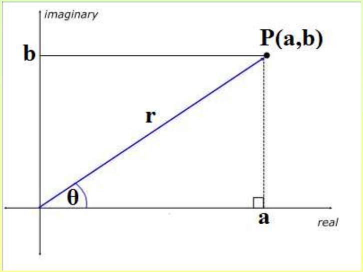
Absolute Value of a Complex Number

Absolute Value of a Complex Number

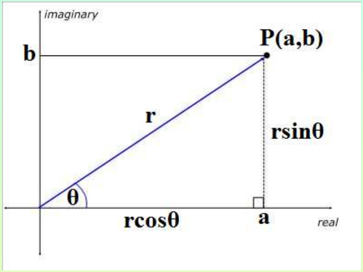
The **absolute value** of a complex number $z = a + bi$, denoted by $|z|$, is

$$|z| = |a + bi| = \sqrt{a^2 + b^2};$$

- Consider the complex number $z = a + bi$;
- If $P(a, b)$ is the point in the complex plane corresponding to z, then $r = |z| = \sqrt{a^2 + b^2}$;



Trigonometric Form of a Complex Number



- $z = a + bi = r\cos\theta + ri\sin\theta = r(\cos\theta + i\sin\theta)$;

Trigonometric or Polar Form of a Complex Number

The complex number $z = a + bi$ can be written in **trigonometric form** as

$$z = r(\cos\theta + i\sin\theta) = r\mathrm{cis}\theta,$$

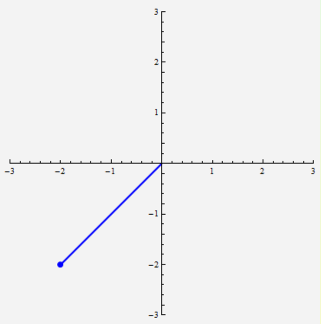
where $a = r\cos\theta$, $b = r\sin\theta$ and $r = \sqrt{a^2 + b^2}$, $\tan\theta = \frac{b}{a}$.

Writing a Complex Number in Trigonometric Form

- Write $z = -2 - 2i$ in trigonometric form;
  Recall the form $z = r(\cos\theta + i\sin\theta)$;
  First calculate $r = |z| = \sqrt{a^2 + b^2} = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$;

  Now factor r out in the rectangular form:

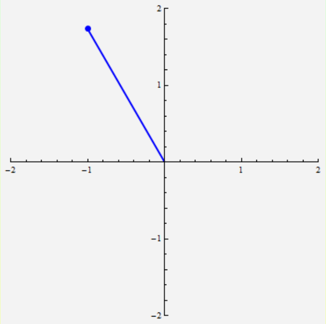
$$\begin{aligned} z &= -2 - 2i \\ &= 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) \\ &= 2\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) \\ &= 2\sqrt{2}\operatorname{cis}\frac{5\pi}{4}; \end{aligned}$$



Writing a Complex Number in Standard Form

- Write $z = 2\text{cis}120^\circ$ in standard form;

$$\begin{aligned} z &= 2\text{cis}120^\circ \\ &= 2(\cos 120^\circ + i \sin 120^\circ) \\ &= 2(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) \\ &= -1 + i\sqrt{3}; \end{aligned}$$



The Product Property of Complex Numbers

- Assume $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ are two complex numbers in polar form;
- We multiply them:

$$\begin{aligned} z_1 z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2(\cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + \\ &\qquad i\sin\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \\ &\qquad i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \\ &= r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)); \end{aligned}$$

The Product Property of Complex Numbers

If $z_1 = r_1 \text{cis}\theta_1$ and $z_2 = r_2 \text{cis}\theta_2$, then

$$z_1 z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2);$$

Example of the Product Property

- Find the product of $z_1 = -1 + i\sqrt{3}$ and $z_2 = -\sqrt{3} + i$ by using the trigonometric forms of the complex numbers; Then write the answer in standard form;

$$\begin{aligned} z_1 &= -1 + i\sqrt{3} \\ &= 2(-\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}) \\ &= 2(\cos\tfrac{2\pi}{3} + i\sin\tfrac{2\pi}{3}); \end{aligned}$$

$$\begin{aligned} z_2 &= -\sqrt{3} + i \\ &= 2(-\tfrac{\sqrt{3}}{2} + i\tfrac{1}{2}) \\ &= 2(\cos\tfrac{5\pi}{6} + i\sin\tfrac{5\pi}{6}); \end{aligned}$$

Therefore, $z_1 z_2 = 2 \cdot 2(\cos(\frac{2\pi}{3} + \frac{5\pi}{6}) + i\sin(\frac{2\pi}{3} + \frac{5\pi}{6})) = 4(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}) = 4(0 + i(-1)) = -4i$;

The Quotient Property of Complex Numbers

The Product Property of Complex Numbers

If $z_1 = r_1 \text{cis}\theta_1$ and $z_2 = r_2 \text{cis}\theta_2$, then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2);$$

- Example: Find the quotient of $z_1 = -1 + i$ and $z_2 = \sqrt{3} - i$ by using trigonometric forms. Write the answer in standard form;

$$z_1 = -1 + i = \sqrt{2}(-\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i) = \sqrt{2}(\cos\tfrac{3\pi}{4} + i\sin\tfrac{3\pi}{4});$$

$$z_2 = \sqrt{3} - i = 2(\tfrac{\sqrt{3}}{2} - \tfrac{1}{2}i) = 2(\cos(-\tfrac{\pi}{6}) + i\sin(-\tfrac{\pi}{6}));$$

$$\tfrac{z_1}{z_2} = \tfrac{\sqrt{2}}{2}(\cos(\tfrac{3\pi}{4} - (-\tfrac{\pi}{6})) + i\sin(\tfrac{3\pi}{4} - (-\tfrac{\pi}{6}))) = \tfrac{\sqrt{2}}{2}(\cos\tfrac{11\pi}{12} + i\sin\tfrac{11\pi}{12}) \approx 0.707(-0.966 + 0.259i) \approx -0.683 + 0.183i;$$

Subsection 3

De Moivre's Theorem

De Moivre's Theorem

De Moivre's Theorem

If $z = r\text{cis}\theta$ and n is a positive integer, then

$$z^n = r^n \text{cis} n\theta.$$

- Example: Find $(2\text{cis}30^\circ)^5$; Write the answer in standard form;

$$(2\text{cis}30^\circ)^5 = 2^5\text{cis}(5 \cdot 30^\circ) = 32\text{cis}150^\circ =$$
$$32(\cos 150^\circ + i \sin 150^\circ) = 32(-\tfrac{\sqrt{3}}{2} + i\tfrac{1}{2}) = -16\sqrt{3} + 16i;$$

- Example: Find $(1+i)^8$ using De Moivre's Theorem; Write the answer in standard form;

$$(1+i)^8 = (\sqrt{2}(\text{cis}\tfrac{\pi}{4}))^8 = \sqrt{2}^8\text{cis}(8 \cdot \tfrac{\pi}{4}) = 16\text{cis}(2\pi) = 16;$$

Using De Moivre's Theorem to Find Roots

De Moivre's Theorem for Finding Roots

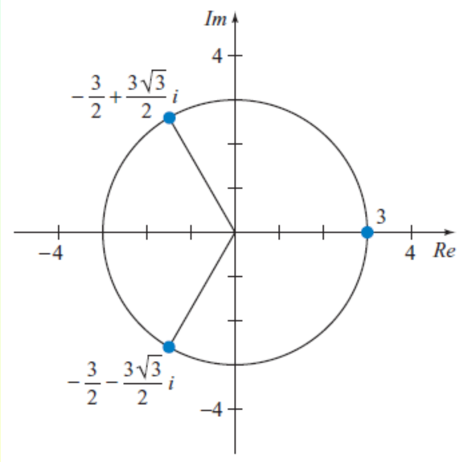
If $z = r\text{cis}\theta$ is a complex number, then there exist n distinct n-th roots of z, given by

$$w_k = r^{1/n}\text{cis}\frac{\theta + 360^\circ k}{n}, \ k = 0, 1, 2, \ldots, n-1;$$

- Find the three cube roots of 27;

$$27 = 27\text{cis}0^\circ;$$
$$w_k = 27^{1/3}\text{cis}\frac{0^\circ + 360^\circ k}{3}, \ k = 0, 1, 2;$$
$$w_0 = 27^{1/3}\text{cis}0 = 3;$$
$$w_1 = 27^{1/3}\text{cis}\frac{360^\circ}{3} = 3\text{cis}120^\circ = 3(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i) = -\tfrac{3}{2} + \tfrac{3\sqrt{3}}{2}i;$$
$$w_2 = 27^{1/3}\text{cis}\frac{720^\circ}{3} = 3\text{cis}240^\circ = 3(-\tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i) = -\tfrac{3}{2} - \tfrac{3\sqrt{3}}{2}i;$$

The Three Cube Roots of 27

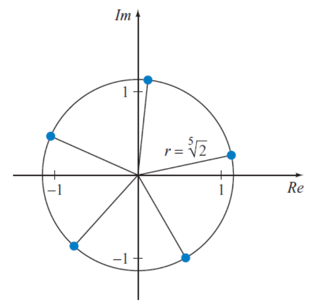


Another Example

- Find all fifth roots of $z = 1 + i\sqrt{3}$;

$$1 + i\sqrt{3} = 2\text{cis}60^\circ;$$
$$w_k = 2^{1/5}\text{cis}\frac{60^\circ + 360^\circ k}{5}, \; k = 0, 1, 2, 3, 4;$$

$$w_0 = \sqrt[5]{2}\text{cis}\frac{60^\circ}{5} = \sqrt[5]{2}\text{cis}12^\circ;$$
$$w_1 = \sqrt[5]{2}\text{cis}\frac{420^\circ}{5} = \sqrt[5]{2}\text{cis}84^\circ;$$
$$w_2 = \sqrt[5]{2}\text{cis}\frac{780^\circ}{5} = \sqrt[5]{2}\text{cis}156^\circ;$$
$$w_3 = \sqrt[5]{2}\text{cis}\frac{1140^\circ}{5} = \sqrt[5]{2}\text{cis}228^\circ;$$
$$w_4 = \sqrt[5]{2}\text{cis}\frac{1500^\circ}{5} = \sqrt[5]{2}\text{cis}300^\circ;$$



Properties of the n-th Roots of z

- **Geometric Property**: All n-th roots of z are equally spaced on a circle with center $(0,0)$ and a radius of $|z|^{1/n}$;
- **Absolute Value Properties**:
  1. If $|z| = 1$, then each n-th root of z has absolute value 1;
  2. If $|z| > 1$, then each n-th root has an absolute value of $|z|^{1/n}$, where $|z|^{1/n}$ is greater than 1 but less than $|z|$;
  3. If $|z| < 1$, then each n-th root has an absolute value of $|z|^{1/n}$, where $|z|^{1/n}$ is less than 1 but greater than $|z|$;
- **Argument Property**: If the argument of z is θ, then the argument of w_0 is $\frac{\theta}{n}$ and the arguments of the remaining roots can be determined by adding multiples of $\frac{360^\circ}{n}$ to $\frac{\theta}{n}$;