College Trigonometry

George Voutsadakis[1]

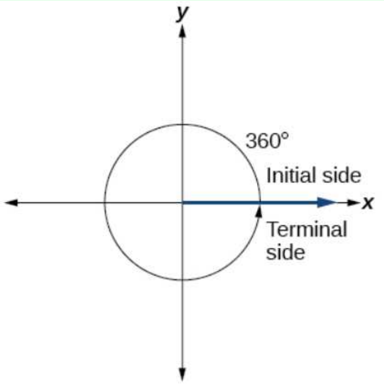
[1]Mathematics and Computer Science
Lake Superior State University

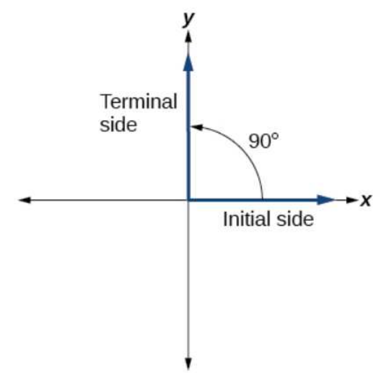
LSSU Math 131

Subsection 1

Angles

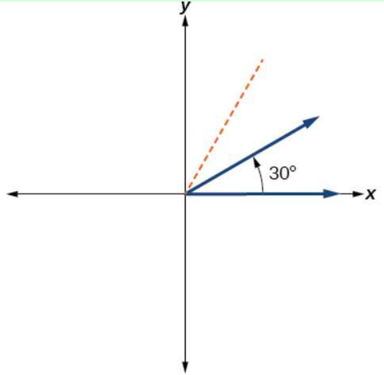
Angles in Degrees in Standard Position

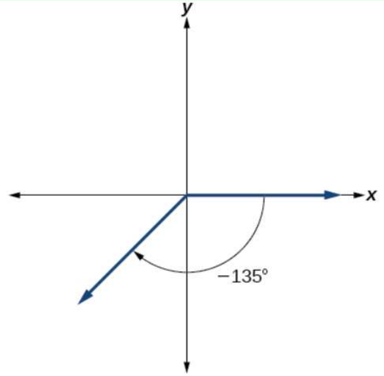




Drawing an Angle in Degrees in Standard Position

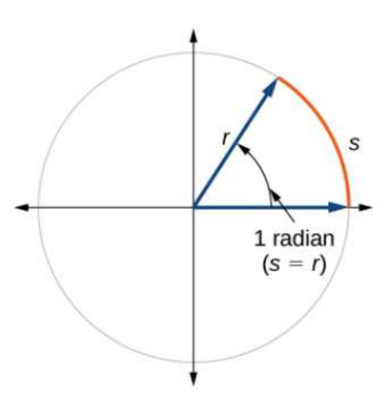
- Sketch an angle of 30° in standard position.
  Note that $30^\circ = \frac{1}{3}90^\circ$.

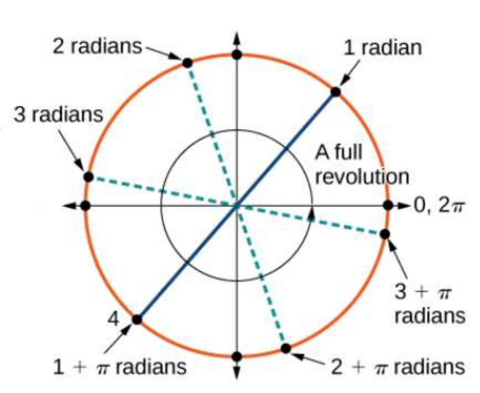




- Sketch an angle of -135° in standard position.
  Note that $-135^\circ = -90^\circ - 45^\circ$.

Angles in Radians in Standard Position





Finding a Radian Measure

- Find the radian measure of one-third of a full rotation.

  A full rotation corresponds to 2π radians.

  Hence, a third of a full rotation equals $\frac{2\pi}{3}$ radians.

Converting Radians to Degrees

- Convert each radian measure to degrees.
  (a) $\frac{\pi}{6}$
  (b) 3

(a) Find the fraction of the circle represented:

$$\frac{\frac{\pi}{6}}{2\pi} = \frac{1}{12}.$$

Then find how much in degrees is $\frac{1}{12}$th of the circle:

$$\frac{1}{12} \cdot 360^\circ = 30^\circ.$$

(b) Same process:

$$\frac{3}{2\pi} \cdot 360^\circ = \left(\frac{540}{\pi}\right)^\circ.$$

Converting Degrees to Radians

- Convert 15 degrees to radians.

  Find the fraction of the circle represented:

$$\frac{15^\circ}{360^\circ} = \frac{1}{24}.$$

  Then find how much in radians is $\frac{1}{24}$th of the circle:

$$\frac{1}{24} \cdot 2\pi = \frac{\pi}{12}.$$

Coterminal and Reference Angles

- **Coterminal angles** are two angles in standard position that have the same terminal side.
- An angle's **reference angle** is the size of the smallest acute angle, t', formed by the terminal side of the angle t and the horizontal axis.

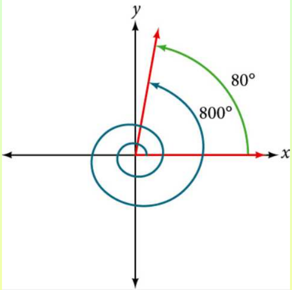
Finding an Angle Coterminal with an Angle $> 360^\circ$

- Find the least positive angle θ that is coterminal with an angle measuring 800°, where $0^\circ \leq \theta < 360^\circ$.

  Find how many times 360 goes into 800: 2 times.

  Subtract these many multiples of 360 from 800:

$$800^\circ - 2 \cdot 360^\circ = 80^\circ.$$

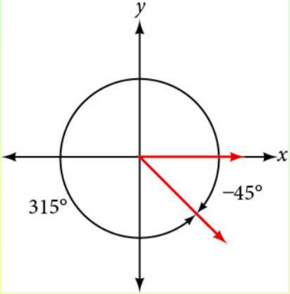


Finding an Angle Coterminal with an Angle $< 0^\circ$

- Show the angle with measure -45° on a circle and find a positive coterminal angle α such that $0^\circ \leq \alpha < 360^\circ$.

Add as many multiples of 360 as needed to get an angle between 0° and 360°.

$$-45^\circ + 360^\circ = 315^\circ.$$



Finding Coterminal Angles Using Radians

- Find an angle β that is coterminal with $\frac{19\pi}{4}$, where $0 \leq \beta < 2\pi$.

  Find how many times 2π goes into $\frac{19\pi}{4}$:

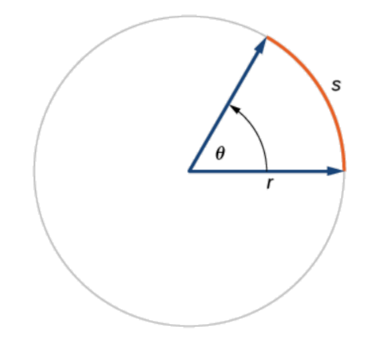
  2 times.

  Subtract this many multiples of 2π from $\frac{19\pi}{4}$:

$$\frac{19\pi}{4} - 2 \cdot 2\pi = \frac{19\pi}{4} - \frac{16\pi}{4} = \frac{3\pi}{4}.$$

Finding the Length of an Arc

- In a circle of radius r, the length of an arc s subtended by an angle with measure θ in radians, shown in



is

$$s = r\theta.$$

Careful: When using this equation, θ must be in radians!

Finding the Length of an Arc

- Find the arc length along a circle of radius 10 units subtended by an angle of 215°.

  First convert the measure of the angle to radians:

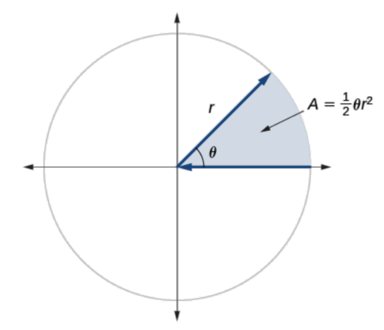
$$\frac{215^\circ}{360^\circ} \cdot 2\pi = \frac{43\pi}{36}.$$

  Then use

$$s = r\theta = 10 \cdot \frac{43\pi}{36} = \frac{430\pi}{36} = \frac{215\pi}{18} \text{ units.}$$

Area of a Sector

- The area of a sector of a circle with radius r subtended by an angle θ, measured in radians,

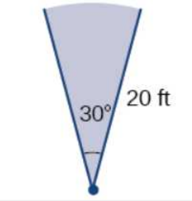


is
$$A = \frac{1}{2}\theta r^2.$$

Careful: When using this equation, θ must be in radians!

Finding the Area of a Sector

- An automatic lawn sprinkler sprays a distance of 20 feet while rotating 30 degrees. What is the area of the sector of grass the sprinkler waters?



Convert degrees to radians:

$$\frac{30^\circ}{360^\circ} \cdot 2\pi = \frac{\pi}{6}.$$

Now we get

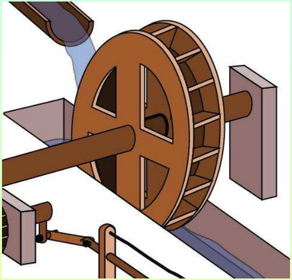
$$A = \frac{1}{2}\theta r^2 = \frac{1}{2} \cdot \frac{\pi}{6} \cdot 20^2 = \frac{400\pi}{12} = \frac{100\pi}{3} \text{ feet}^2.$$

Angular Speed and Linear Speed

- **Angular Speed**:
$$\omega = \frac{\theta}{t}.$$
- **Linear Speed**:
$$v = \frac{s}{t}.$$
- Recall that $s = r\theta$.
- So we get
$$v = \frac{s}{t} = \frac{r\theta}{t} = r \cdot \frac{\theta}{t} = r\omega.$$

Finding Angular Speed

- A water wheel completes 1 rotation every 5 seconds. Find the angular speed in radians per second.



The wheel turns 2π radians in 5 seconds.

So we get

$$\omega = \frac{\theta}{t} = \frac{2\pi}{5} \text{ radians/second.}$$

Finding a Linear Speed

- A bicycle has wheels 28 inches in diameter. A tachometer determines the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road.

  First compute the angular speed of the wheel:

  $$\omega = \frac{\theta}{t} = \frac{180 \cdot 2\pi \text{ rad}}{1 \text{ min}} = 360\pi \text{ rad/min}.$$

  Now compute the linear speed:

  $$v = r\omega = 14 \text{ in} \cdot 360\pi \text{ rad/min} = 5040\pi \text{ in/min}.$$

Subsection 2

Right Triangle Trigonometry

Trigonometric Functions in a Right Triangle

- The six trigonometric functions are defined as follows:

$$\sin t = \frac{\text{opposite}}{\text{hypotenuse}}$$

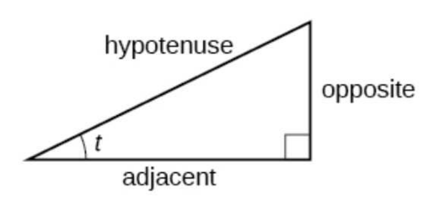
$$\cos t = \frac{\text{adjecent}}{\text{hypotenuse}}$$

$$\tan t = \frac{\text{opposite}}{\text{adjacent}}$$

$$\csc t = \frac{\text{hypotenuse}}{\text{opposite}}$$

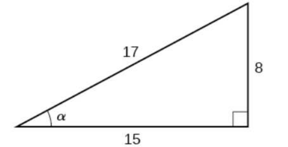
$$\sec t = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\cot t = \frac{\text{adjacent}}{\text{opposite}}$$



Evaluating a Trigonometric Function of a Right Triangle

- Given the triangle shown, find the value of $\cos\alpha$.

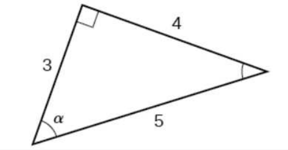


We have by definition

$$\cos\alpha = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{15}{17}.$$

Evaluating Trigonometric Functions of Angles

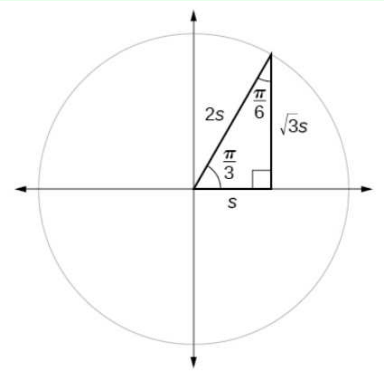
- Using the triangle shown, evaluate $\sin\alpha$, $\cos\alpha$, $\tan\alpha$, $\sec\alpha$, $\csc\alpha$ and $\cot\alpha$.

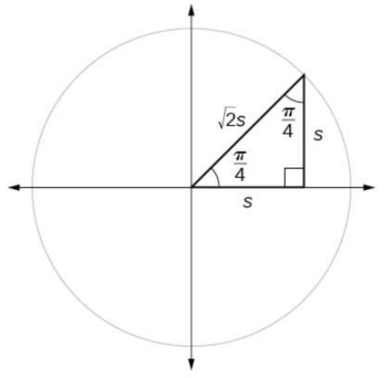


We use again the definitions:

$$\sin\alpha = \frac{4}{5}, \quad \cos\alpha = \frac{3}{5}, \quad \tan\alpha = \frac{4}{3},$$
$$\sec\alpha = \frac{5}{3}, \quad \csc\alpha = \frac{5}{4}, \quad \cot\alpha = \frac{3}{4}.$$

Right Triangles With Special Angles





Evaluating Trigonometric Functions of Special Angles

- Find the exact value of the trigonometric functions of $\frac{\pi}{3}$ using side lengths.

  Recall the right triangle configuration.

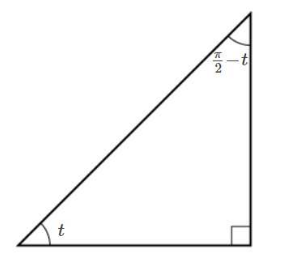
  We have

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}s}{2s} = \frac{\sqrt{3}}{2}, \quad \cos\frac{\pi}{3} = \frac{s}{2s} = \frac{1}{2}, \quad \tan\frac{\pi}{3} = \frac{\sqrt{3}s}{s} = \sqrt{3},$$

$$\csc\frac{\pi}{3} = \frac{2s}{\sqrt{3}s} = \frac{2\sqrt{3}}{3}, \quad \sec\frac{\pi}{3} = \frac{2s}{s} = 2, \quad \cot\frac{\pi}{3} = \frac{s}{\sqrt{3}s} = \frac{\sqrt{3}}{3}.$$

Cofunction Identities

- The cofunction identities in radians:



$$\cos t = \sin\left(\frac{\pi}{2} - t\right), \quad \sin t = \cos\left(\frac{\pi}{2} - t\right),$$

$$\tan t = \cot\left(\frac{\pi}{2} - t\right), \quad \cot t = \tan\left(\frac{\pi}{2} - t\right),$$

$$\sec t = \csc\left(\frac{\pi}{2} - t\right), \quad \csc t = \sec\left(\frac{\pi}{2} - t\right).$$

Using Cofunction Identities

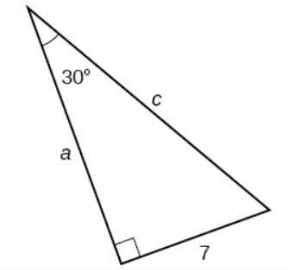
- If $\sin t = \frac{5}{12}$, find $\cos\left(\frac{\pi}{2} - t\right)$.

  We get

$$\cos\left(\frac{\pi}{2} - t\right) = \sin t = \frac{5}{12}.$$

Finding Missing Side Lengths Using Trigonometric Ratios

- Find the unknown sides of the triangle.



We get

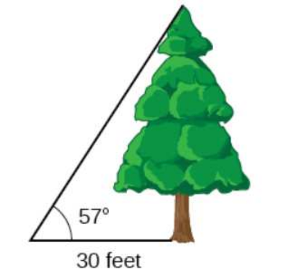
$$\sin 30^\circ = \frac{7}{c} \Rightarrow \frac{1}{2} = \frac{7}{c} \Rightarrow c = 14.$$

Moreover,

$$\cos 30^\circ = \frac{a}{c} \Rightarrow \frac{\sqrt{3}}{2} = \frac{a}{14} \Rightarrow a = 7\sqrt{3}.$$

Measuring a Distance Indirectly

- To find the height of a tree, a person walks to a point 30 feet from the base of the tree. She measures an *angle of elevation* between a line of sight to the top of the tree and the ground, as shown in the figure. Find the height of the tree.



We get

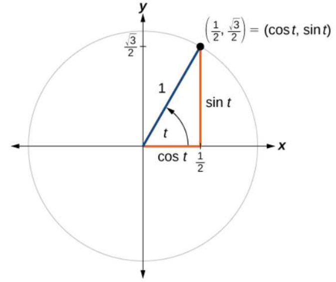
$$\tan(57^\circ) = \frac{h}{30} \Rightarrow h = 30\tan(57^\circ) \Rightarrow h \approx 46.2 \text{ feet.}$$

Subsection 3

Unit Circle

Finding Function Values for Sine and Cosine

- Point P is a point on the unit circle corresponding to an angle of t, as shown in the figure. Find $\cos(t)$ and $\sin(t)$.



  Recall that $\cos t = \dfrac{\text{adjacent}}{\text{hypotenuse}}$.

  But on the unit circle, hypotenuse $= 1$. So

$$\cos t = \frac{x}{1} = x = \frac{1}{2}.$$

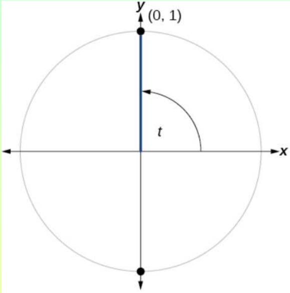
  Similarly,

$$\sin t = y = \frac{\sqrt{3}}{2}.$$

Calculating Sines and Cosines along an Axis

- Find $\cos(90^\circ)$ and $\sin(90^\circ)$.

  Draw the angle on the unit circle.



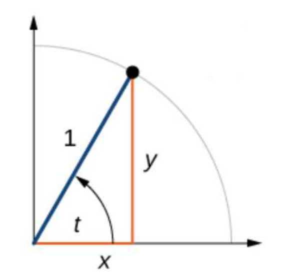
Then we have

$$\cos(90^\circ) = x = 0 \quad \text{and} \quad \sin(90^\circ) = y = 1.$$

The Pythagorean Identity

- The **Pythagorean Identity** states that, for any real number t,

$$\cos^2 t + \sin^2 t = 1.$$



Looking at the figure,

$$\cos^2 t + \sin^2 t = x^2 + y^2 = 1.$$

Finding a Cosine from a Sine or a Sine from a Cosine

- If $\sin(t) = \frac{3}{7}$ and t is in the second quadrant, find $\cos(t)$.
  By the Pythagorean Identity,

$$\sin^2 t + \cos^2 t = 1$$

$$\frac{9}{49} + \cos^2 t = 1$$

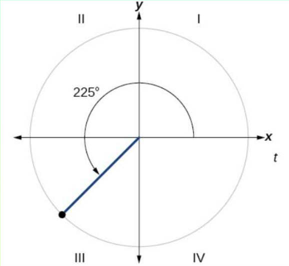
$$\cos^2 t = 1 - \frac{9}{49} = \frac{40}{49}$$

$$\cos t = \pm\sqrt{\frac{40}{49}} = \pm\frac{2\sqrt{10}}{7}.$$

  Since $\frac{\pi}{2} < t < \pi$, $\cos t < 0$.
  Therefore, $\cos t = -\frac{2\sqrt{10}}{7}$.

Finding a Reference Angle

- Find the reference angle of 225°.



Recall that an angle t's *reference angle* is the size of the smallest acute angle, t', formed by the terminal side of the angle t and the horizontal axis.

So we have

$$225^\circ - 180^\circ = 45^\circ.$$

Using Reference Angles to Find Sine and Cosine

- Using a reference angle, find the exact value of $\cos(150^\circ)$ and $\sin(150^\circ)$.

  The angle 150° has reference angle 30°.

  So we get

  $$\sin(150^\circ) = \pm\sin(30^\circ) = \pm\tfrac{1}{2},$$

  $$\cos(150^\circ) = \pm\cos(30^\circ) = \pm\tfrac{\sqrt{3}}{2}.$$

  Since $90^\circ < 150^\circ < 180^\circ$,

  $$\sin(150^\circ) = +\frac{1}{2} \quad \text{and} \quad \cos(150^\circ) = -\frac{\sqrt{3}}{2}.$$

Using Reference Angles to Find Sine and Cosine

- Using the reference angle, find $\cos \frac{5\pi}{4}$. and $\sin \frac{5\pi}{4}$.

  The angle $\frac{5\pi}{4}$ has reference angle $\frac{\pi}{4}$.

  So we get

  $$\sin \frac{5\pi}{4} = \pm \sin \frac{\pi}{4} = \pm \frac{\sqrt{2}}{2},$$

  $$\cos \frac{5\pi}{4} = \pm \cos \frac{\pi}{4} = \pm \frac{\sqrt{2}}{2}.$$

  Since $\pi < \frac{5\pi}{4} < \frac{3\pi}{2}$,

  $$\sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2} \quad \text{and} \quad \cos \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}.$$

Using the Unit Circle to Find Coordinates

- Find the coordinates of the point on the unit circle at an angle of $\frac{7\pi}{6}$.
  Since the point is on the unit circle,

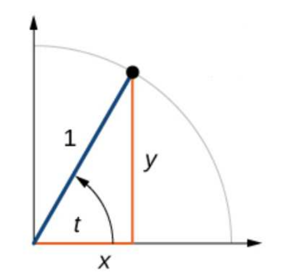
$$\begin{aligned} (x,y) &= (\cos t, \sin t) \\ &= (\cos\tfrac{7\pi}{6}, \sin\tfrac{7\pi}{6}) \\ &= (-\cos\tfrac{\pi}{6}, -\sin\tfrac{\pi}{6}) \\ &= (-\tfrac{\sqrt{3}}{2}, -\tfrac{1}{2}). \end{aligned}$$

Subsection 4

The Other Trigonometric Functions

The Trigonometric Functions on the Unit Circle

- On the unit circle

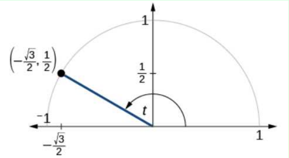


$$\sin t = y, \quad \cos t = x, \quad \tan t = \frac{y}{x},$$

$$\sec t = \frac{1}{x}, \quad \csc t = \frac{1}{y}, \quad \cot t = \frac{x}{y}.$$

Finding Trigonometric Functions from a Point

- The point $(-\frac{\sqrt{3}}{2}, \frac{1}{2})$ is on the unit circle, as shown. Find $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$ and $\cot t$.



We have

$$\sin t = \frac{1}{2}, \qquad \cos t = -\frac{\sqrt{3}}{2}, \qquad \tan t = -\frac{\sqrt{3}}{3},$$

$$\sec t = -\frac{2\sqrt{3}}{3}, \qquad \csc t = 2, \qquad \cot t = -\sqrt{3}.$$

Finding the Trigonometric Functions of an Angle

- Find $\sin t$, $\cos t$, $\tan t$, $\sec t$, $\csc t$ and $\cot t$, when $t = \frac{\pi}{6}$.
  We have

$$\sin\frac{\pi}{6} = \frac{1}{2}, \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \qquad \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3},$$

$$\sec\frac{\pi}{6} = \frac{2\sqrt{3}}{3}, \qquad \csc\frac{\pi}{6} = 2, \qquad \cot\frac{\pi}{6} = \sqrt{3}.$$

Using Reference Angles to Find Trigonometric Functions

- Use reference angles to find all six trigonometric functions of $-\frac{5\pi}{6}$.

  The reference angle of $-\frac{5\pi}{6}$ is $\frac{\pi}{6}$.

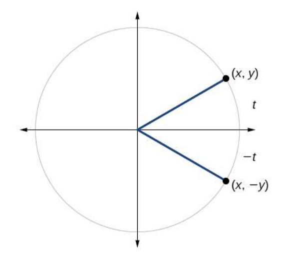
  Moreover $-\frac{5\pi}{6}$ is in Quadrant III.

  So we get:

$$\sin\left(-\frac{5\pi}{6}\right) = -\frac{1}{2}, \quad \cos\left(-\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}, \quad \tan\left(-\frac{5\pi}{6}\right) = \frac{\sqrt{3}}{3},$$

$$\sec\left(-\frac{5\pi}{6}\right) = -\frac{2\sqrt{3}}{3}, \quad \csc\left(-\frac{5\pi}{6}\right) = -2, \quad \cot\left(-\frac{5\pi}{6}\right) = \sqrt{3}.$$

Even and Odd Trigonometric Functions



- From the figure we can tell that

$$\sin(-t) = -\sin t, \quad \cos(-t) = \cos t, \quad \tan(-t) = -\tan t,$$

$$\sec(-t) = \sec t, \quad \csc(-t) = -\csc t, \quad \cot(-t) = -\cot t.$$

- So $\cos t$ and $\sec t$ are the only even trig functions.

Using Even and Odd Properties of Trigonometric Functions

- If the secant of angle t is 2, what is the secant of $-t$?
  Since $\sec t$ is even,
  $$\sec(-t) = \sec t = 2.$$

Fundamental Identities

$$\tan t = \frac{\sin t}{\cos t}$$

$$\sec t = \frac{1}{\cos t}$$

$$\csc t = \frac{1}{\sin t}$$

$$\cot t = \frac{1}{\tan t} = \frac{\cos t}{\sin t}.$$

Using Identities to Evaluate Trigonometric Functions

- Given $\sin(45^\circ) = \frac{\sqrt{2}}{2}$, $\cos(45^\circ) = \frac{\sqrt{2}}{2}$, evaluate $\tan(45^\circ)$.

$$\tan(45^\circ) = \frac{\sin(45^\circ)}{\cos(45^\circ)} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1.$$

- Given $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$, $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$, evaluate $\sec\left(\frac{5\pi}{6}\right)$.

$$\sec\left(\frac{5\pi}{6}\right) = \frac{1}{\cos\left(\frac{5\pi}{6}\right)} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3}.$$

Using Identities to Simplify Trigonometric Expressions

- Simplify $\frac{\sec t}{\tan t}$.

$$\begin{aligned} \frac{\sec t}{\tan t} &= \frac{\frac{1}{\cos t}}{\frac{\sin t}{\cos t}} \\ &= \frac{1}{\cos t} \frac{\cos t}{\sin t} \\ &= \frac{1}{\sin t} \\ &= \csc t. \end{aligned}$$

Alternate Forms of the Pythagorean Identity

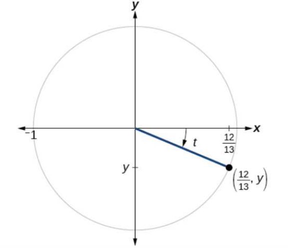
$$\begin{aligned} 1+\tan^2 t &= \sec^2 t \\ \cot^2 t + 1 &= \csc^2 t. \end{aligned}$$

- Let us see why the first holds:

$$\begin{aligned} 1+\tan^2 t &= 1+\frac{\sin^2 t}{\cos^2 t} \\ &= \frac{\cos^2 t}{\cos^2 t}+\frac{\sin^2 t}{\cos^2 t} \\ &= \frac{\cos^2 t+\sin^2 t}{\cos^2 t} \\ &= \frac{1}{\cos^2 t} \\ &= \sec^2 t. \end{aligned}$$

Using Identities to Relate Trigonometric Functions

- If $\cos(t) = \frac{12}{13}$ and t is in quadrant IV, as shown, find the values of the other five trigonometric functions.



By the Pythagorean Identity $\sin^2 t + \cos^2 t = 1$, we get

$$\sin t = -\sqrt{1-\cos^2 t} = -\sqrt{1-\frac{144}{169}} = -\sqrt{\frac{25}{169}} = -\frac{5}{13}.$$

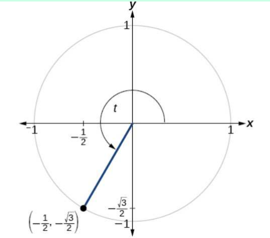
All other trigonometric functions are now easy to compute:

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{5}{13}}{\frac{12}{13}} = -\frac{5}{12}, \quad \sec t = \frac{1}{\cos t} = \frac{1}{\frac{12}{13}} = \frac{13}{12},$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{5}{13}} = -\frac{13}{5}, \quad \cot t = \frac{1}{\tan t} = \frac{1}{-\frac{5}{12}} = -\frac{12}{5}.$$

Finding the Values of Trigonometric Functions

- Find the values of the six trigonometric functions of angle t based on the figure.



The sine and cosine can be determined by the figure:

$$\sin t = -\frac{\sqrt{3}}{2} \quad \text{and} \quad \cos t = -\frac{1}{2}.$$

The other numbers follow easily:

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}, \quad \sec t = \frac{1}{\cos t} = \frac{1}{-\frac{1}{2}} = -2,$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{-\frac{\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3}, \quad \cot t = \frac{1}{\tan t} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

Finding the Value of Trigonometric Functions

- If $\sin(t) = -\frac{\sqrt{3}}{2}$ and $\cos(t) = \frac{1}{2}$, find $\sec(t)$, $\csc(t)$, $\tan(t)$, $\cot(t)$.

  We get

$$\sec t = \frac{1}{\cos t} = \frac{1}{\frac{1}{2}} = 2,$$

$$\csc t = \frac{1}{\sin t} = \frac{1}{\frac{-\sqrt{3}}{2}} = -\frac{2\sqrt{3}}{3},$$

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = -\sqrt{3},$$

$$\cot t = \frac{1}{\tan t} = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}.$$