College Trigonometry

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LSSU Math 131

Subsection 1

Exponential Functions and Applications

Definition of Exponential Functions

Definition of an Exponential Function

The **exponential function with base** b is defined by

$$f(x) = b^x,$$

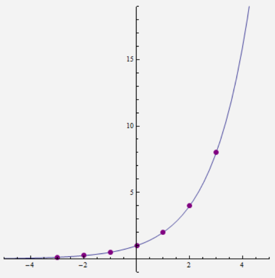
where $0 < b \neq 1$ and x is a real number.

- Example: Evaluate $f(x) = 3^x$ at $x = 2$, $x = -4$ and $x = \pi$;

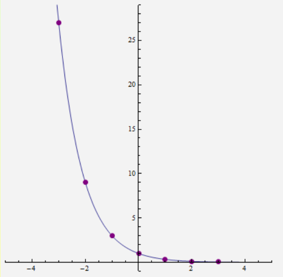
$$f(2) = 3^2 = 9;$$
$$f(-4) = 3^{-4} = \frac{1}{3^4} = \frac{1}{81};$$
$$f(\pi) = 3^\pi \approx 31.544;$$

Graphs of Exponential Functions

x	$y = 2^x$
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

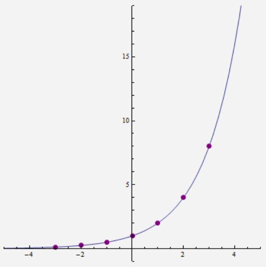


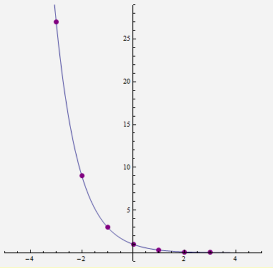
x	$y = (\frac{1}{3})^x$
-3	27
-2	9
-1	3
0	1
1	$\frac{1}{3}$
2	$\frac{1}{9}$
3	$\frac{1}{27}$



Properties of $f(x) = b^x$

- For $0 < b \neq 1$, the exponential function defined by $f(x) = b^x$ has the following properties:
  1. Its domain is the set of all real numbers and its range is the set of all positive real numbers;
  2. The graph is a smooth continuous curve with a y-intercept at $(0, 1)$ and passing through $(1, b)$;
  3. The function f is one-to-one (its graph passes the horizontal line test);
  4. If $b > 1$, f is increasing and has the negative x-axis as a horizontal asymptote; If $0 < b < 1$, f is decreasing and has the positive x-axis as a horizontal asymptote;

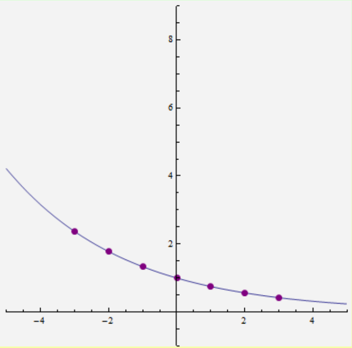




Graphing an Exponential Function

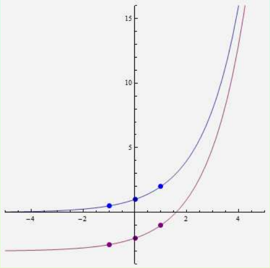
Graph the exponential function $f(x) = \left(\frac{3}{4}\right)^x$;

x	$y = (\frac{3}{4})^x$
-3	$\frac{64}{27}$
-2	$\frac{16}{9}$
-1	$\frac{4}{3}$
0	1
1	$\frac{3}{4}$
2	$\frac{9}{16}$
3	$\frac{27}{64}$

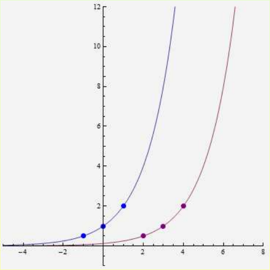


Using Translations to Graph Exponential Functions

x	$y = 2^x$	$y = 2^x - 3$
-2	$\frac{1}{4}$	$-\frac{11}{4}$
-1	$\frac{1}{2}$	$-\frac{5}{2}$
0	1	-2
1	2	-1
2	4	1

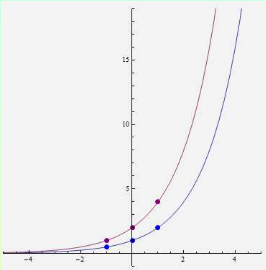


x	$y = 2^x$	x	$y = 2^{x-3}$
-2	$\frac{1}{4}$	1	$\frac{1}{4}$
-1	$\frac{1}{2}$	2	$\frac{1}{2}$
0	1	3	1
1	2	4	2
2	4	5	4

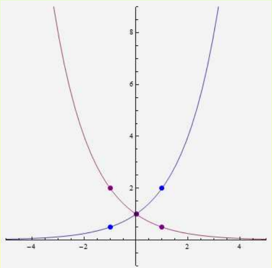


Using Stretching and Reflections

x	$y = 2^x$	$y = 2 \cdot 2^x$
-2	$\frac{1}{4}$	$\frac{1}{2}$
-1	$\frac{1}{2}$	1
0	1	2
1	2	4
2	4	8

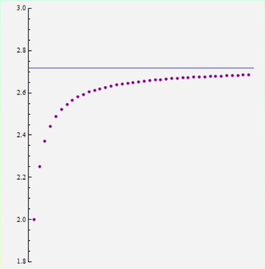


x	$y = 2^x$	$y = 2^{-x}$
-2	$\frac{1}{4}$	4
-1	$\frac{1}{2}$	2
0	1	1
1	2	$\frac{1}{2}$
2	4	$\frac{1}{4}$

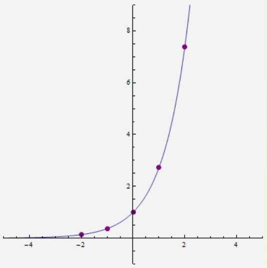


The Natural Exponential Function $f(x) = e^x$

The **number** $e \approx 2.718$ is defined as the number that $\left(1 + \frac{1}{n}\right)^n$ approaches as n increases without a bound:



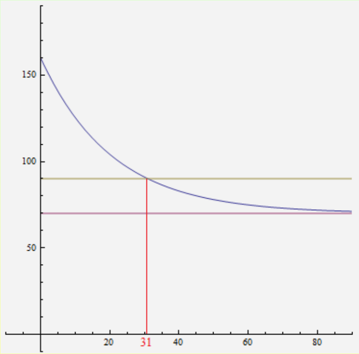
The function $f(x) = e^x$ is called the **natural exponential function**:



Application: Cooling

- A cup of coffee is heated to 160°F and placed in a room that maintains a temperature of 70°F; The temperature T of the coffee in degrees Fahrenheit after t minutes is given by $T = 70 + 90e^{-0.0485t}$;
  - Find the temperature of the coffee 20 minutes after its is placed into the room; $T(20) = 70 + 90e^{-0.0485 \cdot 20} \approx 104.1°F$;
  - Use a graphing utility to determine when the temperature of the coffee will reach 90°F;

    This will happen after about 31 minutes;



Subsection 2

Logarithmic Functions and Applications

Definition of Logarithms and Logarithmic Functions

Definition of a Logarithm and a Logarithmic Function

If $0 < b \neq 1$ and $x > 0$, then

$$y = \log_b x \quad \text{if and only if} \quad b^y = x.$$

The expression $\log_b x$ is read the **logarithm base b of x**; The function defined by $f(x) = \log_b x$ is the **logarithmic function with base b**; It is the inverse function of the exponential $g(x) = b^x$;

- Because of the inverse relationship between the exponential function and the logarithmic function with base b, we get

$$b^{\log_b x} = x \quad \text{and} \quad \log_b b^x = x;$$

Exponential and Logarithmic Forms

The **exponential form** of $y = \log_b x$ is $b^y = x$;
The **logarithmic form** of $b^y = x$ is $y = \log_b x$;

Switching Between Logarithmic and Exponential Forms

- Write each equation in its exponential form:
  - $3 = \log_2 8 \iff 2^3 = 8$;
  - $2 = \log_{10}(x+5) \iff 10^2 = x+5$;
  - $\log_e x = 4 \iff e^4 = x$;
  - $\log_b b^3 = 3 \iff b^3 = b^3$;
- Write each equation in its logarithmic form:
  - $3^2 = 9 \iff 2 = \log_3 9$;
  - $5^3 = x \iff 3 = \log_5 x$;
  - $a^b = c \iff b = \log_a c$;
  - $b^{\log_b 5} = 5 \iff \log_b 5 = \log_b 5$;

Basic Properties of Logarithmic Functions

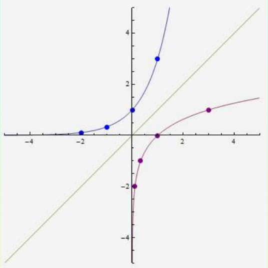
Basic Logarithmic Properties

1. $\log_b b = 1$;
2. $\log_b 1 = 0$;
3. $\log_b (b^x) = x$;
4. $b^{\log_b x} = x$;

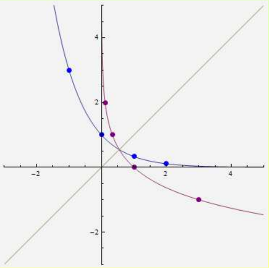
- Example: Evaluate each of the following logarithms:
  - $\log_8 1 = 0$;
  - $\log_5 5 = 1$;
  - $\log_2 (2^4) = 4$;
  - $3^{\log_3 7} = 7$;

Graphs of Logarithmic Functions

x	$y = 3^x$	x	$y = \log_3 x$
-2	$\frac{1}{9}$	$\frac{1}{9}$	-2
-1	$\frac{1}{3}$	$\frac{1}{3}$	-1
0	1	1	0
1	3	3	1
2	9	9	2

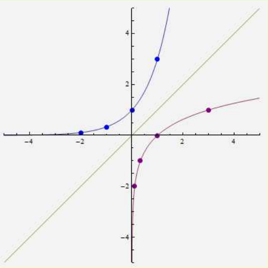


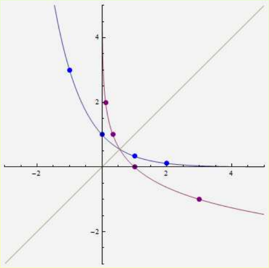
x	$y = (\frac{1}{3})^x$	x	$y = \log_{1/3} x$
-2	9	9	-2
-1	3	3	-1
0	1	1	0
1	$\frac{1}{3}$	$\frac{1}{3}$	1
2	$\frac{1}{9}$	$\frac{1}{9}$	2



Properties of the Graphs

- For $0 < b \neq 1$, the function $f(x) = \log_b x$ has the following properties:
  1. The domain is the set of positive real numbers and its range is the set of all real numbers;
  2. The graph has an x-intercept at $(1, 0)$ and passing through $(b, 1)$;
  3. If $b > 1$, f is increasing and has the negative y-axis as a vertical asymptote; If $0 < b < 1$, f is decreasing and has the positive y-axis as a vertical asymptote;





Domains of Logarithmic Functions

Domain of $f(x) = \log_b x$

The domain of the function $f(x) = \log_b x$ is $\{x : x > 0\}$ or in interval notation $(0, \infty)$.

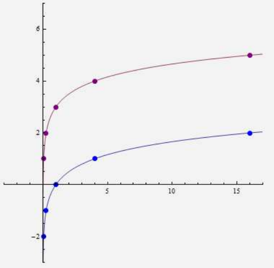
- Example: Find the domain of each of the following logarithmic functions:
  - $f(x) = \log_6 (x - 3)$;
    $$x - 3 > 0 \quad \Rightarrow \quad x > 3;$$
    In interval notation $\mathrm{Dom}(f) = (3, \infty)$;
  - $g(x) = \log_2 |x + 2|$;
    $$|x + 2| > 0 \quad \Rightarrow \quad x \neq -2;$$
    In interval notation $\mathrm{Dom}(g) = (-\infty, -2) \cup (-2, \infty)$;
  - $h(x) = \log_5 (\frac{x}{8-x})$;
    We use the sign table method:

	$x < 0$	$0 < x < 8$	$x > 8$
$\frac{x}{8-x}$	$-$	$+$	$-$

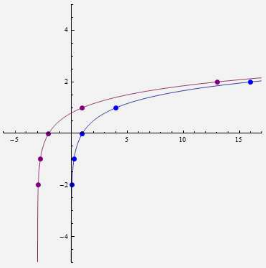
Therefore, $0 < x < 8$ or in interval notation $\mathrm{Dom}(h) = (0, 8)$;

Using Translations to Graph Logarithmic Functions

x	$y = \log_4 x$	$y = \log_4 x + 3$
$\frac{1}{16}$	-2	1
$\frac{1}{4}$	-1	2
1	0	3
4	1	4
16	2	5



x	$y = \log_4 x$	x	$y = \log_4 (x+3)$
$\frac{1}{16}$	-2	$-\frac{47}{16}$	-2
$\frac{1}{4}$	-1	$-\frac{11}{4}$	-1
1	0	-2	0
4	1	1	1
16	2	13	2



Common and Natural Logarithms

Definition of Common and Natural Logarithms

The function $f(x) = \log_{10} x$ is called the **common logarithmic function;** It is usually written as $f(x) = \log x$;
The function $f(x) = \log_e x$ is called the **natural logarithmic function;** It is usually written as $f(x) = \ln x$;

- Note that the definitions of logarithmic functions give

$$y = \log x \quad \text{if and only if} \quad 10^y = x;$$

- Similarly,

$$y = \ln x \quad \text{if and only if} \quad e^y = x;$$

Application: Physiology

- As the population of a city increases, the average walking speed of a pedestrian also increases; A relation between the average pedestrian walking speed s in miles per hour and the population x is thousands is given by

$$s(x) = 0.37 \ln x + 0.05;$$

  - What is the average walking speed in San Francisco, which has a population of $780,000$ people?

$$s(780) = 0.37 \cdot \ln 780 + 0.05 \approx 2.51 \text{ mph};$$

  - Estimate the population of a city where the average walking speed is 3.1 mph;

$$3.1 = 0.37 \ln x + 0.05 \quad \Rightarrow \quad 3.05 = 0.37 \ln x$$
$$\Rightarrow \quad \ln x = \frac{3.05}{0.37} \quad \Rightarrow \quad x = e^{\frac{3.05}{0.37}} \approx 3801.85;$$

    Thus, the population is about $3,800,000$;

Subsection 3

Properties of Logarithms and Logarithmic Scales

Properties of Logarithms

Properties of Logarithms

In the following properties b, M and N are positive real numbers ($b \neq 1$).

- **Product Property**: $\log_b(MN) = \log_b M + \log_b N$;
- **Quotient Property**: $\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$;
- **Power Property**: $\log_b(M^p) = p \log_b M$;
- **Logarithm-of-Each-Side**: $M = N$ implies $\log_b M = \log_b N$;
- **One-to-One Property**: $\log_b M = \log_b N$ implies $M = N$;

Expanding and Condensing Logarithmic Expressions

- Expand the following logarithmic expressions, assuming that all variables represent positive numbers; When possible, evaluate logarithmic expressions;
  - $\log_3(xy^2) = \log_3 x + \log_3(y^2) = \log_3 x + 2\log_3 y$;
  - $\ln\left(\frac{e\sqrt{y}}{z^3}\right) = \ln(e\sqrt{y}) - \ln(z^3) = \ln e + \ln(y^{1/2}) - 3\ln z = 1 + \frac{1}{2}\ln y - 3\ln z$;
- Rewrite the following expressions as single logarithms with coefficient 1; All variables represent positive numbers;
  - $2\ln x + \frac{1}{2}\ln(x+4) = \ln(x^2) + \ln[(x+4)^{1/2}] = \ln[x^2\sqrt{x+4}]$;
  - $\log_5(x^2-4) + 3\log_5 y - \log_5(x-2)^2 = \log_5(x^2-4) + \log_5(y^3) - \log_5(x-2)^2 = \log_5[y^3(x^2-4)] - \log_5(x-2)^2 = \log_5\left[\frac{y^3(x^2-4)}{(x-2)^2}\right] = \log_5\left[\frac{y^3(x+2)(x-2)}{(x-2)^2}\right] = \log_5\left[\frac{y^3(x+2)}{x-2}\right]$;

Change-of-Base Formula

Change-of-Base Formula

If x, a and b are positive real numbers, with $a, b \neq 1$, then

$$\log_b x = \frac{\log_a x}{\log_a b};$$

In particular, if x, b are positive numbers, with $b \neq 1$,

$$\log_b x = \frac{\log x}{\log b} = \frac{\ln x}{\ln b};$$

- Example: Use both Change-of-Base and calculators to compute to the nearest thousandth:
  - $\log_3 18 = \frac{\ln 18}{\ln 3} \approx 2.631;$
  - $\log_{12} 400 = \frac{\ln 400}{\ln 12} \approx 2.411;$

Logarithmic Scales: The Richter Scale

The Richter Scale Magnitude of an Earthquake

An earthquake with intensity of I has a **Richter scale magnitude** of

$$M = \log\left(\frac{I}{I_0}\right),$$

where I_0 is the measure of the intensity of a zero-level (smallest seismographically measurable) earthquake;

- Example: Find the Richter scale magnitude of a 1999 Joshua Tree, CA, earthquake whose intensity was $I = 12,589,254 I_0$;

$$M = \log\left(\frac{I}{I_0}\right) = \log\left(\frac{12,589,254 I_0}{I_0}\right) = \log 12,589,254 \approx 7.1;$$

- Example: Find the intensity of the 1999 Taiwan earthquake, which measured 7.6 on the Richter scale;
$M = \log\left(\frac{I}{I_0}\right) \Rightarrow 7.6 = \log\left(\frac{I}{I_0}\right) \Rightarrow \frac{I}{I_0} = 10^{7.6} \Rightarrow I = 10^{7.6} I_0 = 39,810,717 I_0;$

Logarithmic Scales: Comparing Earthquake Intensities

- Example: The 1960 Chile earthquake had a Richter scale magnitude of 9.5, whereas the 1989 San Francisco earthquake a Richter scale magnitude of 7.1; Compare the intensities of the two earthquakes;

$$M_{\text{C}} = \log\left(\frac{I_{\text{C}}}{I_0}\right) \quad \Rightarrow \quad \frac{I_{\text{C}}}{I_0} = 10^{M_{\text{C}}} \quad \Rightarrow \quad I_{\text{C}} = 10^{M_{\text{C}}} I_0;$$

$$M_{\text{SF}} = \log\left(\frac{I_{\text{SF}}}{I_0}\right) \quad \Rightarrow \quad \frac{I_{\text{SF}}}{I_0} = 10^{M_{\text{SF}}} \quad \Rightarrow \quad I_{\text{SF}} = 10^{M_{\text{SF}}} I_0;$$

$$\frac{I_{\text{C}}}{I_{\text{SF}}} = \frac{10^{M_{\text{C}}} I_0}{10^{M_{\text{SF}}} I_0} = 10^{M_{\text{C}} - M_{\text{SF}}} = 10^{9.5-7.1} = 10^{2.4} \approx 251$$

$$\Rightarrow \quad I_{\text{C}} \approx 251 I_{\text{SF}};$$

Logarithmic Scales: pH of a Solution

Definition of the pH of a Solution

The **pH of a solution** with a hydronium anion concentration of H^+ moles per liter is given by
$$\text{pH} = -\log{[H^+]};$$

- Example: Find the pH of each liquid:
  - Orange juice with $H^+ = 2.8 \times 10^{-4}$ mole/liter;
  $$\text{pH} = -\log{[H^+]} = -\log{(2.8 \times 10^{-4})} \approx 3.55;$$
  - Milk with $H^+ = 3.97 \times 10^{-7}$ mole/liter;
  $$\text{pH} = -\log{[H^+]} = -\log{(3.97 \times 10^{-7})} \approx 6.4;$$
  - Baking soda with $H^+ = 3.98 \times 10^{-9}$ mole/liter;
  $$\text{pH} = -\log{[H^+]} = -\log{(3.98 \times 10^{-9})} \approx 8.4;$$
- Example: A sample blood has pH of 7.3. Find the hydronium ion concentration of the blood;
  $\text{pH} = -\log{[H^+]} \Rightarrow \log{[H^+]} = -\text{pH} \Rightarrow H^+ = 10^{-\text{pH}} = 10^{-7.3} \approx 5 \times 10^{-8}$ moles/liter;

Subsection 4

Exponential and Logarithmic Equations

Solving Exponential Equations

Equality of Exponents Theorem

If $b^x = b^y$, then $x = y$, provided $0 < b \neq 1$.

- Example: Use the Equality of Exponents to solve $2^{3x-7} = 32$;

$$2^{3x-7} = 32 \quad \Rightarrow \quad 2^{3x-7} = 2^5$$
$$\Rightarrow \quad 3x - 7 = 5 \quad \Rightarrow \quad 3x = 12 \quad \Rightarrow \quad x = 4;$$

The Exponential-Logarithmic Correspondence

$$y = \log_b x \quad \text{if and only if} \quad b^y = x.$$

- Example: Solve the exponential equation $5^x = 40$;

$$5^x = 40 \quad \Rightarrow \quad x = \log_5 40;$$

Solving by Taking Logarithms of Both Sides

- Solve $3^{2x-1} = 5^{x+2}$;

$$\begin{aligned} 3^{2x-1} = 5^{x+2} \quad &\Rightarrow \quad \ln 3^{2x-1} = \ln 5^{x+2} \\ &\Rightarrow \quad (2x-1)\ln 3 = (x+2)\ln 5 \\ &\Rightarrow \quad 2x\ln 3 - \ln 3 = x\ln 5 + 2\ln 5 \\ &\Rightarrow \quad 2x\ln 3 - x\ln 5 = 2\ln 5 + \ln 3 \\ &\Rightarrow \quad x(2\ln 3 - \ln 5) = 2\ln 5 + \ln 3 \\ &\Rightarrow \quad x = \frac{2\ln 5 + \ln 3}{2\ln 3 - \ln 5} \approx 7.3; \end{aligned}$$

Exponential Equations Having Two Solutions

- Solve the equation $\dfrac{2^x + 2^{-x}}{2} = 3$;

$$\frac{2^x + 2^{-x}}{2} = 3 \quad \Rightarrow \quad 2^x + 2^{-x} = 6 \quad \Rightarrow \quad 2^x(2^x + 2^{-x}) = 6 \cdot 2^x$$
$$\Rightarrow \quad (2^x)^2 + 1 = 6 \cdot 2^x \quad \Rightarrow \quad (2^x)^2 - 6 \cdot 2^x + 1 = 0$$
$$\overset{y=2^x}{\Rightarrow} \quad y^2 - 6y + 1 = 0 \quad (\text{Recall } y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a})$$
$$\Rightarrow \quad y = \frac{6 \pm \sqrt{36 - 4}}{2} = \frac{6 \pm 4\sqrt{2}}{2} = 3 \pm 2\sqrt{2};$$

Therefore,

$$2^x = 3 \pm 2\sqrt{2} \quad \Rightarrow \quad x = \log_2(3 \pm 2\sqrt{2});$$

Solving Logarithmic Equations

The Exponential-Logarithmic Correspondence

$$y = \log_b x \quad \text{if and only if} \quad b^y = x.$$

- Example: Solve the logarithmic equation $\log(3x-5) = 2$;

$$\log(3x-5) = 2 \quad \Rightarrow \quad 3x - 5 = 10^2$$
$$\Rightarrow \quad 3x = 105 \quad \Rightarrow \quad x = 35;$$

- Example: Solve the logarithmic equation $\log 2x - \log(x-3) = 1$;

$$\log 2x - \log(x-3) = 1 \quad \Rightarrow \quad \log \frac{2x}{x-3} = 1$$
$$\Rightarrow \quad \frac{2x}{x-3} = 10^1 \quad \Rightarrow \quad 2x = 10(x-3)$$
$$\Rightarrow \quad 2x = 10x - 30 \quad \Rightarrow \quad 8x = 30 \quad \Rightarrow \quad x = \frac{15}{4};$$

Example

- Example: Solve the logarithmic equation $\log_3 x + \log_3 (x+6) = 3$;

$$\begin{aligned} &\log_3 x + \log_3 (x+6) = 3 \\ \Rightarrow \quad & \log_3 [x(x+6)] = 3 \\ \Rightarrow \quad & x(x+6) = 3^3 \\ \Rightarrow \quad & x^2 + 6x - 27 = 0 \\ \Rightarrow \quad & (x+9)(x-3) = 0 \\ \Rightarrow \quad & x = -9 \text{ or } x = 3; \end{aligned}$$

Only $x = 3$ is an admissible solution!

Applying the One-to-One Property

Equality of Logarithms Theorem

If $\log_b x = \log_b y$, then $x = y$, provided $0 < b \neq 1$.

- Example: Solve $\ln(3x+8) = \ln(2x+2) + \ln(x-2)$;

$$\begin{aligned} & \ln(3x+8) = \ln(2x+2) + \ln(x-2) \\ \Rightarrow \quad & \ln(3x+8) = \ln[(2x+2)(x-2)] \\ \Rightarrow \quad & 3x+8 = (2x+2)(x-2) \\ \Rightarrow \quad & 3x+8 = 2x^2 - 2x - 4 \\ \Rightarrow \quad & 2x^2 - 5x - 12 = 0 \\ \Rightarrow \quad & (2x+3)(x-4) = 0 \\ \Rightarrow \quad & x = -\frac{3}{2} \text{ or } x = 4 \end{aligned}$$

Only $x = 4$ is an admissible solution!

One More Example

- Example: Solve $\ln x = \frac{1}{2}\ln(2x+\frac{5}{2}) + \frac{1}{2}\ln 2$;

$$\begin{aligned} & \ln x = \tfrac{1}{2}\ln(2x+\tfrac{5}{2}) + \tfrac{1}{2}\ln 2 \\ \Rightarrow \quad & 2\ln x = \ln(2x+\tfrac{5}{2}) + \ln 2 \\ \Rightarrow \quad & \ln(x^2) = \ln[2(2x+\tfrac{5}{2})] \\ \Rightarrow \quad & x^2 = 4x+5 \\ \Rightarrow \quad & x^2-4x-5=0 \\ \Rightarrow \quad & (x+1)(x-5)=0 \\ \Rightarrow \quad & x=-1 \text{ or } x=5 \end{aligned}$$

Only $x = 5$ is an admissible solution!

Subsection 5

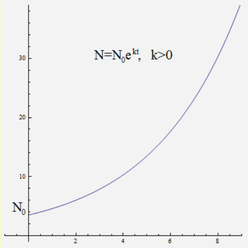
Exponential Growth and Decay

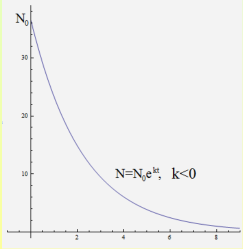
Exponential Growth and Decay

Exponential Growth and Decay Functions

If a quantity N increases or decreases at a rate proportional to the amount present at time t, then the quantity can be modeled by $N(t) = N_0 e^{kt}$, where N_0 is the value of N at time $t = 0$, and k is a constant called the **growth rate constant**.

- If $k > 0$, N increases as t increases and $N(t) = N_0 e^{kt}$ is called an **exponential growth function**;
- If $k < 0$, N decreases as t increases and $N(t) = N_0 e^{kt}$ is called an **exponential decay function**;





Example: Population Growth

- The population of a city is growing exponentially and it was 16,400 in 1995 and 20,200 in 2005.
  - Find the exponential growth function that models the population growth of the city;
    Let $N(t) = N_0 e^{kt}$ model the population of the city t years since 1995;
    Then, we get

$$N_0 = 16400;$$
$$20200 = 16400e^{10k} \quad \Rightarrow \quad e^{10k} = \frac{20200}{16400}$$
$$\Rightarrow \quad 10k = \ln \frac{20200}{16400} \quad \Rightarrow \quad k = \frac{1}{10} \ln \frac{20200}{16400} \approx 0.0208;$$

    Therefore, $N(t) = 16400e^{0.0208t}$;
  - Predict to the nearest 100, the population of the city in 2020;

$$N(25) = 16400e^{0.0208 \cdot 25} \approx 27,600;$$

Example: Radioactive Decay

- Find the exponential decay function for the amount of phosphorus (^{32}P) that remains in a sample after t days, given that the half-life of phosphorus is 14 days;

  Let $N(t) = N_0 e^{kt}$ model the amount of phosphorus remaining in a sample after t days; Then, we get

$$\frac{1}{2}N_0 = N_0 e^{14k}$$
$$\Rightarrow \quad e^{14k} = \frac{1}{2}$$
$$\Rightarrow \quad 14k = \ln \frac{1}{2}$$
$$\Rightarrow \quad k = \frac{1}{14} \ln \frac{1}{2} \approx -0.0495;$$

Therefore, $N(t) = N_0 e^{-0.0495t}$;

Example: Carbon Dating

- Estimate the age of a bone if it now has 85% of the carbon-14 that it had at time $t = 0$, given that the half-life of carbon-14 (^{14}C) is 5730 years;

  Let $N(t) = N_0 e^{kt}$ model the amount of carbon-14 remaining in the bone after t years; Then, we get

$$\frac{1}{2}N_0 = N_0 e^{5730k} \quad \Rightarrow \quad e^{5730k} = \frac{1}{2}$$
$$\Rightarrow \quad 5730k = \ln\frac{1}{2} \quad \Rightarrow \quad k = \frac{1}{5730}\ln\frac{1}{2} \approx -0.00012;$$

  Therefore, $N(t) = N_0 e^{-0.00012t}$;
  Therefore, if the bone has $N = 0.85N_0$, we get

$$0.85N_0 = N_0 e^{-0.00012t} \quad \Rightarrow \quad 0.85 = e^{-0.00012t}$$
$$\Rightarrow \quad -0.00012t = \ln 0.85 \quad \Rightarrow \quad t = -\frac{1}{0.00012}\ln 0.85$$
$$\approx 1343.486 \text{ years};$$

Compound Interest

The Compound Interest Formula

If a principal P is invested at an annual rate r, expressed as a decimal, and compounded n times per year for t years, it produces the balance $A = P(1+\frac{r}{n})^{nt}$.

- Example: Find the balance if $1,000 is invested at an annual interest rate of 10% for 2 years compounded monthly;

$$A = 1000(1+\frac{0.1}{12})^{12\cdot 2} = 1000\cdot 1.0083^{24} \approx \$1,220.39$$

- Example: Find the balance if $1,000 is invested at an annual interest rate of 10% for 2 years compounded daily;

$$A = 1000(1+\frac{0.1}{365})^{365\cdot 2} = 1000\cdot 1.000274^{730} \approx \$1,221.37;$$

Continuous Compounding

Continuous Compounding Interest Formula

If an account with principal P and annual interest rate r is compounded continuously for t years, then the balance is $A = Pe^{rt}$.

- Example: Find the balance after 4 years on \$3,000 invested at an annual rate of 4% compounded continuously;

$$A = Pe^{rt} = 3000e^{0.04 \cdot 4} = 3000e^{0.16} = \$3,520.53;$$

- Example: Find the time required for money invested at an annual rate of 5% to double in value if the investment is compounded semiannually; Do the same for continuous compounding;

$$\begin{aligned} A &= P(1 + \tfrac{r}{n})^{nt} \\ 2P &= P(1 + \tfrac{0.05}{2})^{2t} \\ 2 &= 1.025^{2t} \\ 2t &= \log_{1.025} 2 \\ t &= \tfrac{\ln 2}{2 \ln 1.025} \approx 14.04; \end{aligned}$$

$$\begin{aligned} A &= Pe^{rt} \\ 2P &= Pe^{0.05t} \\ 2 &= e^{0.05t} \\ 0.05t &= \ln 2 \\ t &= 20 \ln 2 \approx 13.86; \end{aligned}$$

Restricted Growth and the Logistic Model

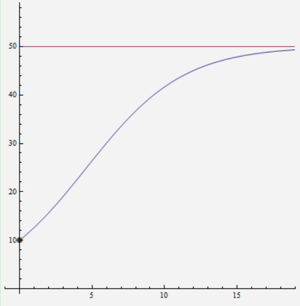
- A **restricted growth model**, unlike the exponential growth model, takes into account the effects of *limited resources*;

The Logistic Model

The magnitude of a population at time $t \geq 0$ is given by

$$P(t) = \frac{c}{1 + ae^{-bt}},$$

where c is the **carrying capacity** (max population supported) and b is a positive constant called the **growth rate constant**; The **initial population** is $P_0 = P(0)$ and the relation between a, c and P_0 is $a = \frac{c - P_0}{P_0}$;



Example: Coyote Population

At the beginning of 2005 the coyote population in a restricted desert area was estimated at 200; By the beginning of 2007, it had increased to 250; The estimate for the carrying capacity of the area is 500 coyotes;

- What is the growth rate constant of the logistic model of the coyote population?

$$c = 500, \quad P_0 = 200, \quad a = \frac{c - P_0}{P_0} = \frac{500-200}{200} = \frac{3}{2};$$

$$P(t) = \frac{500}{1+1.5e^{-bt}};$$

$$P(2) = 250 \quad \Rightarrow \quad \frac{500}{1+1.5e^{-2b}} = 250 \quad \Rightarrow \quad 1 + 1.5e^{-2b} = 2$$

$$\Rightarrow \ 1.5e^{-2b} = 1 \ \Rightarrow \ e^{-2b} = \frac{2}{3} \ \Rightarrow \ b = -\frac{1}{2}\ln\frac{2}{3} \approx 0.203;$$

- When will the population reach 400 coyotes?

$$P(t) = 400 \ \Rightarrow \ \frac{500}{1+1.5e^{-0.203t}} = 400 \ \Rightarrow \ 1 + 1.5e^{-0.203t} = 1.25$$

$$\Rightarrow \quad e^{-0.203t} = \frac{0.25}{1.5} \quad \Rightarrow \quad t = -\frac{1}{0.203}\ln\frac{0.25}{1.5} \approx 8.8 \text{ years};$$