The Real Heart of Mathematics

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Contents

Chapter 1

Linear Equations and Systems

1.1 First-Degree Equations

An **equation** is a statement that two mathematical expressions are equal; for example

$$3x + 5y = 2, \quad 2x + 4 = y, \quad 3x - 7 = 3y + 8x - 3$$

are examples of equations. The letters in each equation are called **variables**.

A **first-degree equation** is one that can be written in the form

$$ax + b = c,$$

where a, b and c are constants (real numbers) and $a \neq 0$. For instance $4x + 5 = 7$ is a first degree equation but $x^3 = 13$ or $x^2 - 5x + 2 = 9$ are not first-degree equations since one of the variables appears cubed and squared, respectively.

A **solution** of an equation is a number that, when substituted for the variable in the equation produces a true statement. For instance, consider the first-degree equation $3x+5 = 11$. The number 2 is a solution to this equation because, if 2 is substituted for x in the equation, we get $3 \cdot 2 + 5 = 11$, which is a true statement. On the other hand 1 is not a solution of this equation since $3 \cdot 1 + 5 = 8 \neq 11$, i.e., substituting 1 for x in the equation does not produce a true statement. To **solve** a first-degree equation means to find its solutions.

There is a specific method one can follow in order to solve a first-degree equation. This method is based on the following two **properties of equality**:

1. The same number may be added to or subtracted from both sides of an equation:

$$\text{If } a = b, \text{ then } a + c = b + c, \text{ and } a - c = b - c.$$

2. Both sides of an equation may be multiplied or divided by the same nonzero number:

$$\text{If } a = b \text{ and } c \neq 0, \text{ then } ac = bc \text{ and } \frac{a}{c} = \frac{b}{c}.$$

Here are some examples of how these properties may be used to solve first-degree equations.

Example 1 *Solve the equation* $6x + 5 = -7$.

Solution:
We start from the given equation $6x+5 = -7$. By subtracting 5 from both sides, we get

$$6x + 5 - 5 = -7 - 5, \text{ i.e., } 6x = -12.$$

Now divide both sides of the equation by 6 to get

$$\frac{6x}{6} = \frac{-12}{6}, \text{ i.e., } x = -2.$$

We finally substitute -2 for x in the original equation to verify that it produces a true statement:

$$6 \cdot (-2) + 5 = -12 + 5 = -7.$$

■

Now solve the following exercises to get used to the process of solving first-degree equations.

Exercise 1 *Solve the equation* $4x - 2 = 18$.

Exercise 2 *Solve the equation* $2x + 7 = 3x - 4$.

Exercise 3 *Solve the equation* $5x - 8 = 2x + 1$.

Why are first-degree equations so important? They help us to solve problems from all areas of life and science. Some examples follow:

Example 2 (Physics) *Town A and town B are 450 miles apart and connected via a straight railroad. A train moves along the straight track from A to B at a speed of 50 mph and is already 100 miles away from town A. In how many hours will it reach town B?*

Solution:
First, we pick a variable to represent the unknown quantity. Let us pick t for the **t**ime that will be needed to arrive at town B. Since the train is already 100 miles away from A and is moving at a speed of 50 mph, it will be $100 + 50t$ miles away from town A in t hours. Therefore to find t we have to solve the first-degree equation

$$100 + 50t = 450.$$

Subtract 100 from both sides to obtain

$$100 + 50t - 100 = 450 - 100, \text{ i.e., } 50t = 350.$$

Now divide both sides by 50 to get

$$\frac{50t}{50} = \frac{350}{50}, \text{ i.e., } t = 7.$$

Check that 7 is the correct solution by substituting 7 for t in the original equation and verifying that it produces a true statement. ■

Example 3 (Business) *A small business producing a certain item needs \$ 1,000 for supplies. It then takes the company \$ 40 to produce and sell each item. If each item sells for \$ 60 find how many items the company has to produce and sell so that its revenue becomes equal to its expenditures.*

Solution:

Let x represent the number of items that the company has to produce and sell. Since each item sells for \$ 60, the revenue of the company will be $60x$. On the other hand, to produce x items the company has to spend $1,000 + 40x$. Thus, the revenue becomes equal to the expenditures when

$$60x = 1,000 + 40x.$$

Subtract $40x$ from both sides to get

$$60x - 40x = 1,000 + 40x - 40x, \text{ i.e., } 20x = 1,000.$$

Now divide both sides by 20 to get

$$\frac{20x}{20} = \frac{1,000}{20}, \text{i.e., } x = 50.$$

Thus, the company needs to produce and sell 50 items to have revenue equaling its expenditures.

Verify that 50 is the right solution by substituting 50 for x in the original equation and seeing that it produces a true statement. ■

Example 4 (Geometry) *If the length of a side of a square is increased by 3 inches, the new perimeter is 40 inches more than twice the length of the side of the original square. Find the length of the side of the original square.*

Solution:

Draw a figure to convince yourself that, if x is the length of the side of the original square, then we must have

$$4(x + 3) = 2x + 40.$$

This gives

$$4x + 12 = 2x + 40.$$

Subtract $2x$ from both sides to get

$$4x + 12 - 2x = 2x + 40 - 2x, \text{ i.e., } 2x + 12 = 40.$$

Now subtract 12 from both sides to get

$$2x + 12 - 12 = 40 - 12, \text{ i.e., } 2x = 28.$$

Finally, divide both sides by 2 to get

$$\frac{2x}{2} = \frac{28}{2}, \text{ i.e., } x = 14.$$

Now check that this is the correct answer by substituting 14 for x in the original equation.

■

Now try to solve the following exercises to see whether you can set up and solve applied problems involving first-degree equations on your own.

Exercise 4 *Joe bought two plots of land for a total of $ 120,000. On the first plot, he made a profit of 15%. On the second, he lost 10%. His total profit was $5,500. How much did he pay for each piece of land?*

Exercise 5 *Suppose that $20,000 is invested at 5%. How much additional money has to be invested at 3% so that the yield on the entire amount ends up being 4%?*

Exercise 6 *A plane flies nonstop from New York to London which are about 3500 miles apart. After one hour and six minutes in the air, the plane passes through Halifax, Nova Scotia, which is 600 miles from New York. Estimate the flying time from New York to London.*

Exercise 7 *On vacation, Mary averaged 50 mph traveling from Denver to Minneapolis. Returning by a different route that covered the same number of miles, she averaged 55 mph. What is the distance between the two cities, if her total traveling time was 32 hours?*

Exercise 8 *The length of a rectangular label is 3 centimeters less than twice the width. The perimeter is 54 centimeters. Find the width.*

Exercise 9 *A puzzle piece in the shape of a triangle has a perimeter of 30 centimeters. Two sides of the triangle are each twice as long as the shortest side. Find the length of the shortest side.*

1.2 Ordered Pairs, Cartesian Systems of Coordinates

A pair of real numbers a, b taken in a specific order, first a and then b, is called an **ordered pair** and denoted by (a, b) or, sometimes, by $\langle a, b\rangle$. For instance $(3, 6.1)$ and $(\sqrt{2}, \frac{5}{\sqrt{7}})$ as well as $(6, 2)$ are ordered pairs of real numbers. The first number in the pair is called the **first coordinate** and the second the **second coordinate**.

A direct consequence of the definition is that, if $a \neq b$, then the pairs (a, b) and (b, a) are not considered to be the same. In fact, **equality** between ordered pairs *is defined by*

$$(a, b) = (c, d) \quad \text{if and only if} \quad a = c \text{ and } b = d.$$

Example 5 *Find x and y if $(x, 3) = (4, 2y)$.*

Solution:
By the definition of equality between ordered pairs, we have

$$(x, 3) = (4, 2y) \quad \text{implies} \quad x = 4 \text{ and } 3 = 2y.$$

Therefore

$$x = 4 \quad \text{and} \quad y = \frac{3}{2}.$$

■

Example 6 *Find x if*

1. $(x, 6) = (3, 2x)$.
2. $(x + 1, 7) = (4, 2x + 3)$.

Solution:

1. We have
$$(x, 6) = (3, 2x) \quad \text{implies} \quad x = 3 \text{ and } 6 = 2x.$$
Thus, we obtain $x = 3$ and $x = 3$, which, *together*, give us the condition $x = 3$.

2. We have
$$(x + 1, 7) = (4, 2x + 3) \quad \text{implies} \quad x + 1 = 4 \text{ and } 7 = 2x + 3.$$
These give
$$x = 3 \quad \text{and} \quad 2x = 4 \quad \text{i.e.,} \quad x = 3 \text{ and } x = 2.$$
The condition $x = 3$ *and* $x = 2$ is clearly **inconsistent**. (x cannot be both 3 and 2 simultaneously.) Hence no x that makes the required equality valid exists. Or, in other words, the equality has no solutions. ■

Now take up the following exercises to make sure that you understand ordered pairs and equality of ordered pairs.

Exercise 10 *Find x and y if $(x, 8) = (y + 1, y - 2)$.*

Exercise 11 *Find x if*

1. $(2x + 1, 4) = (11, x - 1)$.
2. $(3x + 2, 7) = (13, x + 3)$.

A **number line** is a straight line on which, there is a distinguished point O, called the **origin**, a distinguished direction, called the **positive direction**, and a distinguished length, called the **unit length**. The opposite of the positive direction is termed the **negative direction**.

Given a number line, every point on the line may be associated with a unique real number: it is the positive distance to the point from the origin if the origin-point segment points to the positive direction and it is the negative of the distance to the point from the origin in case the origin-point segment points to the negative direction.

Conversely, given any real number, there is a point on the number line associated with it. It is the point whose distance from the origin equals the absolute value of the number and that lies in the positive direction with respect to the origin, if the real number is positive, and to the negative direction, if the real number is negative.

The above statements mean that real numbers are in one-to-one correspondence with points on the number line. The following exercise asks you to perform these back-and-forth steps for specific numbers.

Example 7 *Construct a number line on the plane.*

1. *Pick a point on the line and find the real number that is associated to that point.*
2. *Find the points associated to the real numbers -4.5 and 2.8.*

Solution:

1. The number line is shown in Figure 1.1. Note that, except for the line, a positive direction and a unit length have been picked. These are necessary components. Also on the line an arbitrary point has been picked. The point lies in the negative direction with respect to the origin and its distance from the origin is measured to be 2.5 times the given unit length. Therefore the real number associated to that point is the number -2.5.

2. The points associated to -4.5 and 2.8 are constructed by taking segments that are 4.5 times the length of the unit length in the negative and 2.8 times the length of the unit in the positive direction, respectively. See Figure 1.2. These are depicted in Figure 1.2. ■

Figure 1.1: The number line with a chosen point.

Figure 1.2: The number line with the points associated with -4.5 and 2.8.

Figure 1.3: The Cartesian Plane.

Exercise 12 *Construct a number line on the plane. Do not miss any of the three requirements. Take a point on the line. Find which real number is associated to that point. Explain the process.*

Exercise 13 *Find the points on the number line that you constructed corresponding to the numbers 2, 3.5 and -4.6. Explain the process.*

Following a similar process to the construction of the number line, one may construct the real plane. Now instead of the one-to-one correspondence between real numbers and points on the number line, a one-to-one correspondence between ordered pairs and points on the plane is to be established. Here is the way:

A **Cartesian coordinate system** on the plane consists of two number lines which intersect in a right angle at their origins. The point of intersection is called the **origin**. The unit lengths of the two lines are usually taken to be equal. Also, by convention, one of the lines is placed horizontally and is called the x-**axis** and the other vertically and is called the y-**axis**. Moreover east is taken to be the positive direction on the x-axis and north the positive direction on the y-axis. Note that these are all conventions, but that the only requirement imposed by the definition is that the two number lines intersect at their origins forming a right angle. Figure 1.3 is depicting the situation graphically.

Given a Cartesian coordinate system, every point on the plane may be associated with a unique ordered pair of real numbers: it is the pair having as its first coordinate the real number associated with the projection of the point on the horizontal axis in the horizontal number line and as its second coordinate the real number associated with the projection of the point on the vertical axis on the vertical number line.

Conversely, given any ordered pair of real numbers, there is a point on the plane associated with it. It is the point of intersection of the vertical line that passes through the point associated with its first coordinate on the horizontal number line with the horizontal line that passes through the point associated with its second coordinate on the vertical number line.

Figure 1.4: The plane with a point.

The above statements mean that pairs of real numbers are in one-to-one correspondence with points on the plane. The following exercise asks you to perform these back-and-forth steps for specific pairs of real numbers.

Example 8 *Construct a Cartesian coordinate system on the plane.*

1. *Pick a point on the plane and find the ordered pair that is associated to that point.*
2. *Find the point associated to the ordered pairs* $(-2, 5), (-1, -2)$ *and* $(2, 3)$*.*

Solution:

1. The coordinate system is shown in Figure 1.4. Note that, except for the two lines, a positive direction and a unit length have been picked on each line. These are necessary components. Also on the plane an arbitrary point A has been picked. The projections A' and A'' of the point A on the x-axis and the y-axis, respectively, have been drawn. A' is associated with 2 in the horizontal number line and A'' is associated with -1.5 in the vertical number line. Therefore the point A is associated to the ordered pair $(2, -1.5)$ in the Cartesian system.
2. The point associated to $(-2, 5)$ is found by, first, identifying the point associated with -2 in the horizontal number system and drawing the vertical line that passes through that point, then identifying the point associated to 5 in the vertical number system and drawing the horizontal line passing through that point and, finally, taking the point of intersection of these two lines. The points associated to $(-1, -2)$ and $(2, 3)$ have been similarly constructed. These are depicted in Figure 1.5. ■

Exercise 14 *Construct a Cartesian coordinate system on the plane. Do not miss any of the requirements. Take a point on the plane. Find which ordered pair of real numbers is associated to that point. Explain the process.*

Figure 1.5: The plane with the points associated with $(-2,5),(-1,-2)$ and $(2,3)$.

Exercise 15 *Find the points on the Cartesian plane that you constructed corresponding to the ordered pairs* $(-2,5)$ *and* $(2,-1)$*. Explain the process.*

1.3 Linear Equations in 2 Variables: Graphs and Intercepts

A **linear equation in two variables** x and y is an equation of the form

$$ax + by = c,$$

where a, b, c are constants (real numbers). For instance, $3x + 5y = 8$ is a linear equation in two variables. $3x^2 - 5y = 6$ is not a linear equation, because one of the variables appears squared. Similarly $x - y^3 = 5x + 1$ is not a linear equation since y appears cubed.

A **solution** of a linear equation in two variables x and y is an ordered pair of real numbers, such that substitution of the first coordinate for x and of the second for y in the equation produces a true statement. For instance $(5, 2)$ is a solution for the linear equation $3x - 2y = 11$, since substituting 5 for x and 2 for y in the equation gives $3 \cdot 5 - 2 \cdot 2 = 11$, which is a true statement. On the other hand $(3, -2)$ is not a solution of the equation since substituting 3 for x and -2 for y gives $3 \cdot 3 - 2 \cdot (-2) = 11$, which is not a true statement, since $3 \cdot 3 - 2 \cdot (-2) = 9 + 4 = 13 \neq 11$. Note that $(3, -1)$ is also a solution of the given equation. Verify this fact!!

Exercise 16 *Consider the linear equation in two variables* $-2x + y = 2$. *Give three solutions of this equation. Find the corresponding points on a Cartesian coordinate system. What do you observe?*

To **solve** a linear equation in two variables means to find its solutions. As the example above suggests a linear equation in two variables may have more than one solutions.

Example 9 *Solve the linear equation in two variables* $3x - y = 1$.

Solution:

The given equation has infinitely many solutions. This may be seen by observing that, for every real number s that is substituted for x in the equation, one may find a number t that can be substituted for y that makes the equation true. Given $x = s$, $3s - y = 1$ gives $y = 3s - 1$. So, if we set $t = 3s - 1$, the ordered pair $(s, 3s - 1)$ satisfies this equation no matter what s is. Hence this equation has the infinitely many solutions

$$(s, 3s - 1), \quad s \text{ an arbitrary real number.}$$

Most often s and t are not used and the letters x and y are still used to denote an arbitrary solution. In that case one would have

$$(x, y) = (x, 3x - 1), \quad x \text{ an arbitrary real number.}$$

■

Exercise 17 *Solve the equation* $2x - 3y = 5$.

Figure 1.6: The graph of $y - 2x = 1$.

Exercise 18 *Solve the equation* $-5x + 2y + 1 = 3x - 2y + 6$.

The **graph** of a linear equation in two variables x and y is the set of points in the plane whose coordinates (ordered pairs) are solutions of the equation. As we saw before, a linear equation may have infinitely many solutions. And as you may have discovered by doing one of the exercises, the corresponding points on the plane all lie on a straight line. This makes it possible to obtain the graph of the equation by only determining two points on the graph, i.e., two solutions of the equation, and then joining them by a straight line.

Example 10 *Plot the graph of the equation* $y - 2x = 1$.

Solution:

Now that we know that the graph is a straight line, it suffices to determine two solutions of the equation. Working as before, it is not very difficult to see that the complete set of solutions is given by

$$(x, y) = (x, 2x + 1), \quad x \text{ an arbitrary real number.}$$

To obtain, thus, two solutions, it suffices to choose two real numbers, say 0 and 1 and substitute them for x in the general form of the solution given above. This will give us the two solutions

$$(0, 2 \cdot 0 + 1) \text{ and } (1, 2 \cdot 1 + 1), \quad \text{i.e.,} \quad (0, 1) \text{ and } (1, 3).$$

Now find the points on the Cartesian plane corresponding to these ordered pairs and join them via a straight line. The resulting graph is depicted in Figure 1.6. ■

Usually, when the graph of the linear equation is to be drawn, the two points that are chosen to guide the plotting are the points where the straight line intersects the x- and the y-axis. The first is called the x-**intercept** and the second the y-**intercept**. To find the intercepts the following steps are useful:

Figure 1.7: The graph of $y - 2x = 1$.

x-intercept The x-intercept is, by definition the point of intersection with the x-axis. At that point, then, the y-coordinate is equal to 0. Thus, to determine the x-intercept, we set $y = 0$ and solve the resulting linear equation in one variable for x.

y-intercept The y-intercept is, by definition the point of intersection with the y-axis. At that point, then, the x-coordinate is equal to 0. Thus, to determine the y-intercept, we set $x = 0$ and solve the resulting linear equation in one variable for y.

Example 11 *Plot the graph of the equation* $3y + 2x = 5$.

Solution:
First determine the two intercepts by following the process outlined above.
Set $y = 0$. Then we obtain

$$2x = 5, \quad \text{i.e.,} \quad x = \frac{5}{2}.$$

Thus the x-intercept is the point $(\frac{5}{2}, 0)$.

Next, set $x = 0$. Then

$$3y = 5, \quad \text{i.e.,} \quad y = \frac{5}{3}.$$

Thus, the y-intercept is the point $(0, \frac{5}{3})$.

Now find the points on the Cartesian plane corresponding to these ordered pairs and join them via a straight line. The resulting graph is depicted in Figure 1.7. ■

Now work the following exercises to make sure that you understand intercepts and the concept of a graph of a linear equation in two variables.

Exercise 19 *Find the intercepts of* $3x + y = 4$.

Exercise 20 *Find the intercepts of* $-7x + 3y = 2 + 4y + 3x$.

Exercise 21 *Consider the equation* $x = 2y + 3$.

1. *Solve this equation.*

2. *Find its intercepts.*

3. *Sketch its graph.*

Exercise 22 *Consider the equation* $2x + 7y = 14$.

1. *Solve this equation.*

2. *Find its intercepts.*

3. *Sketch its graph.*

1.4 Slope and Equations of Lines

A straight line is determined by two of its points. If (x_1, y_1) and (x_2, y_2) are two points on a straight line, then the **slope** m of the line is defined to be

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope, in other words, gives the change in y per unit change of x.

Example 12 *Find the slope of the line going through the points* $(-2, 1)$ *and* $(3, 4)$.

Solution:
Let $(x_1, y_1) = (-2, 1)$ and $(x_2, y_2) = (3, 4)$ and apply the definition to obtain

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 1}{3 - (-2)} = \frac{3}{5}.$$

Thus, the slope of the given line is $m = \frac{3}{5}$. ■

Example 13 *Using the definition, find the slope of the line with equation* $3x - 2y = 3$.

Solution:
The definition requires two points on the line. So, we first have to determine two solutions of the equation $3x - 2y = 3$. We choose to determine the intercepts. Setting $y = 0$, we get the x-intercept $(1, 0)$. Setting $x = 0$, we get the y-intercept $(0, -\frac{3}{2})$. Thus, using $(x_1, y_1) = (1, 0)$ and $(x_2, y_2) = (0, -\frac{3}{2})$, we obtain

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-\frac{3}{2} - 0}{0 - 1} = \frac{3}{2}.$$

■

Now try to solve the following exercises to make sure that you understand the definition of the slope.

Exercise 23 *Find the slope of the line that goes through the points* $(-3, 5)$ *and* $(2, 7)$.

Exercise 24 *Find the slope of the line that goes through* $(-5, -2)$ *and* $(-4, 11)$.

Exercise 25 *Find the slope of the line given by the equation* $4x - 3y = 24$.

The slope of a horizontal line is 0 and the slope of a vertical line is undefined. Can you explaine, *based on the definition*, why this is the case?

Slope-Intercept Form: If a line has slope m and y-intercept $(0, b)$, then the line is the graph of the equation

$$y = mx + b.$$

This form of the equation of a line is called the **slope-intercept form**.

Can you see, *based on the definition*, why a line with slope m and y-intercept at the point $(0, b)$ is the graph of the equation $y = mx + b$?

Example 14 *Find the equation of the line with slope $m = 5$ and y-intercept $b = 7$.*

Solution:
An application of the slope-intercept form gives directly the equation $y = mx+b = 5x+7$. ■

Example 15 *Find the equation of the line with slope $m = 2$ passing through the point* $(0, -1)$.

Solution:
Since the line has slope $m = 2$, the slope-intercept form gives an equation $y = 2x + b$, where b is the y-intercept of the line. But, since $(0, -1)$ is a point on the line lying on the y-axis, the y-intercept of the line is $b = -1$. Hence the equation is $y = 2x - 1$. ■

Example 16 *Find the equation of the line with slope $m = -3$ going through the point* $(-2, 5)$.

Solution:
By the slope intercept form, since the line has slope $m = -3$, it must have an equation $y = -3x + b$, where b is its y-intercept. But, since $(-2, 5)$ is a point on the line, its coordinates must be solutions of the equation of the line. This means that, if you substitute -2 for x and 5 for y in the equation of the line you would obtain a true statement. Hence $5 = -3 \cdot (-2) + b$. Hence $5 = 6 + b$, i.e., $b = 1 -$. The equation of the line is, therefore, $y = -3x - 1$. ■

Example 17 *Find the equation of the line going through the points $(-2, -3)$ and $(5, -1)$.*

Solution:
We first set $(x_1, y_1) = (-2, -3)$ and $(x_2, y_2) = (5, -1)$, and compute the slope m of that line, using the definition:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1 - (-3)}{5 - (-2)} = \frac{2}{7}.$$

Now, the slope-intercept form of the line yields an equation $y = \frac{2}{7}x + b$. But the line passes through the point $(-2, -3)$, whence $(-2, -3)$ must be a solution of the equation of the line. Therefore $-3 = \frac{2}{7}(-2) + b$, whence $-3 = -\frac{4}{7} + b$, i.e., $b = -\frac{17}{7}$. The equation of the line is therefore

$$y = \frac{2}{7}x - \frac{17}{7}.$$

■

Another type of problem asks you, given an equation whose graph is a straight line to determine the slope of that line.

Example 18 *Find the slope of the line which is the graph of the equation $y = 7x - 3$.*

Solution:

Compare the given equation with the slope-intercept form. This comparison gives slope $m = 7$ and y-intercept $b = -3$. ■

Example 19 *Find the slope of the line with equation* $4x - 7y = 13$.

Solution:

The given equation is not in the slope-intercept form. So we cannot read the information regarding the slope and y-intercept directly off the equation. However, we may easily solve for y to obtain the slope-intercept form. We have $4x - 7y = 13$ implies $7y = 4x - 13$, i.e., $y = \frac{4}{7}x - \frac{13}{7}$. Hence, by comparison with the slope-intercept form, we obtain slope $m = \frac{4}{7}$ and y-intercept $b = -\frac{13}{7}$. ■

Now try to solve the following problems to make sure that you have a good grasp of the slope-intercept form of the equation of a line.

Exercise 26 *Find the equation of the line with slope* $m = -\frac{1}{2}$ *and* y*-intercept* $b = -3$.

Find the equation of the line with slope $m = -3$ passing through the point $(0, 5)$.

Exercise 27 *Find the equation of the line with slope* $m = 2$ *passing through the point* $(-3, 1)$.

Exercise 28 *Find the equation of the line going through the points* $(-2, 2), (3, -5)$.

Exercise 29 *Find the slope and* y*-intercept of the line which is the graph of the equation* $y = -\frac{1}{5}x + \frac{2}{3}$.

Exercise 30 *Find the slope and the* y*-intercept of the line with equation* $3x - 7y = 2$.

Exercise 31 *Find the slope and the* y*-intercept of the line with equation* $-3x + 2y = 12$.

Parallel and Perpendicular Lines: Two non-vertical lines are parallel if they have the same slope. Symbolically, for two lines l_1 and l_2 with slopes m_1 and m_2, respectively, we have

$$l_1 \parallel l_2 \quad \text{if} \quad m_1 = m_2.$$

Two non-vertical lines are perpendicular if the product of their slopes is -1. Symbolically,

$$l_1 \perp l_2 \quad \text{if} \quad m_1 \cdot m_2 = -1.$$

Example 20 *Determine which of the following pairs of lines are parallel and which are perpendicular.*

1. $4x - 3y = 6$ *and* $3x + 4y = 8$,
2. $3x + 2y = 8$ *and* $6y = 5 - 9x$,

3. $4x = 2y + 3$ *and* $2y = 2x + 3$.

Solution:

1. Solve both equations for y so as to transform them to the slope-intercept form and determine their slopes: $4x-3y = 6$ implies $3y = 4x-6$, i.e., $y = \frac{4}{3}x-2$. Thus the slope of the first line is $m_1 = \frac{4}{3}$. $3x + 4y = 8$ implies $4y = -3x + 8$, i.e., $y = -\frac{3}{4}x + 2$. Thus, the slope of the second line is $m_2 = -\frac{3}{4}$. Now, note that $m_1 \cdot m_2 = \frac{4}{3} \cdot (-\frac{3}{4}) = -1$. Hence the two given lines are perpendicular.

2. Following the same process, we get $3x+2y = 8$ implies $2y = -3x+8$, i.e., $y = -\frac{3}{2}x+4$, whence $m_1 = -\frac{3}{2}$. Also $6y = 5 - 9x$ implies $y = -\frac{3}{2}x + \frac{5}{6}$, whence $m_2 = -\frac{3}{2}$. Now, observe again that $m_1 = -\frac{3}{2} = m_2$, whence the two given lines are parallel.

3. $4x = 2y + 3$ implies $2y = 4x - 3$, whence $y = 2x - \frac{3}{2}$. Hence $m_1 = 2$. On the other hand $2y = 2x + 3$ implies $y = x + \frac{3}{2}$. Hence $m_2 = 1$. So we have neither $m_1 = m_2$ nor $m_1 m_2 = -1$. Therefore the two given lines are neither parallel nor perpendicular. ■

Example 21 *Find the equation of the line that is parallel to* $y = 2x+2002$ *and goes through the point* $(0, 6)$.

Solution:

Since the unknown line is parallel to the line $y = 2x + 2002$, its slope must be $m = 2$. Thus, its equation must have the form $y = 2x + b$, where b is its y-intercept. But $(0, 6)$ is a point of the line lying on the y-axis, i.e., 6 is its y-intercept. Thus, we obtain $y = 2x + 6$ as the equation of the line. ■

Example 22 *Find the equation of the line that is perpendicular to the line with equation* $y = -5x + 2003$ *and goes through the point* $(1, -\frac{4}{5})$.

Solution:

Since the unknown line is perpendicular to the line $y = -5x + 2003$, it must have slope $m = \frac{1}{5}$. Thus, its equation has the form $y = \frac{1}{5}x + b$, where b is its y-intercept. The point $(1, -\frac{4}{5})$ is by assumption a point on this line, whence we must have $-\frac{4}{5} = \frac{1}{5} + b$, whence $b = -1$. Therefore, the equation of our line is $y = \frac{1}{5}x - 1$. ■

Now work out the following exercises to make sure that you have a solid understanding of parallel and perpendicular lines.

Exercise 32 *Say whether the following pairs of lines are parallel or perpendicular*

1. $2x - 5y = 7$ *and* $15y - 5 = 6x$,

2. $x - 3y = 4$ *and* $y = 1 - 3x$,

3. $2x - y = 6$ *and* $x - 2y = 4$.

Exercise 33 *Find the equation of the line that is parallel to* $y = 7x + 6$ *and goes through the point* $(-2, 3)$.

Exercise 34 *Find the equation of the line that is perpendicular to the line* $y = 2x + 1$ *and goes through the point* $(1, \frac{3}{2})$.

Point-Slope Form: If a line has slope m and passes through the point (a, b), then it is the graph of the equation

$$y - b = m(x - a).$$

This is called the **point-slope form** of the equation of the line.

Can we see why a line with slope m passing through (a, b) must have equation $y - b = m(x - a)$?

Example 23 *Find the equation of the line that has slope* $m = 2$ *and goes through the point* $(1, 5)$.

Solution:

By the point slope form, we obtain $y - 5 = 2(x - 1)$, i.e., $y - 5 = 2x - 2$, and, therefore $y = 2x + 3$. ■

Example 24 *Find the equation of the line that is parallel to the line* $2x - 3y = 5$ *and goes through the point* $(1, -\frac{1}{3})$.

Solution:

First solve the given equation for y to determine its slope. We have $2x - 3y = 5$ implies $3y = 2x - 5$, i.e., $y = \frac{2}{3}x - \frac{5}{3}$. Hence its slope is $m = \frac{2}{3}$. The unknown line is parallel to this, and so it has the same slope. Since it also goes through the point $(1, -\frac{1}{3})$, its equation is given by the point-slope form

$$y - (-\frac{1}{3}) = \frac{2}{3}(x - 1), \quad \text{i.e.,} \quad y + \frac{1}{3} = \frac{2}{3}x - \frac{2}{3}.$$

Hence $y = \frac{2}{3}x - 1$. ■

Example 25 *Find the equation of the line that is perpendicular to the line* $y = -7x + 2001$ *and goes through the point* $(1, \frac{8}{7})$.

Solution:

Since the unknown line is perpendicular to the given line, its slope must be $m = \frac{1}{7}$. Since, in addition, it goes through the point $(1, \frac{8}{7})$, its equation is given by the point-slope form

$$y - \frac{8}{7} = \frac{1}{7}(x - 1) \quad \text{i.e.,} \quad y - \frac{8}{7} = \frac{1}{7}x - \frac{1}{7}.$$

This gives $y = \frac{1}{7}x + 1$. ■

Example 26 *Find the equation of the line that goes through the points* $(-2,3)$ *and* $(4,-1)$.

Solution:
First determine the slope by the definition

$$m = \frac{-1-3}{4-(-2)} = \frac{-4}{6} = -\frac{2}{3}.$$

Then, use the point-slope form

$$y - 3 = -\frac{2}{3}(x-(-2)) \quad \text{i.e.,} \quad y - 3 = -\frac{2}{3}x - \frac{4}{3}.$$

Hence $y = -\frac{2}{3}x + \frac{5}{3}$. ■

Now work through the following exercises.

Exercise 35 *Find the equation of the line that has slope* $m = 9$ *and goes through the point* $(1,4)$.

Exercise 36 *Find the equation of the line that is parallel to the line* $2x - y = 7$ *and goes through the point* $(6,3)$.

Exercise 37 *Find the equation of the line that is perpendicular to the line* $5y - x = 2003$ *and goes through the point* $(2,14)$.

Exercise 38 *Find the equation of the line going through the points* $(2,4)$ *and* $(5,-6)$.

Finally, the equation of the **vertical line** with x-intercept $(a,0)$ is given by $x = a$ and the equation of the **horizontal line** with y-intercept $(0,b)$ is given by $y = b$.

Example 27 *Find the equation of the horizontal line passing through* $(2,5)$.

Solution:
This line is horizontal and has y-intercept 5. Hence, its equation is $y = 5$. ■

Example 28 *Find the equation of the vertical line through* $(1,-4)$.

Solution:
This line is vertical and has x-intercept 1. Hence, its equation is $x = 1$. ■

Example 29 *Find the equation of the line that is parallel to* $x = 3$ *and goes through* $(-2,6)$.

Solution:
The given line is vertical. Hence the unknown line is vertical and has x-intercept -2. Thus, its equation is $x = -2$. ■

Example 30 *Find the equation of the line perpendicular to* $x = -1$ *that goes through* $(-7, -2002)$.

Solution:
The given line is vertical. Thus, the unknown line is horizontal and has y-intercept -2002. Thus, its equation is $y = -2002$. ■

Exercise 39 *Find the equations of the horizontal and of the vertical lines that go through the point* $(2002, -2003)$.

Exercise 40 *Find the equation of the line that is parallel to* $y = -3$ *and goes through the point* $(5, 13)$.

Exercise 41 *Find the equation of the line that is perpendicular to the line* $x = 5$ *and goes through* $(-2, 5)$.

1.5 Systems of 2 Linear Equations in 2 Variables

A **system of 2 linear equations in 2 variables**, or a 2×2 **system** for short, is a system of the form

$$\left\{ \begin{array}{rcl} a_{11}x + a_{12}y & = & b_1 \\ a_{21}x + a_{22}y & = & b_2 \end{array} \right\},$$

where $a_{11}, a_{12}, a_{21}, a_{22}, b_1$ and b_2 are constants.

An example of a 2×2 system is the following

$$\left\{ \begin{array}{rcl} 2x + 3y & = & 10 \\ -x + y & = & 5 \end{array} \right\}.$$

In this example, we have $a_{11} = 2, a_{12} = 3, a_{21} = -1, a_{22} = 1$ and $b_1 = 10, b_2 = 5$.

Geometrically a 2×2 system corresponds to a system of 2 straight lines in the plane.

A **solution** of a 2×2 system is an ordered pair (a, b), such that, (a, b) is a solution of **both** linear equations in 2 variables constituting the system. Geometrically, this means that a solution of the 2×2 system is a point lying in both straight lines which are graphs of the linear equations of the system.

Consider the 2×2 system given above. The ordered pair $(2, 2)$ is **not** a solution of the system. Despite the fact that $2 \cdot 2 + 3 \cdot 2 = 10$, $(2, 2)$ is not a solution because $-2 + 2 = 0 \neq 5$. I.e., $(2, 2)$ is a solution of the first linear equation of the system **only**. The point $(-1, 4)$ is a solution because $2 \cdot (-1) + 3 \cdot 4 = 10$ and $-(-1) + 4 = 5$.

Example 31 *Consider the* 2×2 *system*

$$\left\{ \begin{array}{rcl} x + 3y & = & 7 \\ -x + y & = & 1 \end{array} \right\}.$$

Is the ordered pair $(4, 1)$ *a solution of this system? How about the pair* $(1, 2)$*?*

Solution:

$(4, 1)$ is a solution of the first equation but not of the second. Therefore, it is not a solution of the system.

$(1, 2)$ is a solution of both equations. Therefore it is a solution of the given system. ■

Example 32 *Consider the* 2×2 *system*

$$\left\{ \begin{array}{rcl} x + 3y & = & 7 \\ -2x - 6y & = & -14 \end{array} \right\}.$$

Is the ordered pair $(4, 1)$ *a solution of this system? How about the pair* $(1, 2)$*?*

Solution:

$(4, 1)$ is a solution of both equations. It is therefore a solution of the system. $(1, 2)$ also satisfies both equations. It is therefore a solution of the given system as well. ■

Example 33 *Consider the* 2×2 *system*

$$\left\{ \begin{array}{lcl} 2x - y & = & 7 \\ -6x + 3y & = & 9 \end{array} \right\}.$$

Is the ordered pair $(2, -3)$ *a solution of this system? How about the pair* $(1, 5)$*?*

Solution:

$(2, -3)$ satisfies only the first of the two equations. It is therefore not a solution of the system. $(1, 5)$, on the other hand, satisfies only the second equation. It is therefore not a solution of the given system either. ■

Exercise 42 *Consider the* 2×2 *system*

$$\left\{ \begin{array}{lcl} -2x + 5y & = & 14 \\ -7x + y & = & 16 \end{array} \right\}.$$

Is the ordered pair $(3, 4)$ *a solution of this system? How about the pair* $(-2, 2)$*?*

Exercise 43 *Consider the* 2×2 *system*

$$\left\{ \begin{array}{lcl} x - 5y & = & 7 \\ 2x - 10y & = & 14 \end{array} \right\}.$$

Is the ordered pair $(2, -1)$ *a solution of this system? How about the pair* $(7, 0)$*?*

Exercise 44 *Consider the* 2×2 *system*

$$\left\{ \begin{array}{lcl} -3x - y & = & 7 \\ 6x + 2y & = & 3 \end{array} \right\}.$$

Is the ordered pair $(0, -7)$ *a solution of this system? How about the pair* $(-\frac{1}{2}, 3)$*?*

To **solve a** 2×2 system means to find **all** its solutions.

To solve a 2×2 system one usually follows the following steps:

1. Solve one of the 2 equations for one of the variables (pick the one that has the easiest solution).

2. Substitute the expression you obtained in 1 for the same variable in the other equation and solve for the other variable.

3. Now back-substitute into the expression obtained in 1.

Example 34 *Solve the* 2×2 *system*

$$\left\{ \begin{array}{lcl} 2x - 5y & = & -1 \\ -5x + 8y & = & -2 \end{array} \right\}.$$

Solution:

$$\left\{\begin{array}{lcl} 2x-5y & = & -1 \\ -5x+8y & = & -2 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lcl} x-\frac{5}{2}y & = & -\frac{1}{2} \\ -5x+8y & = & -2 \end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{lcl} x & = & \frac{5}{2}y-\frac{1}{2} \\ -5(\frac{5}{2}y-\frac{1}{2})+8y & = & -2 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lcl} x & = & \frac{5}{2}y-\frac{1}{2} \\ -\frac{25}{2}y+\frac{5}{2}+8y & = & -2 \end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{lcl} x & = & \frac{5}{2}y-\frac{1}{2} \\ -\frac{9}{2}y & = & -\frac{9}{2} \end{array}\right\} \Longrightarrow \left\{\begin{array}{lcl} x & = & \frac{5}{2}y-\frac{1}{2} \\ y & = & 1 \end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{lcl} x & = & \frac{5}{2}\cdot 1-\frac{1}{2} \\ y & = & 1 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lcl} x & = & 2 \\ y & = & 1 \end{array}\right\}$$

■

Example 35 *Solve the* 2×2 *system*

$$\left\{\begin{array}{lcl} -x+2y & = & 8 \\ 2x-4y & = & -16 \end{array}\right\}.$$

Solution:

$$\left\{\begin{array}{lcl} -x+2y & = & 8 \\ 2x-4y & = & -16 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lcl} x & = & 2y-8 \\ 2x-4y & = & -16 \end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{lcl} x & = & 2y-8 \\ 2(2y-8)-4y & = & -16 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lcl} x & = & 2y-8 \\ 4y-16-4y & = & -16 \end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{lcl} x & = & 2y-8 \\ 0 & = & 0 \end{array}\right\}$$

Therefore, the solution set of this system is

$$(2y-8, y) \quad y \text{ any real number.}$$

■

Example 36 *Solve the* 2×2 *system*

$$\left\{\begin{array}{lcl} 3x-y & = & 5 \\ 6x-2y & = & 9 \end{array}\right\}.$$

Solution:

$$\left\{\begin{array}{lcl} 3x-y & = & 5 \\ 6x-2y & = & 9 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lcl} y & = & 3x-5 \\ 6x-2y & = & 9 \end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{lcl} y & = & 3x-5 \\ 6x-2(3x-5) & = & 9 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lcl} y & = & 3x-5 \\ 6x-6x+10 & = & 9 \end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{lcl} y & = & 3x-5 \\ 0 & = & -1 \end{array}\right\}.$$

So the given system is inconsistent. ■

Exercise 45 *Solve the* 2×2 *systems*

$$\left\{ \begin{array}{lcl} 7x + y & = & -1 \\ -21x - 3y & = & 3 \end{array} \right\}, \quad \left\{ \begin{array}{lcl} -5x + 2y & = & 3 \\ x + 4y & = & 17 \end{array} \right\} \quad \textit{and} \quad \left\{ \begin{array}{lcl} 2x - 5y & = & -1 \\ 6x - 15y & = & 3 \end{array} \right\}.$$

There is another method of 2×2 systems which is more useful than the simple method presented above because it applies equally well when one wants to solve systems of more than 2 linear equations in more than 2 variables. This method is called **Gauss elimination**.

The operations that we are allowed to performed at each step of applying this method are the following:

1. Change the order in which the equations appear in the system.
2. Multiply both sides of an equation in the system by a nonzero constant.
3. Add to an equation a multiple of another equation side-by-side.

Here follows an example of how these 3 operations may be applied to solve a 2×2 system.

Example 37 *Solve the* 2×2 *system*

$$\left\{ \begin{array}{lcl} 5x + y & = & 61 \\ -x + 2y & = & 12 \end{array} \right\}.$$

Solution:

The given system is

$$\left\{ \begin{array}{lcl} 5x + y & = & 61 \\ -x + 2y & = & 12 \end{array} \right\}.$$

First, apply operation 1 to change the order of the equations.

$$\left\{ \begin{array}{lcl} -x + 2y & = & 12 \\ 5x + y & = & 61 \end{array} \right\}.$$

Then multiply the first equation by -1.

$$\left\{ \begin{array}{lcl} x - 2y & = & -12 \\ 5x + y & = & 61 \end{array} \right\}.$$

Next add to the second equation the first multiplied by -5. We do this to eliminate the variable x from the second equation. That is why the method is called Gauss *elimination*.

$$\left\{ \begin{array}{lcl} x - 2y & = & -12 \\ 11y & = & 121 \end{array} \right\}.$$

Now multiply both sides of the second equation by $\frac{1}{11}$.

$$\left\{ \begin{array}{lcl} x - 2y & = & -12 \\ y & = & 11 \end{array} \right\}.$$

Finally, add to the first equation the second multiplied by 2. We do this to eliminate the variable y from the first equation.

$$\left\{\begin{array}{lll} x & = & 10 \\ y & = & 11 \end{array}\right\}.$$

Thus $(10, 11)$ is a solution of the given 2×2 system. ■

We summarize the method as follows

Gauss Elimination:

1. Multiply both sides of one of the equations by a nonzero constant so as to make the coefficient of x equal to 1.
2. Change the order of the equations so as to make this equation your first equation.
3. Now apply operation 3 to eliminate x from the second equation of the system.
4. Next multiply the second equation by the inverse of the coefficient of y so as to solve it for y.
5. Finally, use operation 3 once more to eliminate y from the first equation.

Important Remarks:

1. If at any point during the above process, the equation $0 = 0$ appears, throw it away and solve the remaining linear equation in two variables.
2. If at any point during the above process an equation of the form $a = b$ appears with a, b constants and $a \neq b$, then this means that your system is **inconsistent**, i.e., a system with no solutions.

You have to keep the above two remarks in mind during the Gauss elimination process.

Example 38 *Apply Gauss elimination to solve the* 2×2 *system*

$$\left\{\begin{array}{lll} x - 3y & = & -11 \\ -x + 2y & = & 7 \end{array}\right\}.$$

Solution:

$$\left\{\begin{array}{lll} x - 3y & = & -11 \\ -x + 2y & = & 7 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lll} x - 3y & = & -11 \\ -y & = & -4 \end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{lll} x - 3y & = & -11 \\ y & = & 4 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lll} x & = & 1 \\ y & = & 4 \end{array}\right\}.$$

Thus $(1, 4)$ is a solution of the above equation. ■

Example 39 *Apply Gauss elimination to solve the* 2×2 *systems*

$$\left\{\begin{array}{lll} -x+2y & = & 3 \\ 2x-4y & = & -8 \end{array}\right\}, \quad \textit{and} \quad \left\{\begin{array}{lll} 3x+2y & = & -12 \\ 9x+6y & = & -36 \end{array}\right\}.$$

Solution:

$$\left\{\begin{array}{lll} -x+2y & = & 3 \\ 2x-4y & = & -8 \end{array}\right\}.$$

Multiply the first equation by -1.

$$\left\{\begin{array}{lll} x-2y & = & -3 \\ 2x-4y & = & -8 \end{array}\right\}.$$

Add to the second equation the first multiplied by -2.

$$\left\{\begin{array}{lll} x-2y & = & -3 \\ 0 & = & -2 \end{array}\right\}.$$

The equation $0=-2$ appeared in the system. Thus the system is inconsistent.

$$\left\{\begin{array}{lll} 3x+2y & = & -12 \\ 9x+6y & = & -36 \end{array}\right\}.$$

Multiply the first equation by $\frac{1}{3}$.

$$\left\{\begin{array}{lll} x+\frac{2}{3}y & = & -4 \\ 9x+6y & = & -36 \end{array}\right\}.$$

Add to the second equation the first multiplied by -9.

$$\left\{\begin{array}{lll} x+\frac{2}{3}y & = & -4 \\ 0 & = & 0 \end{array}\right\}.$$

The equation $0=0$ appeared in the system. Thus the system is equivalent to the single linear equation $x+\frac{2}{3}y=-4$. This gives $x=-\frac{2}{3}y-4$. Thus the solutions are

$$(-\frac{2}{3}y-4, y), \quad y \text{ any real number.}$$

■

Now solve the following exercises to make sure you have a good understanding of Gauss elimination.

Exercise 46 *Solve the following* 2×2 *systems using Gauss elimination*

$$\left\{\begin{array}{lll} -2x+5y & = & 1 \\ 5x+3y & = & 13 \end{array}\right\}, \quad \left\{\begin{array}{lll} 3x-y & = & -10 \\ -5x+3y & = & 18 \end{array}\right\} \quad \textit{and} \quad \left\{\begin{array}{lll} 3x-y & = & 5 \\ -6x+2y & = & -10 \end{array}\right\}$$

Exercise 47 *Solve the following* 2×2 *systems using Gauss elimination*

$$\left\{\begin{array}{lll} -2x+6y & = & 3 \\ 5x-15y & = & -5 \end{array}\right\}, \quad \left\{\begin{array}{lll} -2x-8y & = & 4 \\ -5x+3y & = & -13 \end{array}\right\} \quad \textit{and} \quad \left\{\begin{array}{lll} 7x-y & = & 9 \\ -x+2y & = & -5 \end{array}\right\}$$

1.6 Applications of Linear Systems

Why studying systems of linear equations in many variables? They are very useful in solving problems encountered in different areas of science and real life.

Example 40 *Fifty-six biscuits are to be fed to ten pets. Each pet is either a cat or a dog. Each dog is to get six biscuits and each cat to get five. How many dogs are there?*

Solution:
Suppose that there are x dogs and y cats. Then, since the number of pets is 10 and, since there are 56 biscuits available, the following two linear equations in the variables x and y must hold:

$$\left\{ \begin{array}{lcl} x+y & = & 10 \\ 6x+5y & = & 56 \end{array} \right\}$$

We solve the system by the substitution method:

$$\left\{ \begin{array}{lcl} x+y & = & 10 \\ 6x+5y & = & 56 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{lcl} x & = & 10-y \\ 6x+5y & = & 56 \end{array} \right\} \Longrightarrow$$

$$\left\{ \begin{array}{lcl} x & = & 10-y \\ 6(10-y)+5y & = & 56 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{lcl} x & = & 10-y \\ 60-6y+5y & = & 56 \end{array} \right\} \Longrightarrow$$

$$\left\{ \begin{array}{lcl} x & = & 10-y \\ y & = & 4 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{lcl} x & = & 6 \\ y & = & 4 \end{array} \right\}.$$

■

Example 41 *A friend of yours has invested \$250 in two companies A and B. The stocks of A currently sell for \$3 a share and the stock of B sells for \$5 a share. If the stock of A triples and the stock of B doubles, then her stocks will be worth \$650. Can you determine how many shares she own from each stock?*

Solution:
Suppose that she owns x shares from stock A and y shares from stock B. Then the following two equations hold

$$\left\{ \begin{array}{lcl} 3x+5y & = & 250 \\ 9x+10y & = & 650 \end{array} \right\}.$$

We solve this 2×2 system

$$\left\{ \begin{array}{lcl} 3x+5y & = & 250 \\ 9x+10y & = & 650 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{lcl} 9x+15y & = & 750 \\ 9x+10y & = & 650 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{lcl} 5y & = & 100 \\ 9x+10y & = & 650 \end{array} \right\} \Longrightarrow$$

$$\left\{ \begin{array}{lcl} y & = & 20 \\ 9x+10y & = & 650 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{lcl} y & = & 20 \\ 9x+200 & = & 650 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{lcl} y & = & 20 \\ 9x & = & 450 \end{array} \right\} \Longrightarrow$$

$$\left\{\begin{array}{lll} y & = & 20 \\ x & = & 50 \end{array}\right\}.$$

■

Example 42 *A flight leaves New York at 8pm and arrives in Paris at 9am (Paris time). This 13-hour difference includes the flight time plus the change in time zones. The return flight leaves Paris at 1pm and arrives in New York at 3pm (New York time). This 2-hour difference includes the flight time minus the time zones, plus an extra hour due to the fact that flying westward is against the wind. Find the actual flight time eastward and the difference in time zones.*

Solution:
We set x the actual flight time eastward and y the difference in time zones. Then

$$\left\{\begin{array}{lll} x+y & = & 13 \\ x-y+1 & = & 2 \end{array}\right\}.$$

Then we have

$$\left\{\begin{array}{lll} x+y & = & 13 \\ x-y+1 & = & 2 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lll} x+y & = & 13 \\ 2x+1 & = & 15 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lll} x+y & = & 13 \\ 2x & = & 14 \end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{lll} x+y & = & 13 \\ x & = & 7 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lll} y & = & 5 \\ x & = & 7 \end{array}\right\}.$$

Thus, the actual time flight is 7 hours and the time-zone difference is 5 hours. ■

Example 43 *If 20 lbs of rice and 10 lbs of potatoes cost \$9 and 30 lbs of rice and 12 lbs of potatoes cost \$ 12, then how much do 10 lbs of rice and 50 lbs of potatoes cost?*

Solution:
Set x the price per lb of rice and y the price per lb of potatoes. Then we have

$$\left\{\begin{array}{lll} 20x+10y & = & 9 \\ 30x+12y & = & 12 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lll} x+\frac{1}{2}y & = & \frac{9}{20} \\ 30x+12y & = & 12 \end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{lll} x & = & \frac{9}{20}-\frac{1}{2}y \\ 30(\frac{9}{20}-\frac{1}{2}y)+12y & = & 12 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lll} x & = & \frac{9}{20}-\frac{1}{2}y \\ \frac{27}{2}-15y+12y & = & 12 \end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{lll} x & = & \frac{9}{20}-\frac{1}{2}y \\ -3y & = & -\frac{3}{2} \end{array}\right\} \Longrightarrow \left\{\begin{array}{lll} x & = & \frac{9}{20}-\frac{1}{2}y \\ y & = & \frac{1}{2} \end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{lll} x & = & \frac{9}{20}-\frac{1}{2}\frac{1}{2} \\ y & = & \frac{1}{2} \end{array}\right\} \Longrightarrow \left\{\begin{array}{lll} x & = & \frac{9}{20}-\frac{1}{2}\frac{1}{2} \\ y & = & \frac{1}{2} \end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{lll} x & = & \frac{9}{20}-\frac{5}{20} \\ y & = & \frac{1}{2} \end{array}\right\} \Longrightarrow \left\{\begin{array}{lll} x & = & \frac{1}{5} \\ y & = & \frac{1}{2} \end{array}\right\}.$$

So to find the cost of 10 lbs of rice and 50 lbs of potatoes we calculate

$$10 \cdot \frac{1}{5} + 50 \cdot \frac{1}{2} = 27.$$

■

Example 44 *A liquor store sells two brands of wine A and B. A sells for \$10 a bottle and B for \$8 a bottle. The entire stock is worth \$820. Sales are slow and only half of the bottles of A and $\frac{3}{4}$ of the bottles of B are sold for a total of \$490. How many bottles of each wine are left in the store?*

Solution:

Suppose that the store has initially x bottles of A and y bottles of B. Then the following two equations must hold

$$\left\{\begin{array}{lcl} 10x + 8y & = & 820 \\ 10\frac{x}{2} + 8\frac{3y}{4} & = & 490 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lcl} 10x + 8y & = & 820 \\ 5x + 6y & = & 490 \end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{lcl} x + \frac{4}{5}y & = & 82 \\ 5x + 6y & = & 490 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lcl} x + \frac{4}{5}y & = & 82 \\ 2y & = & 80 \end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{lcl} x + \frac{4}{5}y & = & 82 \\ y & = & 40 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lcl} x + \frac{4}{5}40 & = & 82 \\ y & = & 40 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lcl} x & = & 50 \\ y & = & 40 \end{array}\right\}.$$

Therefore, the bottles of A remaining are 25 and the bottles of B remaining are 10. ■

Example 45 *A rectangle has perimeter 14 feet and its length is 1 unit less than 3 times its width. Find the length of each of its sides.*

Solution:

Let l denote the length and w the width of the rectangle. Then we have

$$\left\{\begin{array}{lcl} 2l + 2w & = & 14 \\ l & = & 3w - 1 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lcl} 2(3w-1) + 2w & = & 14 \\ l & = & 3w - 1 \end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{lcl} 6w - 2 + 2w & = & 14 \\ l & = & 3w - 1 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lcl} 8w & = & 16 \\ l & = & 3w - 1 \end{array}\right\} \Longrightarrow$$

$$\left\{\begin{array}{lcl} w & = & 2 \\ l & = & 3w - 1 \end{array}\right\} \Longrightarrow \left\{\begin{array}{lcl} w & = & 2 \\ l & = & 5 \end{array}\right\}.$$

■

Example 46 *Two runners start moving towards each other on a straight line from two points that are 1000 feet apart. When they meet the first runner has covered 200 feet less that twice the distance that the second runner has covered? How much distance has each of the two covered until they meet?*

Solution:

Let x denote the distance that the first runner has covered and y the distance that the second runner has covered until they meet. Then

$$\left\{ \begin{array}{lcl} x+y & = & 1000 \\ x & = & 2y-200 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{lcl} 2y-200+y & = & 1000 \\ x & = & 2y-200 \end{array} \right\} \Longrightarrow$$

$$\left\{ \begin{array}{lcl} 3y & = & 1200 \\ x & = & 2y-200 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{lcl} y & = & 400 \\ x & = & 2y-200 \end{array} \right\} \Longrightarrow$$

$$\left\{ \begin{array}{lcl} y & = & 400 \\ x & = & 2 \cdot 400 - 200 \end{array} \right\} \Longrightarrow \left\{ \begin{array}{lcl} y & = & 400 \\ x & = & 600 \end{array} \right\}.$$

■

Now do the following exercises to get used to setting up applied problems and to solving the resulting 2×2 systems of linear equations.

Exercise 48 *An animal shelter houses 3 dogs and 7 cats. The animal shelter needs 29 units of food per day to feed the 10 animals at the cost of \$44 a day. If the dog food costs \$2 per unit and the cat food \$1 per unit how many units of food does each dog and each cat consume?*

Exercise 49 *If 10 lbs of apples and 5 lbs of oranges cost \$8 and 24 lbs of apples and 15 lbs of oranges cost \$ 21, then how much do 5 lbs of apples and 50 lbs of oranges cost?*

Exercise 50 *A liquor store sells two brands of beer A and B. A sells for \$4 a six-pack and B for \$8 a six-pack. The entire stock is worth \$320. Sales are slow and only half of the packs of A and $\frac{1}{5}$ of the packs of B are sold for a total of \$124. How many six-packs of each beer are left in the store?*

Exercise 51 *An isosceles triangle has perimeter 31 feet and its sides have length two units less than twice the length of its base. What is the length of the base and the length of its sides?*

Exercise 52 *Two runners start moving towards each other on a straight line from two points that are 400 feet apart. When they meet the first runner has covered 50 feet less that twice the distance that the second runner has covered? How much distance has each of the two covered until they meet?*

Chapter 2

Matrices and Systems

2.1 Matrices, Additive Structure

Matrices are very important in management, natural science, engineering and social science as a way to organize data.

An $n \times m$ **matrix** is a rectangular array of numbers consisting of n **rows** and m **columns**. Each number in the array is called an **element** or an **entry** of the matrix. The element that occupies the i-th row and the j-th column of the matrix A is usually denoted by a_{ij}. A itself is sometimes denoted by $A = [a_{ij}]$ to give a notation for its elements.

For an example, consider the 2×3 matrix (2 rows and 3 columns)

$$A = \begin{bmatrix} 2 & 5 & -6 \\ -1 & 4 & 9 \end{bmatrix}.$$

We have $a_{12} = 5$ and $a_{23} = 9$.

$n \times m$ is sometimes referred to as the **dimension** of the $n \times m$ matrix.

Example 47 *A liquor distributor ships six-packs of* Budweiser®, Miller® *and* Coors® *to the grocery stores in a small town. Each brand comes in two versions of regular and light. If the distributor ships 10 Buds and 20 Bud Lights, 15 Millers and 20 Miller Lights and 12 Coors and 30 Coors Lights, construct a 2×3 matrix organizing the shipment of beer.*

Solution:

The data may be depicted by the following table

	Budweiser	Miller	Coors
Regular	10	15	12
Light	20	20	30

This data may be written in the form of a 2×3 matrix

$$B = \begin{bmatrix} 10 & 15 & 12 \\ 20 & 20 & 30 \end{bmatrix}$$

with the understanding that the first row corresponds to regular and the second to light and that the three columns correspond to Budweiser, Miller and Coors, respectively. ■

Exercise 53 *A Ford® dealership ships cars to 4 small towns in the area every month. To town A, they ship 3 Escorts, 2 Contours and 1 Taurus. To town B they ship 5 Escorts, 12 Contours and 3 Tauruses, to town C, 2 Escorts, 2 Contours and 1 Taurus and to town D 10 Escorts, 20 Contours and 7 Tauruses. Create a 4×3 matrix to organize the data. Explain what each row and each column represents.*

A **row matrix** is a matrix having only one row. A **column matrix** is a matrix having only one column. For instance $A = [2 \quad 4 \quad -45]$ is a row matrix. The matrix $B = \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$ is a column matrix. A **square matrix** is a matrix with the same number of rows and columns. An $n \times n$ square matrix is sometimes said to have **dimension** n. For instance $C = \begin{bmatrix} 1 & -3 & -4 \\ 6 & -5 & 4 \\ 2 & 5 & -19 \end{bmatrix}$ is a square matrix of dimension 3.

The **sum** of two $n \times m$ matrices A and B is the $n \times m$ matrix $A+B$ in which each element is the sum of the corresponding elements of A and B. I.e., more formally, if $C = A + B$,

$$c_{ij} = a_{ij} + b_{ij} \quad \text{for all } 1 \leq i \leq n, 1 \leq j \leq m.$$

Example 48 *Compute the sum $A + B$ of the following matrices*

1. $A = \begin{bmatrix} 1 & -2 \\ -6 & 7 \\ 12 & 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 7 & -1 \\ 6 & -5 \\ -2 & 15 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & -3 \\ 6 & -5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -3 & -4 \\ 6 & -5 & 4 \end{bmatrix}$

Solution:

1. We have

$$A + B = \begin{bmatrix} 1 & -2 \\ -6 & 7 \\ 12 & 5 \end{bmatrix} + \begin{bmatrix} 7 & -1 \\ 6 & -5 \\ -2 & 15 \end{bmatrix} = \begin{bmatrix} 1+7 & -2-1 \\ -6+6 & 7-5 \\ 12-2 & 5+15 \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ 0 & 2 \\ 10 & 20 \end{bmatrix}$$

2. The first matrix is a 2×2 matrix whereas the second matrix is a 2×3 matrix. Since the dimensions of the matrices do not match, the two matrices cannot be added, i.e., their sum is not defined. ■

Exercise 54 *Compute the sum $A + B$ of the following matrices*

1. $A = \begin{bmatrix} 2 & -13 \\ 8 & 7 \end{bmatrix}$ *and* $B = \begin{bmatrix} 7 & -9 \\ -13 & 4 \end{bmatrix}$

2. $A = \begin{bmatrix} 1 & -3 & 3 \\ 6 & -5 & 12 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & -3 & -4 \\ 6 & -5 & 4 \end{bmatrix}$

3. $A = \begin{bmatrix} 1 & -3 \\ 6 & -5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 5 & -1 & 0 \\ -17 & -5 & -10 \end{bmatrix}$

Note that the $n \times m$ matrix $\mathbf{0}$ all of whose entries are equal to 0 has the property

$$A + \mathbf{0} = \mathbf{0} + A = A, \quad \text{for every } n \times m \text{ matrix } A.$$

This is the reason why it is said to be the **zero matrix**.

For instance consider the 2×4 matrix $A = \begin{bmatrix} 1 & -3 & 7 & 10 \\ 6 & -5 & 9 & -2 \end{bmatrix}$. We have

$$A + \mathbf{0} = \begin{bmatrix} 1 & -3 & 7 & 10 \\ 6 & -5 & 9 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1+0 & -3+0 & 7+0 & 10+0 \\ 6+0 & -5+0 & 9+0 & -2+0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 7 & 10 \\ 6 & -5 & 9 & -2 \end{bmatrix} = A.$$

The **additive inverse** or **negative** of a matrix A is the matrix $-A$ in which each element is the additive inverse of the corresponding element of A. I.e., more formally, if $B = -A$, then

$$b_{ij} = -a_{ij}, \quad \text{for all} \quad 1 \leq i \leq n, 1 \leq j \leq m.$$

Example 49 *Compute the additive inverse* $-A$ *of the matrix* $A = \begin{bmatrix} 1 & -3 & -4 \\ 6 & -5 & 4 \end{bmatrix}$.

Solution:

We have

$$-A = -\begin{bmatrix} 1 & -3 & -4 \\ 6 & -5 & 4 \end{bmatrix} = \begin{bmatrix} -1 & -(-3) & -(-4) \\ -6 & -(-5) & -4 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 4 \\ -6 & 5 & -4 \end{bmatrix}.$$

■

Exercise 55 *Compute the additive inverses* $-A$ *and* $-B$ *of the following matrices* $A = \begin{bmatrix} 4 & -1 & 0 \\ -6 & -7 & 5 \end{bmatrix}$ *and* $B = \begin{bmatrix} 4 & -1 \\ -6 & -7 \\ -1 & 5 \\ 9 & 0 \end{bmatrix}$.

The negative inverse $-A$ of the matrix A has the important property that, when added to A, the sum is the zero matrix. That is

$$A + (-A) = (-A) + A = \mathbf{0}, \quad \text{for every matrix } A.$$

Consider, for instance, the matrix $A = \begin{bmatrix} -6 & 3 & 0 \\ 5 & -7 & 19 \end{bmatrix}$. We have that

$$-A = \begin{bmatrix} 6 & -3 & 0 \\ -5 & 7 & -19 \end{bmatrix}.$$

Therefore

$$A + (-A) = \begin{bmatrix} -6 & 3 & 0 \\ 5 & -7 & 19 \end{bmatrix} + \begin{bmatrix} 6 & -3 & 0 \\ -5 & 7 & -19 \end{bmatrix} =$$
$$= \begin{bmatrix} -6+6 & 3-3 & 0+0 \\ 5-5 & -7+7 & 19-19 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbf{0}.$$

The subtraction of matrices of the same dimension is defined in a manner comparable to the subtraction of real numbers.

The **difference** $A - B$ of two $n \times m$ matrices A and B is the $n \times m$ matrix in which each element is the difference of the corresponding elements of A and B. I.e., more formally,

$$A - B = A + (-B).$$

Example 50 *Compute the difference $A - B$, where*

1. $A = \begin{bmatrix} 6 & 3 & 13 \\ -5 & 7 & -8 \end{bmatrix}$ *and* $B = \begin{bmatrix} 10 & -5 & 17 \\ 7 & -12 & 6 \end{bmatrix}$

2. $A = \begin{bmatrix} 6 & 3 & 13 \\ -5 & 7 & -8 \end{bmatrix}$ *and* $B = \begin{bmatrix} 10 & -5 \\ 7 & -12 \\ 1 & 8 \\ -9 & -13 \end{bmatrix}$

Solution:

1. We have

$$A - B = A + (-B) = \begin{bmatrix} 6 & 3 & 13 \\ -5 & 7 & -8 \end{bmatrix} + (-\begin{bmatrix} 10 & -5 & 17 \\ 7 & -12 & 6 \end{bmatrix}) =$$
$$= \begin{bmatrix} 6 & 3 & 13 \\ -5 & 7 & -8 \end{bmatrix} + \begin{bmatrix} -10 & 5 & -17 \\ -7 & 12 & -6 \end{bmatrix} =$$
$$= \begin{bmatrix} 6-10 & 3+5 & 13-17 \\ -5-7 & 7+12 & -8-6 \end{bmatrix} = \begin{bmatrix} -4 & 8 & -4 \\ -12 & 19 & -14 \end{bmatrix}.$$

2. The two given matrices are of different dimensions. Thus, their difference is not defined. ■

Exercise 56 *Compute the difference* $A - B$ *where*

1. $A = \begin{bmatrix} -1 & 2 & -4 \\ -5 & 26 & -7 \end{bmatrix}$ *and* $B = \begin{bmatrix} 90 & -46 & 12 \\ 71 & -1 & 0 \end{bmatrix}$,

2. $A = \begin{bmatrix} -1 & 2 \\ -5 & 26 \\ 9 & 8 \end{bmatrix}$ *and* $B = \begin{bmatrix} 9 & -6 \\ 1 & -1 \end{bmatrix}$,

3. $A = \begin{bmatrix} -1 & 2 \\ -5 & 26 \\ 9 & 8 \end{bmatrix}$ *and* $B = \begin{bmatrix} 9 & -6 \\ 1 & -1 \\ -4 & 3 \end{bmatrix}$.

The **product** of a number c and a matrix A is the matrix cA each of whose elements is c times the corresponding element of A. I.e., more formally, if $B = cA$,

$$b_{ij} = c \cdot a_{ij}, \quad \text{for all} \quad 1 \leq i \leq n, 1 \leq j \leq m.$$

Example 51 *Compute the matrix* $3A$ *and the matrix* $2A - 5B$, *where* $A = \begin{bmatrix} 9 & -6 \\ 1 & -1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 3 & -7 \\ -1 & 5 \end{bmatrix}$.

Solution:
We have

$$3A = 3\begin{bmatrix} 9 & -6 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 9 & 3 \cdot (-6) \\ 3 \cdot 1 & 3 \cdot (-1) \end{bmatrix} = \begin{bmatrix} 27 & -18 \\ 3 & -3 \end{bmatrix}.$$

Also

$$2A - 5B = 2\begin{bmatrix} 9 & -6 \\ 1 & -1 \end{bmatrix} - 5\begin{bmatrix} 3 & -7 \\ -1 & 5 \end{bmatrix} =$$

$$= \begin{bmatrix} 18 & -12 \\ 2 & -2 \end{bmatrix} - \begin{bmatrix} 15 & -35 \\ -5 & 25 \end{bmatrix} = \begin{bmatrix} 3 & 23 \\ 7 & -27 \end{bmatrix}.$$

■

Exercise 57 *Compute the matrix* $5A$ *and the matrix* $-2A+3B$, *where* $A = \begin{bmatrix} 9 & -2 & 5 \\ -1 & -1 & 2 \end{bmatrix}$ *and* $B = \begin{bmatrix} -3 & -1 & 7 \\ 5 & 9 & -12 \end{bmatrix}$.

2.2 Multiplicative Structure

Multiplying matrices is a more involved operation than multiplying a number with a matrix. To explain this operation, the product of a row matrix with a column matrix is explained first.

The **product** of a row matrix A with m columns with a column matrix B with m rows is defined to be the square matrix of dimension 1 whose entry is obtained by multiplying the corresponding entries of A and B and adding the results. I.e., more formally, if $C = A \cdot B$,

$$c_{11} = a_{11}b_{11} + a_{12}b_{21} + \ldots + a_{1m}b_{m1}.$$

Example 52 *Find the product of the matrices*

1. $A = \begin{bmatrix} -3 & -1 & 7 \end{bmatrix}$ *and* $B = \begin{bmatrix} -4 \\ 7 \\ 12 \end{bmatrix}$

2. $A = \begin{bmatrix} 3 & -4 & 2 & -6 \end{bmatrix}$ *and* $B = \begin{bmatrix} 6 \\ -2 \\ -1 \\ -5 \end{bmatrix}$

Solution:

1. We have

$$A \cdot B = \begin{bmatrix} -3 & -1 & 7 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 7 \\ 12 \end{bmatrix} = \begin{bmatrix} (-3)\cdot(-4) + (-1)\cdot 7 + 7 \cdot 12 \end{bmatrix} =$$

$$= \begin{bmatrix} 12 - 7 + 84 \end{bmatrix} = \begin{bmatrix} 89 \end{bmatrix}$$

2. We have

$$A \cdot B = \begin{bmatrix} 3 & -4 & 2 & -6 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -2 \\ -1 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \cdot 6 + (-4)\cdot(-2) + 2\cdot(-1) + (-6)\cdot(-5) \end{bmatrix} =$$

$$= \begin{bmatrix} 18 + 8 - 2 + 30 \end{bmatrix} = \begin{bmatrix} 54 \end{bmatrix}$$

■

Exercise 58 *Find the product* $A \cdot B$ *if*

1. $A = \begin{bmatrix} -6 & 2 & -11 \end{bmatrix}$ *and* $B = \begin{bmatrix} 5 \\ -7 \\ 3 \end{bmatrix}$

2. $A = \begin{bmatrix} -3 & -1 \end{bmatrix}$ *and* $B = \begin{bmatrix} -4 \\ 7 \\ 12 \\ 5 \end{bmatrix}$

3. $A = \begin{bmatrix} -3 & 6 & 12 & -1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 5 \\ -2 \\ -3 \\ 8 \end{bmatrix}$

Now, with the multiplication of a row matrix by a column matrix defined, the definition of the product of two matrices will be introduced.

Let A be an $n \times m$ matrix and B an $m \times k$ matrix. The **product matrix** $A \cdot B$ is the $n \times k$ matrix whose entry in the i-th row and j-th column is the entry of the product of the i-th row of A with the j-th column of B. I.e., more formally, if $C = A \cdot B$, then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{im}b_{mj}, \quad \text{for all} \quad 1 \le i \le n, 1 \le j \le k.$$

Example 53 *Compute the product* $A \cdot B$ *if*

1. $A = \begin{bmatrix} -1 & 3 & -2 \\ 2 & -6 & -3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 5 & 2 \\ -1 & 2 \\ 3 & -7 \end{bmatrix}$

2. $A = \begin{bmatrix} -1 & 3 \\ 2 & -6 \end{bmatrix}$ *and* $B = \begin{bmatrix} 5 & 2 \\ -1 & 2 \\ 3 & -7 \end{bmatrix}$

3. $A = \begin{bmatrix} -5 & 3 \\ -2 & -6 \\ 3 & 8 \end{bmatrix}$ *and* $B = \begin{bmatrix} 5 & 2 \\ 3 & -7 \end{bmatrix}$

Solution:

1. We have

$$A \cdot B = \begin{bmatrix} -1 & 3 & -2 \\ 2 & -6 & -3 \end{bmatrix} \cdot \begin{bmatrix} 5 & 2 \\ -1 & 2 \\ 3 & -7 \end{bmatrix} =$$

$$= \begin{bmatrix} (-1) \cdot 5 + 3 \cdot (-1) + (-2) \cdot 3 & (-1) \cdot 2 + 3 \cdot 2 + (-2) \cdot (-7) \\ 2 \cdot 5 + (-6) \cdot (-1) + (-3) \cdot 3 & 2 \cdot 2 + (-6) \cdot 2 + (-3) \cdot (-7) \end{bmatrix} =$$

$$= \begin{bmatrix} -5 - 3 - 6 & -2 + 6 + 14 \\ 10 + 6 - 9 & 4 - 12 + 21 \end{bmatrix} = \begin{bmatrix} -14 & 18 \\ 7 & 13 \end{bmatrix}$$

2. A is a 2×2 matrix and B is a 3×2 matrix. So the number of columns of A is different from the number of rows of B. Therefore the product $A \cdot B$ is not defined.

3. We have

$$A \cdot B = \begin{bmatrix} -5 & 3 \\ -2 & -6 \\ 3 & 8 \end{bmatrix} \cdot \begin{bmatrix} 5 & 2 \\ 3 & -7 \end{bmatrix} =$$

$$= \begin{bmatrix} (-5) \cdot 5 + 3 \cdot 3 & (-5) \cdot 2 + 3 \cdot (-7) \\ (-2) \cdot 5 + (-6) \cdot 3 & (-2) \cdot 2 + (-6) \cdot (-7) \\ 3 \cdot 5 + 8 \cdot 3 & 3 \cdot 2 + 8 \cdot (-7) \end{bmatrix} =$$

$$= \begin{bmatrix} -25 + 9 & -10 - 21 \\ -10 - 18 & -4 + 42 \\ 15 + 24 & 6 - 56 \end{bmatrix} = \begin{bmatrix} -16 & -31 \\ -28 & 38 \\ 39 & -50 \end{bmatrix}$$

■

Now do the following exercise to make sure that you understand the notion of product of an $n \times m$ with an $m \times k$ matrix.

Exercise 59 *Compute the product $A \cdot B$ if*

1. $A = \begin{bmatrix} -1 & 3 \\ 2 & -6 \\ 4 & 7 \end{bmatrix}$ *and* $B = \begin{bmatrix} 5 & 2 & -1 & 0 \\ -1 & 9 & 5 & -1 \end{bmatrix}$

2. $A = \begin{bmatrix} -1 & 3 \\ 2 & -6 \end{bmatrix}$ *and* $B = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

3. $A = \begin{bmatrix} -5 & 3 \\ -2 & -6 \\ 3 & 8 \end{bmatrix}$ *and* $B = \begin{bmatrix} 5 & 2 \\ 3 & -7 \\ -1 & 0 \end{bmatrix}$

Matrix multiplication has some of the nice properties of the multiplication of numbers. For instance, it is *associative* and *distributive*. This means that, if A is an $n \times m$, B is an $m \times k$ and C is a $k \times l$ matrix, then

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C,$$

if A is an $n \times m$ matrix and B and C are two $m \times k$ matrices, then

$$A \cdot (B + C) = A \cdot B + A \cdot C$$

and, if B, C are $n \times m$ matrices and A is an $m \times k$ matrix, then

$$(B + C) \cdot A = B \cdot A + C \cdot A.$$

However, matrix multiplication is *not commutative*. In other words, there exist square matrices of dimension n A and B, such that

$$A \cdot B \neq B \cdot A.$$

Exercise 60 *Find two* 2×2 *matrices* A *and* B*, such that* $A \cdot B \neq B \cdot A$.

The **identity matrix** of dimension n is the square matrix of dimension n whose diagonal entries are equal to 1 and all other entries equal to 0. For instance, the identity matrix of dimension 2 is $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and the identity matrix of dimension 3 is $\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

The identity matrix plays in matrix multiplication the same role that the identity element 1 plays in the multiplication of numbers. It has the property that, if A is an $n \times m$ matrix and $\mathbf{I}_m$ the identity matrix of dimension m, then

$$A \cdot \mathbf{I}_m = A$$

and if $\mathbf{I}_n$ is the identity matrix of dimension n and A is any $n \times m$ matrix, then

$$\mathbf{I}_n \cdot A = A.$$

Exercise 61 *Let* $A = \begin{bmatrix} 5 & -3 \\ 2 & 6 \\ 11 & 8 \end{bmatrix}$. *Compute the products*

1. $\mathbf{I}_3 \cdot A$,

2. $A \cdot \mathbf{I}_2$.

2.3 Augmented Matrices and Systems of Equations

We revisit in this lecture the Gauss elimination method of solving 2×2 systems. Our goal is to, first, recast the systems as matrices, by keeping only the coefficients of the variables and the constants involved in each of the linear equations, and, then reformulate the allowable operations in terms of row operations of the corresponding matrices. This recasting will lead to the so-called Gauss-Jordan method for solving $n\times m$ systems, i.e., systems consisting of n linear equations in m variables.

We first present an example of a solution of a 2×2 system with the Gauss elimination as a reminder of the method.

Example 54 *Solve the following* 2×2 *system using Gauss elimination.*

$$\left\{\begin{array}{rcrcr} 2x & - & 5y & = & -17 \\ -3x & + & 3y & = & 12 \end{array}\right\}.$$

Solution:
First, multiply the first equation by $\frac{1}{2}$. We get

$$\left\{\begin{array}{rcrcr} x & - & \frac{5}{2}y & = & -\frac{17}{2} \\ -3x & + & 3y & = & 12 \end{array}\right\}.$$

Now add to the second equation the first multiplied by 3:

$$\left\{\begin{array}{rcrcr} x & - & \frac{5}{2}y & = & -\frac{17}{2} \\ & & -\frac{9}{2}y & = & -\frac{27}{2} \end{array}\right\}.$$

Next, multiply the second equation by $-\frac{2}{9}$:

$$\left\{\begin{array}{rcrcr} x & - & \frac{5}{2}y & = & -\frac{17}{2} \\ & & y & = & 3 \end{array}\right\}.$$

Finally, add to the first equation the second equation multiplied by $\frac{5}{2}$:

$$\left\{\begin{array}{rcrcr} x & & & = & -1 \\ & & y & = & 3 \end{array}\right\}.$$

Hence our system has the solution $(-1,3)$. ■

Note that the variables x and y are carried over from stage to stage in the solution of the system without really affecting the process. The crucial elements in each stage are the coefficients of the variables and the constants and the way they are manipulated. To organize these data without showing the variables we use the following table

	coefficient of x	coefficient of y	constant term
first equation	2	−5	−17
second equation	−3	3	12

Thus, recasting this table as a matrix we obtain

$$\left[\begin{array}{rr|r} 2 & -5 & -17 \\ -3 & 3 & 12 \end{array}\right]$$

Note the vertical line that separates the coefficients from the constant terms. The square matrix of dimension 2 on the left of the vertical line

$$\left[\begin{array}{rr} 2 & -5 \\ -3 & 3 \end{array}\right]$$

is called the **coefficient matrix** of the system. The 2×3 matrix above is called the **augmented matrix** of the system.

The three allowable operations in the Gauss elimination method translate to the following three allowable operations for manipulating the augmented matrix with the goal of solving the 2×2 system:

Matrix Row Operations for Solving Linear Systems:

1. Interchange two rows of the matrix.
2. Multiply all the entries in a single row of the matrix by a nonzero constant.
3. Add to one row a multiple of another row.

We see how these operations may be applied to solve the given 2×2 system:

Example 55 *Consider the augmented matrix of the system given in Example 54. Apply matrix row operations to solve the system.*

Solution:

$$\left[\begin{array}{rr|r} 2 & -5 & -17 \\ -3 & 3 & 12 \end{array}\right] \xrightarrow{r_1 \cdot \frac{1}{2}} \left[\begin{array}{rr|r} 1 & -\frac{5}{2} & -\frac{17}{2} \\ -3 & 3 & 12 \end{array}\right] \xrightarrow{r_2 \leftarrow r_2 + 3r_1} \left[\begin{array}{rr|r} 1 & -\frac{5}{2} & -\frac{17}{2} \\ 0 & -\frac{9}{2} & -\frac{27}{2} \end{array}\right]$$

$$\xrightarrow{r_2 \cdot (-\frac{2}{9})} \left[\begin{array}{rr|r} 1 & -\frac{5}{2} & -\frac{17}{2} \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 + \frac{5}{2} r_2} \left[\begin{array}{rr|r} 1 & 0 & -1 \\ 0 & 1 & 3 \end{array}\right]$$

The system corresponding to this augmented matrix is the system

$$x = -1 \quad \text{and} \quad y = 3.$$

■

The process followed to solve the 2×2 system above is called the **augmented matrix method** or the **Gauss-Jordan method**. The usefulness of this method lies in the facts that, first, it is applicable for systems with more equations in more variables, and, second, in that it is easily programmable to be executed by a computer.

We outline the steps of this method:

Gauss-Jordan Method:

1. Obtain from the system its augmented matrix A.
2. Use row operation 2 to make $a_{11} = 1$.
3. Use repeatedly row operation 3 to eliminate the remaining entries in the first column.
4. Next make $a_{22} = 1$ using again row operation 2.
5. Use repeatedly row operation 3 to eliminate all other entries in the second column.
6. Repeat the above steps for all other columns.

The Gauss-Jordan method is applied below to some more complicated systems.

Example 56 *Use the Gauss-Jordan method to solve the* 3×3 *system*

$$\left\{\begin{array}{rcrcrcr} x & - & 2y & + & z & = & 0 \\ 3x & + & y & - & 5z & = & 8 \\ -2x & - & 5y & + & z & = & -3 \end{array}\right\}$$

Solution:
The augmented matrix of the given system is

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 3 & 1 & -5 & 8 \\ -2 & -5 & 1 & -3 \end{array}\right]$$

We proceed to apply a series of row operations to solve this system:

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 3 & 1 & -5 & 8 \\ -2 & -5 & 1 & -3 \end{array}\right] \xrightarrow{r_2 \leftarrow r_2 - 3r_1} \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 7 & -8 & 8 \\ -2 & -5 & 1 & -3 \end{array}\right] \xrightarrow{r_3 \leftarrow r_3 + 2r_1} \left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 7 & -8 & 8 \\ 0 & -9 & 3 & -3 \end{array}\right] \xrightarrow{r_2 \cdot \frac{1}{7}}$$

$$\left[\begin{array}{rrr|r} 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{8}{7} & \frac{8}{7} \\ 0 & -9 & 3 & -3 \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 + 2r_2} \left[\begin{array}{rrr|r} 1 & 0 & -\frac{9}{7} & \frac{16}{7} \\ 0 & 1 & -\frac{8}{7} & \frac{8}{7} \\ 0 & -9 & 3 & -3 \end{array}\right] \xrightarrow{r_3 \leftarrow r_3 + 9r_2} \left[\begin{array}{rrr|r} 1 & 0 & -\frac{9}{7} & \frac{16}{7} \\ 0 & 1 & -\frac{8}{7} & \frac{8}{7} \\ 0 & 0 & -\frac{51}{7} & \frac{51}{7} \end{array}\right] \xrightarrow{r_3 \cdot (-\frac{7}{51})}$$

$$\left[\begin{array}{rrr|r} 1 & 0 & -\frac{9}{7} & \frac{16}{7} \\ 0 & 1 & -\frac{8}{7} & \frac{8}{7} \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 + \frac{9}{7}r_3} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & -\frac{8}{7} & \frac{8}{7} \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{r_2 \leftarrow r_2 + \frac{8}{7}r_3} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Thus the system obtained is

$$x = 1, \quad y = 0, \quad \text{and} \quad z = -1.$$

■

Example 57 *Use the Gauss-Jordan method to solve the system*

$$\left\{\begin{array}{rcrcr} x & + & y & = & 3 \\ -2x & + & y & = & 0 \\ 3x & + & 3y & = & 9 \end{array}\right\}$$

Solution:
The augmented matrix of the system is

$$\left[\begin{array}{rr|r} 1 & 1 & 3 \\ -2 & 1 & 0 \\ 3 & 3 & 9 \end{array}\right] \xrightarrow{r_2 \leftarrow r_2 + 2r_1} \left[\begin{array}{rr|r} 1 & 1 & 3 \\ 0 & 3 & 6 \\ 3 & 3 & 9 \end{array}\right] \xrightarrow{r_3 \leftarrow r_3 - 3r_1} \left[\begin{array}{rr|r} 1 & 1 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{r_2 \cdot \frac{1}{3}}$$

$$\left[\begin{array}{rr|r} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 - r_2} \left[\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

The final system gives

$$x = 1 \quad \text{and} \quad y = 2.$$

■

Example 58 *Use the Gauss-Jordan method to solve the system*

$$\left\{\begin{array}{rcrcrcr} x & + & y & - & z & = & 3 \\ 2x & - & y & + & 3z & = & 13 \end{array}\right\}$$

Solution:
Take the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 3 \\ 2 & -1 & 3 & 13 \end{array}\right] \xrightarrow{r_2 \leftarrow r_2 - 2r_1} \left[\begin{array}{rrr|r} 1 & 1 & -1 & 3 \\ 0 & -3 & 5 & 7 \end{array}\right] \xrightarrow{r_2 \cdot (-\frac{1}{3})}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 3 \\ 0 & 1 & -\frac{5}{3} & -\frac{7}{3} \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 - r_2} \left[\begin{array}{rrr|r} 1 & 0 & \frac{2}{3} & \frac{16}{3} \\ 0 & 1 & -\frac{5}{3} & -\frac{7}{3} \end{array}\right]$$

Thus, the equations of the equivalent system are

$$\left\{\begin{array}{rcr} x + \frac{2}{3}z & = & \frac{16}{3} \\ y - \frac{5}{3}z & = & -\frac{7}{3} \end{array}\right\}, \quad \text{i.e.,} \quad \left\{\begin{array}{rcl} x & = & \frac{16}{3} - \frac{2}{3}z \\ y & = & -\frac{7}{3} + \frac{5}{3}z \end{array}\right\}$$

Hence the solutions are given by

$$(-\frac{2}{3}z + \frac{16}{3}, \frac{5}{3}z - \frac{7}{3}, z), \quad z \text{ any real number.}$$

■

Example 59 *Use the Gauss-Jordan method to solve the system*

$$\left\{\begin{array}{rcrcrcr} x & + & 2y & - & 3z & = & 1 \\ -2x & - & 4y & + & 6z & = & -2 \\ -x & + & y & + & 5z & = & 5 \end{array}\right\}$$

Solution:
Take the augmented matrix

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ -2 & -4 & 6 & -2 \\ -1 & 1 & 5 & 5 \end{array}\right] \xrightarrow{r_2 \leftarrow r_2 + 2r_1} \left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 5 & 5 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ -1 & 1 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_2 \leftarrow r_2 + r_1}$$

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & 3 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_2 \cdot \frac{1}{3}} \left[\begin{array}{rrr|r} 1 & 2 & -3 & 1 \\ 0 & 1 & \frac{2}{3} & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 - 2r_2} \left[\begin{array}{rrr|r} 1 & 0 & -\frac{13}{3} & -3 \\ 0 & 1 & \frac{2}{3} & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the equations are

$$\left\{\begin{array}{rcl} x - \frac{13}{3}z & = & -3 \\ y + \frac{2}{3}z & = & 2 \end{array}\right\}, \quad \text{i.e.,} \quad \left\{\begin{array}{rcl} x & = & -3 + \frac{13}{3}z \\ y & = & 2 - \frac{2}{3}z \end{array}\right\}$$

Hence the solutions are given by

$$\left(\frac{13}{3}z - 3, -\frac{2}{3}z + 2, z\right), \quad z \text{ any real number.}$$

■

Example 60 *Use the Gauss-Jordan method to solve the system*

$$\left\{\begin{array}{rcrcrcr} 3x & - & y & + & 5z & = & 4 \\ 2x & + & 5y & - & 3z & = & 9 \\ -4x & - & 10y & + & 6z & = & -7 \end{array}\right\}$$

Solution:
Take the augmented matrix

$$\left[\begin{array}{rrr|r} 3 & -1 & 5 & 4 \\ 2 & 5 & -3 & 9 \\ -4 & -10 & 6 & -7 \end{array}\right] \xrightarrow{r_1 \cdot \frac{1}{3}} \left[\begin{array}{rrr|r} 1 & -\frac{1}{3} & \frac{5}{3} & \frac{4}{3} \\ 2 & 5 & -3 & 9 \\ -4 & -10 & 6 & -7 \end{array}\right] \xrightarrow{r_2 \leftarrow r_2 - 2r_1}$$

$$\left[\begin{array}{rrr|r} 1 & -\frac{1}{3} & \frac{5}{3} & \frac{4}{3} \\ 0 & \frac{17}{3} & -\frac{19}{3} & \frac{19}{3} \\ -4 & -10 & 6 & -7 \end{array}\right] \xrightarrow{r_3 \leftarrow r_3 + 4r_1} \left[\begin{array}{rrr|r} 1 & -\frac{1}{3} & \frac{5}{3} & \frac{4}{3} \\ 0 & \frac{17}{3} & -\frac{19}{3} & \frac{19}{3} \\ 0 & -\frac{34}{3} & \frac{38}{3} & -\frac{5}{3} \end{array}\right] \xrightarrow{r_2 \cdot \frac{3}{17}}$$

$$\left[\begin{array}{rrr|r} 1 & -\frac{1}{3} & \frac{5}{3} & \frac{4}{3} \\ 0 & 1 & -\frac{19}{17} & \frac{19}{17} \\ 0 & -\frac{34}{3} & \frac{38}{3} & -\frac{5}{3} \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 + \frac{1}{3}r_2} \left[\begin{array}{rrr|r} 1 & 0 & \frac{66}{51} & \frac{87}{51} \\ 0 & 1 & -\frac{19}{17} & \frac{19}{17} \\ 0 & -\frac{34}{3} & \frac{38}{3} & -\frac{5}{3} \end{array}\right] \xrightarrow{r_3 \leftarrow r_3 + \frac{34}{3}r_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{66}{51} & \frac{87}{51} \\ 0 & 1 & -\frac{19}{17} & \frac{19}{17} \\ 0 & 0 & 0 & \frac{561}{51} \end{array}\right].$$

The last equation gives $0 = \frac{561}{51}$, which shows that we have an inconsistent system with no solutions. ■

Now try to use the Gauss-Jordan method to solve the following exercises to make sure that you have a good understanding of the method.

Exercise 62 *Use the Gauss-Jordan method to solve the following systems*

1.

$$\left\{\begin{array}{rcrcrcr} x & - & 2y & + & 4z & = & 6 \\ x & + & y & + & 13z & = & 6 \\ -2x & + & 6y & - & z & = & -10 \end{array}\right\} \text{ and } \left\{\begin{array}{rcrcrcr} x & - & y & + & 5z & = & -6 \\ 3x & + & 3y & - & z & = & 10 \\ x & + & 3y & + & 2z & = & -5 \end{array}\right\}$$

2.

$$\left\{\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 0 \\ 3x & - & y & + & z & = & 6 \end{array}\right\} \text{ and } \left\{\begin{array}{rcrcrcr} x & + & 2y & + & z & = & 5 \\ 2x & + & y & - & 3z & = & -2 \\ 3x & + & y & + & 4z & = & -5 \end{array}\right\}$$

3.

$$\left\{\begin{array}{rcrcrcr} 3x & + & 5y & - & z & = & 0 \\ 4x & - & y & + & 2z & = & 1 \\ -6x & - & 10y & + & 2z & = & 0 \end{array}\right\} \text{ and } \left\{\begin{array}{rcrcrcr} 3x & - & 2y & + & z & = & 6 \\ 3x & + & y & - & z & = & -4 \\ -x & + & 2y & - & 2z & = & -8 \end{array}\right\}$$

4.

$$\left\{\begin{array}{rcrcrcr} 2x & + & 3y & + & z & = & 9 \\ 4x & + & y & - & 3z & = & -7 \\ 6x & + & 2y & - & 4z & = & -8 \end{array}\right\} \text{ and } \left\{\begin{array}{rcrcrcr} 4x & + & 3y & + & z & = & 1 \\ -2x & - & y & + & 2z & = & 0 \end{array}\right\}$$

2.4 The Determinant Method

In Lecture 8, the operation of multiplication of an $n \times m$ matrix by an $m \times k$ matrix was defined. It was pointed out that the square matrix $\mathbf{I}_n$ of dimension n, all of whose diagonal entries are equal to 1 and all of whose other entries are 0, has the property that, for every square matrix A of dimension n,

$$A \cdot \mathbf{I}_n = \mathbf{I}_n \cdot A = A.$$

I.e., $\mathbf{I}_n$ plays in the matrix algebra a similar role to the role of the unit element 1 in the algebra of numbers. It is, therefore natural to ask whether matrices have **inverses**. That is, given an $n \times m$ matrix A, does there exist a matrix B, such that

$$A \cdot B = B \cdot A = \mathbf{I}_n?$$

Example 61 *Show that if a matrix A has an inverse B as shown above, then A must necessarily be a square matrix of dimension n and the same is then true for B.*

Solution:
Suppose that A is an $n \times m$ matrix. Then, since the product $A \cdot B$ is defined, B has to be an $m \times k$ matrix, for some k. But the product $B \cdot A$ is also defined, which shows that $k = n$. Hence A is $n \times m$ and B is $m \times n$. But note that in this case $A \cdot B$ would be $n \times n$ whereas $B \cdot A$ would be $m \times m$. Since both are equal to $\mathbf{I}_n$ by assumption, we must have $m = n$. Thus, finally, both A and B have to be square matrices of dimension n. ■

Being a square matrix of dimension n is not enough to have an inverse. Some square matrices have inverses and others do not. Those that do have inverses are said to be **invertible** and those that do not are said to be **singular**.

Example 62 *Show that, if the $n \times n$ matrix A is invertible, then its inverse B is unique.*

Solution:
Suppose that A has two inverses B and C. Then

$$A \cdot B = B \cdot A = \mathbf{I}_n, \quad \text{and} \quad A \cdot C = C \cdot A = \mathbf{I}_n.$$

But in this case

$$\begin{aligned} B &= B \cdot \mathbf{I}_n \\ &= B \cdot (A \cdot C) \\ &= (B \cdot A) \cdot C \\ &= \mathbf{I}_n \cdot C \\ &= C. \end{aligned}$$

Hence $B = C$ and the inverse of A is unique. ■

The unique inverse of A, if it exists, is denoted (by analogy with the inverse of a number) by A^{-1}.

For instance the matrix $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ has as inverse the matrix $B = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$. To check this, note that

$$A \cdot B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} + \frac{2}{5} & \frac{1}{5} - \frac{1}{5} \\ \frac{6}{5} - \frac{6}{5} & \frac{2}{5} + \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2$$

and similarly

$$B \cdot A = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} + \frac{2}{5} & -\frac{3}{5} + \frac{3}{5} \\ -\frac{2}{5} + \frac{2}{5} & \frac{2}{5} + \frac{3}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2$$

Thus we may write $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$

Exercise 63 *Show that the inverse of A^{-1} is A, i.e., $(A^{-1})^{-1} = A$.*

Example 63 *Show that if a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible, then $ad - bc \neq 0$ and $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$*

Solution:

Suppose that A is invertible with inverse matrix $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Then $A \cdot B = \mathbf{I}_2$, which is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This gives

$$\begin{bmatrix} ax + bz & ay + bw \\ cx + dz & cy + dw \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus we must have

$$\left\{ \begin{array}{rcl} ax + bz & = & 1 \\ cx + dz & = & 0 \end{array} \right\}, \quad \text{and} \quad \left\{ \begin{array}{rcl} ay + bw & = & 0 \\ cy + dw & = & 1 \end{array} \right\}.$$

From the first equation of the system in the right, we conclude that not both of a and b can be 0. Suppose that $a \neq 0$. Then solving that equation for y we get $y = -\frac{bw}{a}$. Now substituting this value for y in the second equation of the same system yields $-\frac{bcw}{a} + dw = 1$, i.e., $-bcw + adw = a$, whence $w(-bc + ad) = a$. Since $a \neq 0$, we have $ad - bc \neq 0$ and, furthermore $w = \frac{a}{ad-bc}$. Similarly, one can show that $x = \frac{d}{ad-bc}$, $y = -\frac{b}{ad-bc}$ and $z = -\frac{c}{ad-bc}$. Therefore, the inverse B of A is

$$B = \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

One may handle similarly the case where $b \neq 0$. ■

Now try to prove the easier converse statement.

Exercise 64 *Show that if* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *and* $ad - bc \neq 0$*, then the matrix* $B = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. *is the inverse matrix of* A.

By the previous Example and Exercise, to check whether a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible or not, it suffices to compute the quantity $D = ad - bc$ and check whether it is zero or not. This quantity is called the **determinant** of the matrix A. It is denoted $|A|$ or sometimes $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. So, if the determinant is zero the matrix is singular, whereas, if the determinant is not zero, then, A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Example 64 *Is the matrix* $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$ *invertible? If yes, compute its inverse. Do the same for* $B = \begin{bmatrix} 2 & 3 \\ -1 & -\frac{3}{2} \end{bmatrix}$.

Solution:
We compute the determinant of A:

$$D = ad - bc = 1 \cdot 1 - (-1) \cdot 2 = 1 + 2 = 3 \neq 0.$$

Hence A is invertible and

$$A^{-1} = \frac{1}{D}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix}.$$

For B we have

$$D = ad - bc = 2 \cdot (-\frac{3}{2}) - 3 \cdot (-1) = -3 + 3 = 0.$$

Hence B is singular. ■

Exercise 65 *Find the inverses* A^{-1} *and* B^{-1} *if they exist, where*

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}.$$

Exercise 66 *Find the inverses* A^{-1} *and* B^{-1} *if they exist, where*

$$A = \begin{bmatrix} 3 & -5 \\ 5 & 3 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} -2 & 1 \\ 6 & -4 \end{bmatrix}.$$

The reason why determinants are so important in linear algebra is that they are very helpful in solving systems of linear equations. The two methods that are presented below both use determinants.

First, we present two examples, one for each method:

Example 65 *Solve the system of equations*

$$\left\{ \begin{array}{rcl} 2x + y & = & 1 \\ -x + y & = & -2 \end{array} \right\}.$$

Solution:

Notice that the given system may be written in the form

$$\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}. \tag{2.1}$$

Check whether the matrix of the coefficients $A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ is invertible: $D = ad - bc = 2 \cdot 1 - 1 \cdot (-1) = 2 + 1 = 3 \neq 0$. Now determine its inverse

$$A^{-1} = \frac{1}{D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

Now take equation (2.1) and multiply both sides on the left by A^{-1}. We have

$$A^{-1} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

which gives (since $A^{-1} \cdot A = \mathbf{I}_2$)

$$\mathbf{I}_2 \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

and, since multiplying any matrix by the unit matrix gives the matrix itself,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

i.e.,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \cdot 1 + (-\frac{1}{3}) \cdot (-2) \\ \frac{1}{3} \cdot 1 + \frac{2}{3} \cdot (-2) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

whence $x = 1$ and $y = -1$. ∎

The method given in the preceding example is called the **Inverse Matrix Method**.

Exercise 67 *Check whether the matrix of coefficients of the system*

$$\left\{ \begin{array}{rcl} 5x - y & = & 5 \\ -3x + 2y & = & 4 \end{array} \right\}$$

is invertible and, if yes, then use the inverse matrix method to solve the system.

Exercise 68 *Check whether the matrix of coefficients of the system*

$$\left\{ \begin{array}{lcl} -x+4y & = & -2 \\ 3x-10y & = & 10 \end{array} \right\}$$

is invertible and, if yes, then use the inverse matrix method to solve the system.

The second method, called the **Determinant Method** or **Cramer's Method** works as follows:

Example 66 *Solve the system*

$$\left\{ \begin{array}{lcl} -3x-y & = & -1 \\ 2x-y & = & 9 \end{array} \right\}$$

Solution:
Check whether the determinant of the matrix of the coefficients is zero or not.

$$\left| \begin{array}{cc} -3 & -1 \\ 2 & -1 \end{array} \right| = (-3)\cdot(-1) - (-1)\cdot 2 = 3+2 = 5 \neq 0.$$

Now compute

$$x = \frac{\left| \begin{array}{cc} -1 & -1 \\ 9 & -1 \end{array} \right|}{\left| \begin{array}{cc} -3 & -1 \\ 2 & -1 \end{array} \right|}, \quad \text{and} \quad y = \frac{\left| \begin{array}{cc} -3 & -1 \\ 2 & 9 \end{array} \right|}{\left| \begin{array}{cc} -3 & -1 \\ 2 & -1 \end{array} \right|},$$

i.e., $x = \frac{10}{5} = 2$ and $y = \frac{-25}{5} = -5$. ■

Exercise 69 *Use Cramer's method to solve the system*

$$\left\{ \begin{array}{lcl} -x+2y & = & -2 \\ 3x-7y & = & 2 \end{array} \right\}$$

Exercise 70 *Use Cramer's method to solve the system*

$$\left\{ \begin{array}{lcl} -5x+2y & = & 2 \\ 8x-2y & = & 4 \end{array} \right\}$$

Chapter 3

Propositional Logic

3.1 Statements, Compound Statements, Truth and Falsity

A **statement** or **proposition** is a sentence that is either **true** or **false** but not both.

For instance "2+2 equals 4" is a statement since it is a sentence that is true, but "He is a college student" is *not* a statement because depending of who is meant by the "He" it may be a true or a false statement (but we cannot tell). Similarly, "$x + y > 0$" is *not* a statement (why?).

One uses small letters $p, q, r, \ldots$ or capital letters $P, Q, R, \ldots$ to denote statements.

Three symbols that are used to build more complicated expressions from simpler ones will now be introduced.

- The symbol $\neg$ is used to denote **not**,
- the symbol $\wedge$ is used to denote **and** and
- the symbol $\vee$ is used to denote **or**.

Thus, given a statement P, the sentence "$\neg P$" is read "not P", or "it is not the case that P" and it is called the **negation of** P. Given a second statement Q, the sentence "$P \wedge Q$" is read "P and Q" and is called the **conjunction of** P **and** Q. Finally, the sentence "$P \vee Q$" is read "P or Q" and is called the **disjunction of** P **and** Q.

Important Remark: In an expression like $\neg P \wedge Q$ the negation symbol assumes priority and then come the conjunction and disjunction symbols with equal priority. Thus $\neg P \wedge Q$ is the same as $(\neg P) \wedge Q$, but an expression like $P \wedge Q \vee R$ is ambiguous because, since $\wedge$ and $\vee$ have the same priority, it is not clear whether it means $(P \wedge Q) \vee R$ or $P \wedge (Q \vee R)$.

Example 67 *Let P = "It is hot" and Q = "It is sunny". Translate the following sentences in symbolic logical expressions.*

1. *It is not hot but it is sunny.*
2. *It is neither hot nor sunny.*

Solution:

1. "It is not hot but it is sunny" means really "it is not hot and it is sunny". Therefore, since "It is not hot" is $\neg P$ and "It is sunny" is Q, the given sentence can be written symbolically as $\neg P \wedge Q$.

2. "It is neither hot nor sunny" means "It is not hot and it is not sunny". Therefore, since "It is not hot" is $\neg P$ and "It is not sunny" is $\neg Q$, the given sentence may be written symbolically as $\neg P \wedge \neg Q$. ■

Exercise 71 *Let S = "stocks are increasing" and I = "interest rates are steady". Write the following statement in symbolic form:*

1. *Stocks are increasing but interest rates are steady.*

2. *Neither are stocks increasing nor are interest rates steady.*

Exercise 72 *Let M = "Kate is a math major" and*

$$C = \text{"Kate is a Computer Science major".}$$

Write in symbolic form the statement "Kate is a math major but not a computer science major".

If our compound sentences are to be statements, then they must always be either true or false. The truth value of the compound sentences may be determined by the truth values of the statements out of which they are composed by the following rules:

- If P is a statement variable, the **negation** of P, "not P", is denoted by $\neg P$ and has the opposite truth value from P: $\neg P$ is true if P is false and $\neg P$ is false if P is true. Summarizing as a **truth table**:

P	$\neg P$
F	T
T	F

- If P, Q are statement variables, the **conjunction of** P **and** Q, "P and Q", is denoted by $P \wedge Q$ and it is true only when both P and Q are true. Summarizing as a truth table:

P	Q	$P \wedge Q$
F	F	F
F	T	F
T	F	F
T	T	T

- If P, Q are statement variables, the **disjunction of** P **and** Q, "P or Q", is denoted by $P \vee Q$ and it is true if at least one of P and Q is true and false if both P and Q are false. Summarizing as a truth table:

P	Q	$P \vee Q$
F	F	F
F	T	T
T	F	T
T	T	T

Up to this point, only the expressions $\neg P, P \wedge Q$ and $P \vee Q$ have been assigned truth values. Out of the connectives not, and and or one can compose increasingly complex expressions such as $\neg P \vee Q, (P \vee Q) \wedge \neg(P \wedge Q), (P \wedge Q) \vee R$ and so on. Such expressions are called **statement forms** or **propositional forms**. More precisely, a **statement form** or **propositional form** is an expression made up of statement variables $P, Q, R, \ldots$ and logical connectives (such as $\neg, \wedge, \vee$) that becomes a statement when actual statements are substituted for the component statement variables. The **truth table** for a given statement form displays the truth values that correspond to the different combinations of truth values for the variables.

Example 68 *Construct the truth table for the connective* $\oplus$ *defined by*

$$P \oplus Q = (P \vee Q) \wedge \neg(P \wedge Q).$$

Solution:

P	Q	$P \vee Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$(P \vee Q) \wedge \neg(P \wedge Q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

■

$\oplus$ is called the "XOR" or **exclusive or** connective and its meaning in English is "P or Q but not both".

Exercise 73 *Construct the truth table for* $(P \wedge Q) \vee \neg R$.

Exercise 74 *Construct the truth table for* $(P \wedge Q) \vee \neg(P \vee Q)$.

3.2 Logical Equivalence, Tautologies and Contradictions

Consider the two compound statements $P \wedge Q$ and $Q \wedge P$. They have the following truth tables

P	Q	$P \wedge Q$	$Q \wedge P$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

We see that they are both true or both false at exactly the same combination of truth values for their variables. Such statement forms are said to be **logically equivalent**.

More precisely, two *statement forms* are said to be **logically equivalent** if, and only if, they have identical truth values for each possible substitution of statements for their statement variables. Logical equivalence of P and Q is denoted by $P \equiv Q$. Two *statements* are **logically equivalent** if, and only if, when the same statement variables are used to represent identical component statements, their forms are logically equivalent.

Testing for the logical equivalence of two statement forms P and Q:

1. Construct the truth table for P.
2. Construct the truth table for Q using the same statement variables for identical component statements.
3. Check each combination of truth values of the statement variables to see whether the truth value of P is the same as the truth value of Q.
   (a) If in each row the truth value of P is the same as the truth value of Q, then P and Q are logically equivalent.
   (b) If in some row P and Q assume different truth values, then P and Q are not logically equivalent.

Example 69 *Show that $\neg(\neg P) \equiv P$.*

Solution:
We have

P	$\neg P$	$\neg(\neg P)$
T	F	T
F	T	F

Thus, P and $\neg(\neg P)$ agree on all rows of their truth tables, and, therefore, they are logically equivalent. ■

Example 70 *Show that $(\neg(P \wedge Q)$ and $\neg P \wedge \neg Q$ are not logically equivalent.*

Solution:

Method 1: Construct the truth tables and show that there exists at least one line where the two statement forms assume different truth values.

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P \wedge \neg Q$
T	T	F	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	F	T	T

Since, at the second and third lines the truth tables differe $\neg(P \wedge Q) \not\equiv \neg P \wedge \neg Q$.

Method 2: Find an example to show that the two statement forms are not equivalent. To find an example, you must find two statements, such that when you substitute one for P and the other for Q in the two forms, you will obtain a new statement that is true in the first case and false in the second or vice-versa. To illustrate this, let

$$P = \text{"}0 < 1\text{"} \quad \text{and} \quad Q = \text{"}1 < 0\text{"}.$$

Then $\neg(P \wedge Q)$ is the statement "It is not both the case that $0 < 1$ and $1 < 0$" which is a true statement. On the other hand $\neg P \wedge \neg Q$ is the statement "$0 \not< 1$ and $1 \not< 0$", which is false. ■

Example 71 *Prove the* **First De Morgan's Law***:*

$$\neg(P \wedge Q) \equiv \neg P \vee \neg Q.$$

Solution:

We use the truth table method:

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

The entries in all rows of the columns corresponding to $\neg(P \wedge Q)$ and $\neg P \vee \neg Q$ are the same. Hence, $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$. ■

Exercise 75 *Determine whether* $(P \vee Q) \vee (P \wedge R) \equiv (P \vee Q) \wedge R$.

Exercise 76 *Determine whether* $((\neg P \vee Q) \wedge (P \vee \neg R)) \wedge (\neg P \vee \neg Q) \equiv \neg(P \vee R)$.

Example 72 *Use De Morgan's Laws to write the negations of the statements*

1. *John is six feet tall and he weighs at least 200 pounds.*

2. The bus was late or Tom's watch was slow.

Solution:

1. John is not six feet tall or he weighs less than 200 pounds.
2. The bus was not late and Tom's watch was not slow. ■

Exercise 77 *Use De Morgan's laws to write the negation of* $-1 < x \leq 4$.

Exercise 78 *Use De Morgan's Laws to write the negation of "Hal is a math major and Hal's sister is a computer science major".*

A **tautology** is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a tautology is called a **tautological statement**.

A **contradiction** is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a contradiction is called a **contradictory statement**.

Example 73 *Determine whether the statement form* $(P \wedge Q) \vee (\neg P \vee (P \wedge \neg Q))$ *is a tautology or a contradiction.*

Solution:
We construct the truth table

P	Q	$P \wedge Q$	$\neg Q$	$P \wedge \neg Q$	$\neg P$	$\neg P \vee (P \wedge \neg Q)$	$(P \wedge Q) \vee (\neg P \vee (P \wedge \neg Q))$
F	F	F	T	F	T	T	T
F	T	F	F	F	T	T	T
T	F	F	T	T	F	T	T
T	T	T	F	F	F	F	T

Thus, since for all truth values of the statement variables, $(P \wedge Q) \vee (\neg P \vee (P \wedge \neg Q))$ assumes the truth value T, it is a tautology. ■

Exercise 79 *Determine whether the statement form* $(P \wedge \neg Q) \wedge (\neg P \vee Q)$ *is a tautology or a contradiction.*

Exercise 80 *Determine whether the statement form* $((\neg P \wedge Q) \wedge (Q \wedge R)) \wedge \neg Q$ *is a tautology or a contradiction.*

3.3 Conditional Statements

Let P and Q be statements. A sentence of the form "If P then Q" is symbolically denoted as "$P \to Q$". P is the **hypothesis** and Q is the **conclusion**. The sentence "If 48 is divisible by 6, then 48 is divisible by 3" is an example of such a sentence. Such sentences are called **conditionals** because the truth of Q is conditioned on the truth of P.

$\to$ is a connective, like $\wedge$ and $\vee$, and it can be used to combine statements to create new ones. More formally, if P and Q are statement variables, the **conditional of Q by P** is read "If P then Q" or "P implies Q" and is denoted by $P \to Q$. It is false only if P is true and Q is false. It is true for all other combinations of truth values for the variables P and Q. Its truth table is therefore

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

A conditional statement whose hypothesis is false is said to be **vacuously true**. For instance the statement "If the Lakers win the game, they will be champions" is vacuously true if the Lakers do not win the game.

$\to$ has the lowest priority among the connectives. Thus, the statement form $P \vee \neg Q \to \neg P$ should be read as $(P \vee (\neg Q)) \to (\neg P)$.

Example 74 *Construct the truth table for the statement form $P \vee \neg Q \to \neg P$.*

Solution:

P	Q	$\neg P$	$\neg Q$	$P \vee \neg Q$	$P \vee \neg Q \to \neg P$
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

■

Exercise 81 *Construct the truth table for $\neg P \vee Q \to \neg Q$.*

Exercise 82 *Do the same for $P \wedge \neg Q \to R$.*

Exercise 83 *Do the same for $P \vee (\neg P \wedge Q) \to Q$.*

Example 75 *Show that $P \vee Q \to R \equiv (P \to R) \wedge (Q \to R)$.*

Solution:
We construct the truth tables:

P	Q	R	$P \vee Q$	$P \rightarrow R$	$Q \rightarrow R$	$P \vee Q \rightarrow R$	$(P \rightarrow R) \wedge (Q \rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Now the last two columns agree on all rows. Thus $P \vee Q \rightarrow R$ and $(P \rightarrow R) \wedge (Q \rightarrow R)$ are logically equivalent. ■

Exercise 84 *Use the Example to rewrite the statement " If $x > 2$ or $x < -2$, then $x^2 > 4$".*

Example 76 *Show that $P \rightarrow Q \equiv \neg P \vee Q$.*

Solution:
Construct the truth tables

P	Q	$\neg P$	$P \rightarrow Q$	$\neg P \vee Q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	F	T	T

Thus, since all entries in the last two columns are the same $P \rightarrow Q$ and $\neg P \vee Q$ are logically equivalent. ■

Example 77 *Rewrite the following statement in the if-then form: "Either the Red Wings win this game or they end the season last."*

Solution:
Let

$$\neg P = \text{"The Red Wings win this game"}$$

and

$$Q = \text{"They end the season last"}.$$

Then the given statement is the statement $\neg P \vee Q$. Now

$$P = \text{"The Red Wings do not win this game"}.$$

So the equivalent if-then version $P \rightarrow Q$ is "If the Red Wings do not win this game, then they end the season last". ■

Exercise 85 1. *Show that* $\neg(P \to Q) \equiv P \wedge \neg Q$.

2. *Use part 1 to write the negation of "If Sue is Dan's mother, then Ron is his cousin".*

3. *Do the same with "If I study hard, then I will get an A".*

Exercise 86 1. *Show that the following statement forms are all equivalent*

$$P \to Q \vee R, \quad P \wedge \neg Q \to R, \quad P \wedge \neg R \to Q.$$

2. *Use part 1 to write "If* n *is prime, then* n *is odd or* n *is 2" in two different ways.*

The **contrapositive** of a conditional "If P then Q" is the statement "If $\neg Q$ then $\neg P$". In symbols, the contrapositive of $P \to Q$ is $\neg Q \to \neg P$.

It is a fundamental law of logic that a conditional statement is logically equivalent to its contrapositive:

Example 78 *Show that* $P \to Q \equiv \neg Q \to \neg P$.

Solution:
Construct the truth tables:

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$\neg Q \to \neg P$
F	F	T	T	T	T
F	T	T	F	T	T
T	F	F	T	F	F
T	T	F	F	T	T

Thus $P \to Q$ and $\neg Q \to \neg P$ are logically equivalent. ∎

Exercise 87 *Write the contrapositive of "If today is Easter, then tomorrow is Monday".*

Exercise 88 *Use the contrapositive to rewrite the statement "If Terry is in good form, then he will play in the game".*

3.4 Converse, Inverse and Biconditionals

Suppose we are given the conditional statement "If P then Q".

- The **converse** of "if P then Q" is the statement "if Q then P".
- The **inverse** of "if P then Q" is the statement "if $\neg P$ then $\neg Q$".

In symbols we have

- **converse** of $P \to Q$ is $Q \to P$ and
- **inverse** of $P \to Q$ is $\neg P \to \neg Q$.

Example 79 *Write the inverse and converse statements of*

(a) "If the keeper saves this shot, then the team will win".

(b) "If Kate drops this class, she will not be full-time".

Solution:

(a) Let

$$P = \text{"the keeper saves this shot"}$$

and

$$Q = \text{"the team will win"}.$$

Then the given statement is the statement $P \to Q$. Its converse is the statement $Q \to P$, i.e., "if the team wins, then the keeper will save this shot". Its inverse is the statement $\neg P \to \neg Q$. But,

$$\neg P = \text{"the keeper does not save this shot"}$$

and

$$\neg Q = \text{"the team will not win"}.$$

Hence, the inverse statement is "if the keeper does not save this shot, then the team will not win".

(b) Let

$$P = \text{"Kate drops this class"}$$

and

$$Q = \text{"she will not be full-time"}.$$

Then the given statement is the statement $P \to Q$. Its converse is the statement $Q \to P$, i.e., "if Kate is not full-time then she will drop this class". Its inverse is the statement $\neg P \to \neg Q$. But,

$$\neg P = \text{"Kate does not drop this class"}$$

and

$$\neg Q = \text{“she will be full-time”}.$$

Hence, the inverse statement is “if Kate does not drop this class, then she will be full-time”. ■

Exercise 89 *Write the converse and the inverse of the statements*

1. *“If S is a square, then S is a rectangle”,*
2. *“If n is prime, then n is odd or n is 2”,*
3. *“If n is divisible by 6, then n is divisible by 2 and n is divisible by 3”.*

We can show that

- $P \to Q \not\equiv Q \to P$
- $P \to Q \not\equiv \neg P \to \neg Q$
- $Q \to P \equiv \neg P \to \neg Q$

In fact, these may be derived by the following truth table

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$Q \to P$	$\neg P \to \neg Q$
F	F	T	T	T	T	T
F	T	T	F	T	F	F
T	F	F	T	F	T	T
T	T	F	F	T	T	T

Note that the columns under $Q \to P$ and $\neg P \to \neg Q$ are identical but that they both differ from the column corresponding to $P \to Q$. So neither the converse nor the inverse of a statement is logically equivalent with the statement itself but they are equivalent with each other.

If P, Q are statements, then

“P only if Q” means “if not Q then not P” or, equivalently, “If P then Q”.

Example 80 *Use the contrapositive to write in two ways “Terry will play only if he is in top form”.*

Solution:
Let

$$P = \text{“Terry will play” and} \quad Q = \text{“he is in top form”}.$$

Then, the given statement is “P only if Q”. This is, by definition, equivalent to $P \to Q$ and, by the logical equivalence of a statement with its contrapositive, also with $\neg Q \to \neg P$. Hence it may be written equivalently in the following two forms

1. "If Terry plays then he is in top form" and

2. "If Terry is not in top form then he will not play".

■

Exercise 90 *Use the contrapositive to write in two ways "Don will be allowed in the boat only if he is an expert sailor".*

Given statement variables P and Q, the **biconditional of P and Q** is "P if, and only if, Q" and is denoted by $P \leftrightarrow Q$. It is true if both P and Q have the same truth values and false if they have opposite truth values. Sometimes instead of "if, and only if," the abbreviation **"iff"** is used. In summary, the truth table for $\leftrightarrow$ is

P	Q	$P \leftrightarrow Q$
F	F	T
F	T	F
T	F	F
T	T	T

The connective $\leftrightarrow$ has the same priority with $\rightarrow$. Thus, in complicated statement forms containing several connectives, the following table gives the order of strength in decreasing order

order	connectives
1.	$\neg$
2.	$\wedge, \vee$
3.	$\rightarrow, \leftrightarrow$

It is not a coincidence that the biconditional is called "if, and only if,". The following truth table shows that

$$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P).$$

The first of these is "P only if Q" and the second is "P if Q". Hence, "P if, and only if, Q" turns out to be logically equivalent to "P only if Q and P if Q".

P	Q	$P \rightarrow Q$	$Q \rightarrow P$	$P \leftrightarrow Q$	$(P \rightarrow Q) \wedge (Q \rightarrow P)$
F	F	T	T	T	T
F	T	T	F	F	F
T	F	F	T	F	F
T	T	T	T	T	T

Exercise 91 *Prove that* $(P \leftrightarrow Q) \leftrightarrow R \not\equiv (P \leftrightarrow Q) \wedge (Q \leftrightarrow R)$.

Exercise 92 *Write the following statements as conjunctions of two if-then statements*

1. "$x^3 = 1$ *iff* $x = 3$"

2. *"$|x| = x$ iff $x \geq 0$".*

If P and Q are statements, then

- P is a **sufficient condition** for Q means "If P then Q".
- P is a **necessary condition** for Q means "If not P then not Q".

Since $\neg P \to \neg Q \equiv Q \to P$,

P is a **necessary condition** for Q also means "if Q then P".

Therefore

P is a **necessary and sufficient condition** for Q means "P if, and only if, Q".

Example 81 *Write the following statements in the if-then form*

1. *"Mike's studying is a necessary condition for his passing this class".*
2. *"Ryan's being a student is a sufficient condition for using the computer labs on campus".*

Solution:

1. Since "P is a necessary condition for Q" means "if Q then P", the given statement is equivalent to "if Mike studies, then he will pass this class".
2. Since "P is a sufficient condition for Q" means "if P then Q", the given statement is equivalent to "if Ryan is a student, then he may use the computer labs on campus".

■

Exercise 93 *Rewrite the following statements in if-thenform.*

1. *"Starting from home at 8:00am is a sufficient condition for my being at work on time".*
2. *"Having two equal angles is a sufficient condition for a triangle to be isosceles".*
3. *"Being divisible by 3 is a necessary condition for a number to be divisible by 9".*
4. *"The temperature being at or under 32 is a necessary condition for water to turn into ice".*

3.5 Valid Arguments

An **argument** is a sequence of statements. All statements except the final one are called **premises** or **assumptions** or **hypotheses**. The final statement is called the **conclusion**. One uses the symbol $\therefore$, read "therefore", or, sometimes a horizontal line, also read "therefore", to separate the premises from the conclusion.

For instance

$$\begin{array}{l} \text{If Socrates is a human being, then Socrates is mortal} \\ \text{Socrates is a human being} \\ \hline \text{Socrates is mortal} \end{array}$$

is an argument. It has the abstract form

$$\begin{array}{l} \text{If } P \text{ then } Q \\ P \\ \hline Q \end{array} \qquad \text{or also} \qquad \begin{array}{rl} & \text{If } P \text{ then } Q \\ & P \\ \therefore & Q \end{array}$$

An **argument form** is a sequence of sentence forms that yield an argument when all statement variables in the sentence forms are substituted for by actual statements. An argument form is **valid** if no matter what particular statements are substituted for the statement variables in its premises, if the resulting premises are all true, then its conclusion is also true. An argument is **valid** if its argument form is valid.

When an argument is valid and its premises are true, the truth of the conclusion is said to be **inferred** or **deduced** from the truth of the premises.

Test an argument form for validity:

1. Identify the premises and the conclusion.
2. Construct a truth table for all the premises and the conclusion.
3. Find the rows, called the **critical rows** in which all the premises are true.
4. In each critical row determine whether the conclusion of the argument is true.
   (a) If it is, then the argument form is valid.
   (b) If not, the argument is invalid.

Example 82 *Show that the argument form*

$$\begin{array}{l} P \vee (Q \vee R), \neg R \\ \hline P \vee Q \end{array}$$

is valid.

Solution:
We construct the truth table for the premises $P \vee (Q \vee R)$ and $\neg R$ and the conclusion $P \vee Q$.

P	Q	R	$Q \vee R$	$P \vee (Q \vee R)$	$\neg R$	$P \vee Q$	
T	T	T	T	T	F	T	
T	T	F	T	**T**	**T**	T	critical
T	F	T	T	T	F	T	
T	F	F	F	**T**	**T**	T	critical
F	T	T	T	T	F	T	
F	T	F	T	**T**	**T**	T	critical
F	F	T	T	T	F	F	
F	F	F	F	F	T	F	

The second, fourth and sixth rows in the truth table are critical because both assumptions are true. Since in these three rows the conclusion is also true, the given argument form is valid. ■

Example 83 *Show that the argument form*

$$\frac{P \to Q \vee \neg R, Q \to P \wedge R}{P \to R}$$

is invalid.

Solution:
Construct the truth table for the premises and the conclusion

P	Q	R	$\neg R$	$Q \vee \neg R$	$P \wedge R$	$P \to Q \vee \neg R$	$Q \to P \wedge R$	$P \to R$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	
T	F	T	F	F	T	F	T	
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	
F	T	F	T	T	F	T	F	
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

Only the truth values of the conclusion at the critical rows have been computed. Since in one of these (fourth row) the conclusion is false, the argument is invalid. ■

Exercise 94 *Determine whether the following argument forms are valid*

1. $P \to Q, Q \to P \therefore P \vee Q.$

2. $P, P \to Q, \neg Q \vee R \therefore R.$

3. $P \vee Q, P \rightarrow \neg Q, P \rightarrow R \therefore R.$

4. $P \rightarrow Q, P \rightarrow R \therefore P \rightarrow Q \wedge R.$

In what follows, several special valid argument forms that are used very often for deductions are presented.

The argument form

$$\begin{array}{l} \text{If } P \text{ then } Q \\ P \\ \hline Q \end{array}$$

is called **modus ponens**, which is the Latin for "method of affirming". The following truth table shows that it is a valid argument form:

P	Q	$P \rightarrow Q$	P	Q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

Only the first row is critical and in that row the conclusion is true. Hence modus ponens is a valid argument form.

The argument form

$$\begin{array}{l} \text{If } P \text{ then } Q \\ \neg Q \\ \hline \neg P \end{array}$$

is called **modus tollens**, which is the Latin for "method of denying". The following truth table shows that it is a valid argument form:

P	Q	$P \rightarrow Q$	$\neg Q$	$\neg P$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Only the last row is critical and in that row the conclusion is true. Hence modus tollens is a valid argument form.

Example 84 *Use modus ponens or modus tollens to fill in the blank rows so that the following arguments become valid deductions.*

1.

If there are more balls than boxes, then two balls are in the same box.
There are more balls than boxes.

$\therefore$ ______________________________

2.

If this number is divisible by 6, then it is divisible by 2.
This number is not divisible by 2.
$\therefore$ ______________________________

Solution:

1. Two balls are in the same box.

2. This number is not divisible by 6.

■

Exercise 95 (Disjunctive Addition) *1. Show that the following argument forms are valid*

$$\frac{P}{P \vee Q} \quad \text{and} \quad \frac{Q}{P \vee Q}$$

2. Suppose you are compiling a list of all freshmen and sophomores in your class. Explain how the argument forms above allow you to add someone who is a freshman to your list.

Exercise 96 (Conjunctive Simplification) *1. Show that the following argument forms are valid*

$$\frac{\begin{array}{l} P \vee Q \\ \neg Q \end{array}}{P} \quad \text{and} \quad \frac{\begin{array}{l} P \vee Q \\ \neg P \end{array}}{Q}$$

2. Construct an example to show how these argument forms may be used for deductions.

Exercise 97 (Hypothetical Syllogism) *1. Show that the following argument form is valid*

$$\frac{\begin{array}{l} P \to Q \\ Q \to R \end{array}}{P \to R}$$

2. Construct an example to show how this argument form may be used for deductions.

Exercise 98 (Dilemma: Proof by Division into Cases) *1. Show that the following argument form is valid*

$$\frac{\begin{array}{l} P \vee Q \\ P \to R \\ Q \to R \end{array}}{R}$$

2. Show how this argument form may be used to deduce from the premise that every real number is either negative or nonnegative the conclusion $|x|^2 = x^2$.

3.6 Fallacies

A **fallacy** is an error in reasoning that results in an invalid argument.

Example 85 (Converse Error) *Show that the following argument is invalid*

If George is a cheater, then George sits in the back row.
George sits in the back row.

George is a cheater.

Solution:
The general form of the above argument is

$$\begin{array}{l} P \to Q \\ Q \\ \hline P \end{array}$$

The truth table exploring the validity of that form is

P	Q	$P \to Q$	Q	P
F	F	T	F	F
F	T	T	T	F
T	F	F	F	T
T	T	T	T	T

From the table we see that row 2 is a critical row (both premises are true) in which the conclusion is false. Therefore, the given argument form is invalid. ■

Example 86 (Inverse Error) *Show that the following argument is invalid*

If interest rates are going up, then stock market prices will go down.
Interest rates are not going up.

Stock market prices will not go down.

Solution:
The general form of the above argument is

$$\begin{array}{l} P \to Q \\ \neg P \\ \hline \neg Q \end{array}$$

The truth table exploring the validity of that form is

P	Q	$P \to Q$	$\neg P$	$\neg Q$
F	F	T	T	T
F	T	T	T	F
T	F	F	F	T
T	T	T	F	F

From the table we see that row 2 is a critical row (both premises are true) in which the conclusion is false. Therefore, the given argument form is invalid. ■

Exercise 99 *Use symbols to write the logical form of each argument and then use the truth table to test the argument for validity*

1.

If Tom is not on team A, then Evan is on team B.
If Evan is not on team B, then Tom is on team A.
———————————————
Tom is not on team A or Evan is not on team B.

2.

Dave is a math major or Dave is an economics major.
If Dave is a math major, then Dave is required to take Math 341.
———————————————
Dave is an economics major or Dave is not required to take Math 341.

Exercise 100 *Some of the following arguments are valid and some exhibit the converse or the inverse error. Use symbols to write the logical form of each argument. If the argument is valid, identify which valid argument form guarantees its validity. Otherwise state whether the converse or the inverse error has been made.*

1.

If Jim solved this problem correctly, then Jim got the answer 2.
Jim obtained the answer 2.
———————————————
Jim solved the problem correctly.

2.

This real number is rational or it is irrational.
This real number is not rational.
———————————————
This real number is irrational.

3.

If I go to the movies, I will not finish my homework.
If I don't finish my homework, I won't do well on the exam tomorrow.
———————————————
If I go to the movies, I won't do well on the exam tomorrow.

Sometimes **people mix up the ideas of validity and truth**. If an argument seems valid, they accept the conclusion as true. And if an argument seems fishy (really a slang expression for invalid), they think the conclusion must be false. This is **not** correct.

For instance consider the argument

If John Lennon was a rock star, then John Lennon had red hair.
John Lennon was a rock star.
———————————————
John Lennon had red hair.

This argument is a valid argument by modus ponens. But one of its premises is false and so is its conclusion.

Also consider the argument

If Detroit is a big city, then Detroit has big buildings.
Detroit has tall buildings.
Detroit is a big city.

This argument is invalid by the converse error but it has a true conclusion.

Exercise 101 *1. Give an example of a valid argument with a false conclusion.*

2. Give an example of an invalid argument with a true conclusion.

Example 87 (Contradiction Rule) *Show that the following argument form is valid*

$$\frac{\neg P \to C \quad \textit{where } C \textit{ is a contradiction}}{P}$$

Solution:

The following truth table shows the validity of the argument:

P	C	$\neg P$	$\neg P \to C$	P
T	F	F	T	T
F	F	T	F	F

Only the first row is a critical row and in that row the conclusion is also true. ■

Study the following examples and exercises to get an idea on how you would tackle more complex situations based on the argument forms that have been presented so far.

Example 88 (How to use logic to find your glasses) *You are about to leave for school one morning and discover that you don't have your glasses. You know that the following statements are true:*

(a) If my glasses are on the kitchen table then I saw them at breakfast.

(b) I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.

(c) If I was reading the newspaper in the living room, then my glasses are on the coffee table.

(d) I did not see my glasses at breakfast.

(e) If I was reading my book in bed, then my glasses are on the bed table.

(f) If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

Where are the glasses?

Solution:
Here is the reasoning that leads via valid arguments to the conclusion that the glasses are on the coffee table:

1. The glasses are not on the kitchen table, by (a),(d) and modus tollens.
2. I did not read the newspaper in the kitchen, by (f),1 and modus tollens.
3. I read the newspaper in the living room by (b),2 and disjunctive syllogism.
4. My glasses are on the coffee table by (c),3 and modus ponens.

■

Example 89 (Knights and Knaves or Hints for travelling abroad.) *You find yourself on an island containing two types of people: knights who always tell the truth and knaves who always lie. You visit the island and are approached by two natives who speak to you as follows:*

A says: B is a knight.
B says: A and I are of opposite type.

What are A and B?

Solution:
Here is the reasoning that leads via valid arguments to the conclusion that both A and B are knaves:

1. Suppose A is a night.
2. What A says is true by the definition of a knight.
3. B is a night too by 2.
4. What B says is also true, by the definition of a knight.
5. A and B are of opposite types by 4.
6. 1 and 3 and 5 are contradictory. So the assumption 1 is false by the contradiction rule.
7. A is not a knight by 6.
8. A is a knave by 7 and disjunctive syllogism.
9. What A says is false, by the definition of a knave.
10. B is not a night by 9.

11. B is a knave also by disjunctive syllogism.

■

Exercise 102 (How to find a hidden treasure) *You find a note written by a pirate containing five true statements about a hidden treasure.*

(a) If this house is next to a lake, then the treasure is not in the kitchen.

(b) If the tree in the front yard is an elm, then the treasure is in the kitchen.

(c) This house is next to a lake.

(d) The tree in the front yard is an elm or the treasure is buried under the flagpole.

(e) If the tree in the backyard is an oak, then the treasure is in the garage.

Where is the hidden treasure?

Exercise 103 (More hints for travelling abroad.) *You find yourself on an island containing two types of people: knights who always tell the truth and knaves who always lie. You visit the island and are approached by natives who speak to you as follows:*

1.

A says: Both of us are knights.
B says: is a knave.

What are A and B?

2.

C says: Both of us are knaves.
D remains silent.

What are C and D?

3.

E says: F is a knave.
F says: E is a knave.

What are E and F?

4.

U says: None of us is a knight.
V says: At least three of us are knights.
W says: At most three of us are knights.
X says: Exactly five of us are knights.
Y says: Exactly two of us are knights.
Z says: Exactly one of us is a knight.

Which are knights and which are knaves?

Exercise 104 (How to solve crime mysteries) *A detective who is trying to solve a murder mystery determined that*

- *(a) Lord Hazelton, the murdered man, was killed by a blow on the head with a brass candlestick.*
- *(b) Either Lady Hazelton or a maid, Sara, was in the dining room at the time of the murder.*
- *(c) If the cook was in the kitchen at the time of the murder, then the butler killed Lord Hazelton with a fatal dose of strychnine.*
- *(d) if Lady Hazelton was in the dining room at the time of the murder, then the chauffeur killed Lord Hazelton.*
- *(e) If the cook was not in the kitchen at the time of the murder, then Sara was not in the dining room when the murder was committed.*
- *(f) If Sara was in the dining room at the time when the murder was committed, then the wine steward killed Lord Hazelton.*

Assuming only one cause of death, is it possible for the detective to deduce the identity of the murderer from the above facts? If so, who did kill Lord Hazelton?

Chapter 4

Sets

4.1 Basic Definitions and Examples

A **set** is a collection of objects, called its **elements** or **members**. For instance $\{1, 2, 3\}$ is the set containing the elements 1,2 and 3. $\{1, 2, 3, \ldots\}$ denotes the set containing all positive integers.

The **axiom of extension** says that **a set is completely determined by its elements**. This means that, when writing a set, repeating elements or changing the order of elements is irrelevant. The only property that matters is membership of an element in the set. For instance, $\{1, 2, 3, 3\}, \{1, 1, 3, 2\}$ and $\{2, 1, 3\}$ all denote the same set, the set that contains the three elements 1,2 and 3.

Sets are usually denoted by capital letters, such as A, B, C, S, X, Y etc. If x is an element of the set A, then we write $x \in A$, read x "is in" A or x "is an element of" A or x "belongs to" A. To indicate that x is not an element of a set A, we write $x \notin A$.

Sets may be themselves elements of other sets. For instance, the set $\{1, \{1\}\}$ has two elements. It contains the number 1 and it also contains the set that contains the number 1. Do not confuse 1 with $\{1\}$. They are two different mathematical entities.

Example 90 *Which of the following sets are equal?*

$$\{a, b, c, d\}, \{d, e, a, c\}, \{d, b, a, c\}, \{a, a, d, e, c, e\}$$

Solution:

The first and third sets are the same because they both contain the elements a, b, c and d. The difference is just in the order in which they are listed and it is irrelevant by the axiom of extension.

Similarly, the second and fourth sets are the same since they both contain exactly the same elements a, c, d, e. The order in which these elements are listed is different in the two sets and there are repetitions in the fourth set, but these are irrelevant. Only membership matters by the axiom of extension. ■

Given a set S and a property P that elements of S may or may not satisfy, one may construct a new set by aggregating together the elements of S that have the property P. This new set is denoted by

$$\{x \in S : P(x)\}$$

and it is read "the set of all $x \in S$, such that $P(x)$ is true". Before seeing some more examples, let us agree to denote by $\mathbf{R}$ the set of all real numbers, by $\mathbf{Z}$ the set of all integers and by $\mathbf{N}$ the set of all nonnegative integers, or natural numbers. Then

$$\{x \in \mathbf{R} : 0 < x \leq 2\}$$

denotes the set of all real numbers that are positive and less than or equal to 2.

Exercise 105 *Which of the following sets are equal?*

(a) $A = \{0, 1, 2\}$

(b) $B = \{x \in \mathbf{R} : -1 \leq x < 3\}$

(c) $C = \{x \in \mathbf{R} : -1 < x < 3\}$

(d) $D = \{x \in \mathbf{Z} : -1 < x < 3\}$

(e) $E = \{x \in \mathbf{N} : -1 < x < 3\}$

Example 91 *Describe the set* $S = \{n \in \mathbf{Z} : n = (-1)^k,\ \textit{for some integer } k\}$.

Solution:

As k ranges over the set of all integers, $(-1)^k$ only takes the values -1 and 1. Hence $S = \{-1, 1\}$. ■

Exercise 106 *Describe the set* $T = \{m \in \mathbf{Z} : m = 1 + (-1)^i,\ \textit{for some integer } i\}$.

If A, B are sets, A is called a **subset** of B, written $A \subseteq B$, if, and only if, every element of A is also an element of B. The phrases A "is contained in" B and B "contains" A are alternative ways of saying that A is a subset of B. In this case B is also said to be a **superset** of A and one also writes $B \supseteq A$. For instance, $\{1, 2, 3\} \subseteq \{1, 2, 3, 4, 5\}$ because every element in the first set is also an element in the second, and $\{1, 2, 3\} \subseteq \mathbf{N}$ since $1, 2$ and 3 are all contained in $\mathbf{N}$.

If A is not a subset of B, then one writes $A \not\subseteq B$. This is the case if there is at least one element in A that is not an element of B. For instance $\{-3, 1, 2, 5\} \not\subseteq \mathbf{N}$ since $-3 \in \{-3, 1, 2, 5\}$ but $-3 \notin \mathbf{N}$.

Example 92 *Show that, for every set* A, $A \subseteq A$.

Solution:

Every element in A is also an element in A. Therefore, by the definition $A \subseteq A$. ■

Let A and B be sets. A is said to be a **proper subset** of B, denoted $A \subset B$, if, and only if, every element of A is an element of B but there is at least one element of B that is not in A. For instance $\{1,2,3\} \subseteq \{1,2,3\}$, but $\{1,2,3\} \not\subset \{1,2,3\}$ because every element in $\{1,2,3\}$ is also an element in $\{1,2,3\}$ but there exists no element in $\{1,2,3\}$ that is not an element in $\{1,2,3\}$. On the other hand $\mathbf{N} \subset \mathbf{Z}$. Can you explain this?

Example 93 *Let $A = \{c,d,f,g\}, B = \{f,j\}$ and $C = \{d,g\}$. Answer each of the following questions explaining your answers.*

(a) Is $B \subseteq A$?

(b) Is $C \subseteq A$?

(c) Is $C \subseteq C$?

(d) Is $C \subset A$?

Solution:

(a) $B \not\subseteq A$ because $j \in B$ but $j \notin A$.

(b) $C \subseteq A$ because every element in C is also an element of A.

(c) $C \subseteq C$ because every element of C is also an element of C.

(d) $C \subset A$ because every element of C is an element of A and, there is an element of A, namely c, that is not an element of C. ■

Example 94 *Which of the following are true statements?*
(a) $2 \in \{1,2,3\}$ *(b)* $\{2\} \in \{1,2,3\}$ *(c)* $2 \subseteq \{1,2,3\}$
(d) $\{2\} \subseteq \{1,2,3\}$ *(e)* $\{2\} \subseteq \{\{1\},\{2\}\}$ *(f)* $\{2\} \in \{\{1\},\{2\}\}$.

Solution:

(a) $2 \in \{1,2,3\}$ is a true statement since 2 is an element of the set $\{1,2,3\}$.

(b) $\{2\} \in \{1,2,3\}$ is not a true statement, since $\{1,2,3\}$ does not contain the set containing 2.

(c) $2 \subseteq \{1,2,3\}$ is not true because 2 is not even a set.

(d) $\{2\} \subseteq \{1,2,3\}$ is true because every element of $\{2\}$ is also an element of $\{1,2,3\}$.

(e) $\{2\} \subseteq \{\{1\},\{2\}\}$ is not a true statement because $2 \in \{2\}$ but $2 \notin \{\{1\},\{2\}\}$.

(f) $\{2\} \in \{\{1\},\{2\}\}$ is a true statement because $\{2\}$ is an element of $\{1\},\{2\}\}$. ■

Exercise 107 *Answer each of the following questions explaining your answers:*
(a) Is $3 \in \{1, 2, 3\}$*?* *(b) Is* $1 \subseteq \{1\}$*?* *(c) Is* $\{2\} \in \{1, 2\}$*?* *(d) Is* $\{3\} \in \{1, \{2\}, \{3\}\}$*?*
(e) Is $1 \in \{1\}$*?* *(f) If* $\{2\} \subseteq \{1, \{2\}, \{3\}\}$*?* *(g) Is* $\{1\} \subseteq \{1, 2\}$*?* *(h) Is* $1 \in \{\{1\}, 2\}$*?*
(i) Is $\{1\} \subseteq \{1, \{2\}\}$*?* *(j) Is* $\{1\} \subseteq \{1\}$*?*

Given sets A and B, A **equals** B, written $A = B$ if, and only if, every element of A is in B and every element of B is in A. In symbols $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Example 95 *Let*

$$A = \{n \in \mathbf{Z} : n = 2p, \textit{ for some integer } p\}, \quad B = \{m \in \mathbf{Z} : m = 2q - 2, \textit{ for some integer } q\}$$

and $C = \{k \in \mathbf{Z} : k = 3r + 1, \textit{ for some integer } r\}$. *Is* $A = B$*? Is* $A = C$*?*

Solution:

We claim that $A = B$. To prove this, we must show that $A \subseteq B$ and $B \subseteq A$.

So suppose, first, that $n \in A$. Thus, $n = 2p$, for some integer p. Thus $n = 2p + 2 - 2$, whence $n = 2(p + 1) - 2$. Since p is an integer, $p + 1$ is also an integer. Thus $n = 2q - 2$ for the integer $q = p + 1$, which shows that $n \in B$. Thus $A \subseteq B$.

Suppose, conversely, that $m \in B$. Then $m = 2q - 2$, for some integer q. Therefore $m = 2(q - 1)$. But, q being an integer $p = q - 1$ is also an integer, which shows that $m = 2p$ for the integer $p = q - 1$. Therefore $m \in A$ and $B \subseteq A$.

having shown that $A \subseteq B$ and $B \subseteq A$, we may now conclude that $A = B$.

We claim that $A \neq C$. $2 \in A$, since $2 = 2 \cdot 1$ and 1 is an integer. But $2 \notin C$. If it was an element of C, then, there would be an integer r, such that $2 = 3r + 1$. Then $1 = 3r$, whence $r = \frac{1}{3}$, which is not an integer. ■

Exercise 108

Let

$$A = \{m \in \mathbf{Z} : m = 2i - 1, \textit{ for some integer } i\}, B = \{n \in \mathbf{Z} : n = 3j + 2 \textit{ for some integer } j\},$$

$$C = \{p \in \mathbf{Z} : p = 2r + 1 \textit{ for some integer } r\}, D = \{q \in \mathbf{Z} : q = 3s - 1 \textit{ for some integer } s\}.$$

Is $A = B$*? Is* $A = C$*? Is* $A = D$*? Is* $B = D$*?*

Exercise 109 *Let*

$$R = \{x \in \mathbf{Z} : x \textit{ is divisible by } 2\}, S = \{y \in \mathbf{Z} : y \textit{ is divisible by } 3\}$$

and $T = \{z \in \mathbf{Z} : z \textit{ is divisible by } 6\}$. *Is* $R \subseteq T$*? Is* $T \subseteq R$*? Is* $T \subseteq S$*? Explain your answers.*

4.2 Operations on Sets

When we are discussing about sets, all the sets under consideration will be assumed to contain elements that are taken from a large set understood in advance, called the **universe of discourse** or **universal set** and denoted by U. So, fo instance, if all our sets consist of real numbers, we could use $\mathbf{R}$, the set of all real numbers as our universal set. If all our sets consisted of human beings, we could take the set of all human beings as being our universal set.

In the sequel we introduce operations on sets that produce new sets from old ones, much like the matrix operations produce new matrices from old ones and the logical connectives that produce new statement forms from old simpler ones.

Let A, B be sets in U. The **union** $A \cup B$ of A and B is the set of all elements x in U, such that x is in A or x is in B. Note that "or" here is used in the same way as in logic. I.e., it has its inclusive meaning. $A \cup B$ contains all elements that are in A or in B or maybe in both A and B. In symbols

$$A \cup B = \{x \in U : x \in A \quad \text{or} \quad x \in B\}.$$

The **intersection** $A \cap B$ of A and B is the set of all elements x in U, such that $x \in A$ and $x \in B$. In symbols

$$A \cap B = \{x \in U : x \in A \quad \text{and} \quad x \in B\}.$$

The **difference** B **minus** A or **relative complement of** A **in** B, denoted $B - A$, is the set of all elements x in the universe U, such that x is in B but x is not in A. In symbols

$$B - A = \{x \in U : x \in B \quad \text{and} \quad x \notin A\}.$$

The **complement** A^c of A is the set of all elements x in U, such that x is not in A. In symbols

$$A^c = \{x \in U : x \notin A\}.$$

Example 96 *Let* $U = \{a, b, c, d, e, f, g\}, A = \{a, c, e, g\}, B = \{d, e, f, g\}$. *Find* $A \cup B, A \cap B, B - A$ *and* A^c.

Solution:

The union $A \cup B$ contains all elements $x \in U$, such that $x \in A$ or $x \in B$. Thus $A \cup B = \{a, c, d, e, f, g\}$.

The intersection $A \cap B$ consists of all $x \in U$, such that $x \in A$ and $x \in B$. Thus $A \cap B = \{e, g\}$.

The difference $B - A$ consists of all $x \in U$, such that $x \in B$ and $x \notin A$. Therefore $B - A = \{d, f\}$.

Finally, the complement of A consists of all $x \in U$, such that $x \notin A$. Hence $A^c = \{b, d, f\}$. ∎

Example 97 *Let* $U = \mathbf{R}, A = \{x \in \mathbf{R} : -1 < x \leq 0\}$ *and* $B = \{x \in \mathbf{R} : 0 \leq x < 1\}$. *Find* $A \cup B, A \cap B$ *and* A^c.

Solution:

$A \cup B$ consists of all real numbers x such that $x \in A$ or $x \in B$. These are all real numbers x, such that $-1 < x \leq 0$ or $0 \leq x < 1$. Thus

$$A \cup B = \{x \in \mathbf{R} : -1 < x < 1\}.$$

$A \cap B$ contains all real numbers x, such that $x \in A$ and $x \in B$, i.e., $-1 < x \leq 0$ and $0 \leq x < 1$. Hence $A \cap B = \{0\}$.

Finally, the complement of A consists of all real numbers x, such that $x \notin A$. These are all real numbers that do not satisfy $-1 < x \leq 0$. Therefore

$$A^c = \{x \in \mathbf{R} : x \leq -1 \text{ or } x > 0\}.$$

■

Exercise 110 *Let* $U = \{a, b, c, d, e\}, A = \{a, b, c\}, B = \{b, c, d\}$ *and* $C = \{b, c, e\}$.

(a) *Find* $A \cup (B \cap C), (A \cup B) \cap C$ *and* $A \cup B) \cap (A \cup C)$. *Which of these sets are equal?*

(b) *Find* $A \cap (B \cup C), (A \cap B) \cup C$ *and* $(A \cap B) \cup (A \cap C)$. *Which of these sets are equal?*

(c) *Find* $(A - B) - C$ *and* $A - (B - C)$. *Are these two sets equal?*

Exercise 111 *Let* $U = \mathbf{R}, A = \{x \in \mathbf{R} : 0 \leq x < 2\}$ *and* $B = \{x \in \mathbf{R} : 1 < x < 5\}$. *Find* $A \cup B, A \cap B$ *and* A^c.

Exercise 112 *Pick a universe* U *and find in it three sets* A, B *and* C, *that satisfy* $A \subseteq B, C \subseteq B$ *and* A *and* C *have no elements in common.*

Let n be a positive integer and $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) elements of the universe U. The **ordered** n**-tuple** $(x_1, x_2, \ldots, x_n)$ consists of these elements together with the ordering: first x_1, then x_2, then x_3 and so on, up to x_n. An ordered 2-tuple is called an **ordered pair** and an ordered 3-tuple is called an **ordered triple**.

Two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $y_1, y_2, \ldots, y_n)$ are **equal** if, and only if, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$. In symbols

$$(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n) \Longleftrightarrow x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n.$$

Example 98 *Is* $(1, 2) = (2, 1)$*? How about* $(3, (-2)^2, \frac{1}{2}) = (\sqrt{9}, 4, \frac{3}{6})$*?*

Solution:

$$(1, 2) = (2, 1) \Longleftrightarrow 1 = 2,\ 2 = 1.$$

Since the right-hand side is false, the left is also false. On the other hand

$$(3, (-2)^2, \frac{1}{2}) = (\sqrt{9}, 4, \frac{3}{6}) \Longleftrightarrow 3 = \sqrt{9},\ (-2)^2 = 4,\ \frac{1}{2} = \frac{3}{6}.$$

Thus, since the right-hand side is true, the two ordered triples are equal. ■

Given two sets A and B, the **Cartesian product** of A and B, denoted $A \times B$, is the set of all ordered pairs (a, b), with a in A and b in B. In symbols

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Similarly, given n sets $A_1, A_2, \ldots, A_n$, the **Cartesian product** of $A_1, A_2, \ldots, A_n$, denoted $A_1 \times A_2 \times \ldots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$, such that a_1 is in A_1, a_2 is in A_2,..., a_n is in A_n. In symbols

$$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

Example 99 *Let $A = \{x, y\}, B = \{1, 2, 3\}$ and $C = \{a, b\}$. Find $A \times B, (A \times B) \times C$ and $A \times B \times C$.*

Solution:
We have

$$A \times B = \{(x, 1), (x, 2), (x, 3), (y, 1), (y, 2), (y, 3)\}.$$

Similarly,

$$\begin{aligned}(A \times B) \times C = \{&((x, 1), a), ((x, 2), a), ((x, 3), a), ((y, 1), a),\\ &((y, 2), a), ((y, 3), a), ((x, 1), b), ((x, 2), b),\\ &((x, 3), b), ((y, 1), b), ((y, 2), b), ((y, 3), b)\}\end{aligned}$$

Finally,

$$\begin{aligned}(A \times B) \times C = \{&(x, 1, a), (x, 2, a), (x, 3, a), (y, 1, a),\\ &(y, 2, a), (y, 3, a), (x, 1, b), (x, 2, b),\\ &(x, 3, b), (y, 1, b), (y, 2, b), (y, 3, b)\}\end{aligned}$$

■

Exercise 113 *Let $A = \{x, y, z, w\}$ and $B = \{a, b\}$. List the elements in each of $A \times B, B \times A, A \times A$ and $B \times B$.*

Exercise 114 *Let $A = \{1, 2, 3\}, B = \{u, v\}$ and $C = \{m, n\}$. Find $A \times (B \times C), (A \times B) \times C$ and $A \times B \times C$.*

Exercise 115 *(This will be presented in class!) Use* **Venn diagrams** *to depict the following sets: $A \cap B, B \cup C, A^c, A - (B \cup C), (A \cup B)^c$ and $A^c \cap B^c$. Venn diagrams are extremely useful in providing intuition about relations between sets!!*

4.3 Proving Subset Relations

In this section we learn how to prove a relation involving unions, intersections, complements and difference of sets that holds for arbitrary sets.

Example 100 *Show that, for all sets A and B, $A \cap B \subseteq A$ and $A \cap B \subseteq B$.*

Solution:
Recall that, by the definition of the subset relation, $A \cap B \subseteq A$ means that every element in $A \cap B$ is also an element in A. That is what we want to show.

So let's take an arbitrary element x in $A \cap B$. By the definition of intersection, $x \in A \cap B$ means that $x \in A$ and $x \in B$. Thus, $x \in A$, which was what we wanted to show. Now work similarly for $A \cap B \subseteq B$. ■

Example 101 *Show that for all sets A, B, $A \subseteq A \cup B$ and $B \subseteq A \cup B$.*

Solution:
For $A \subseteq A \cup B$, we need to show that every element of A is also an element of $A \cup B$. For $B \subseteq A \cup B$, we need to show that every element of B is also an element of $A \cup B$. We do the first and challenge you to do the second.

Let x be an element in A. This means that $x \in A$ or $x \in B$ is also a true statement. But this means that $x \in A \cup B$. Hence any element of A is also an element of $A \cup B$.

Now prove $B \subseteq A \cup B$ on your own following the same line of thought. ■

The above proofs illustrate the basic technique that is used to prove that one set is a subset of another. We summarize this method below.

Method for proving that one set is a subset of another set: Let X, Y be two given sets. To show that $X \subseteq Y$,

1. suppose that x is a particular but arbitrarily chosen element of X,

2. show that x is also an element of Y.

Exercise 116 *Show that, for all sets A, B, C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.*

The basic properties that are used in applying the method above to show that a given set is a subset of another set are

1. $x \in X \cup Y \longleftrightarrow x \in X$ or $x \in Y$

2. $x \in X \cap Y \longleftrightarrow x \in X$ and $x \in Y$

3. $x \in X - Y \longleftrightarrow x \in X$ and $x \notin Y$

4. $x \in X^c \longleftrightarrow x \notin X$

5. $(x, y) \in X \times Y \longleftrightarrow x \in X$ and $y \in Y$

Example 102 *Show that, for all sets A and B, $A - B \subseteq A$.*

Solution:

We need to show that every element in $A-B$ is also an element of A. We start by letting x be a particular but arbitrarily chosen element of $A - B$. By the definition of $A - B$, we conclude that $x \in A$ and $x \notin B$. In particular $x \in A$, which was what we wanted to show. Thus $A - B \subseteq A$. ■

Example 103 *Show that, for all sets A and B, if $A \subseteq B$, then $A \cup B \subseteq B$.*

Solution:

we are assuming that $A \subseteq B$ and want to show that $A \cup B \subseteq B$. I.e., we want to show that, every element of $A \cup B$ is also an element of B. So let us take a particular but arbitrarily chosen element x of $A \cup B$. By the definition of union, this means that $x \in A$ or $x \in B$. But, by out hypothesis $A \subseteq B$, this implies that $x \in A$ or $x \in A$, whence $x \in A$, which is what we wanted to show. ■

Exercise 117 *Show that, for all sets A and B, if $A \subseteq B$, then $A \subseteq A \cap B$.*

Example 104 *Show that, for all sets A, B and C, if $A \subseteq B$, then $A \cap C \subseteq B \cap C$.*

Solution:

We assume that $A \subseteq B$ and want to show that $A \cap C \subseteq B \cap C$, i.e., that every element of $A \cap C$ is also an element of $B \cap C$. So let x be a particular but arbitrarily chosen element of $A \cap C$. By the definition of intersection, $x \in A$ and $x \in C$. But, by assumption, $A \subseteq B$, whence $x \in B$ and $x \in C$. Hence, again by the definition of intersection, $x \in B \cap C$. We have thus shown that $x \in A \cap C$ implies $x \in B \cap C$. Therefore $A \cap C \subseteq B \cap C$, as was to be shown. ■

Now, follow the steps in the proof above to solve the next exercise.

Exercise 118 *Show that, for all sets A, B and C, if $A \subseteq B$, then $A \cup C \subseteq B \cup C$.*

Example 105 *Show that, for all sets A, B, if $A \subseteq B$, then $B^c \subseteq A^c$.*

Solution:

Suppose that $A \subseteq B$. We need to show that $B \subseteq A^c$, i.e., that every element in B^c is also an element in A^c. So let x be a particular but arbitrarily chosen element in B^c. By the definition of B^c, we get that $x \notin B$. Then $x \notin A$, because, if $x \in A$, $A \subseteq B$ would yield $x \in B$ and we know that $x \notin B$. But $x \notin A$ is, by the definition of A^c the same as saying that $x \in A^c$. Hence, we showed that $x \in B^c$ implies that $x \in A^c$. ■

Exercise 119 *Show that, for all sets A, B, if $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.*

Exercise 120 *Show that, for all sets A, B, if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.*

4.4 Proving Set Identities

In this section we use the technique developed in the previous section to prove several identities between sets, i.e., equality relations between sets that hold for all sets.

Method for proving that two sets are equal: Let X, Y be given sets. To prove that $X = Y$,

1. prove that $X \subseteq Y$
2. prove that $Y \subseteq X$

Example 106 *Show that, for all sets A, B and C, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.*

Solution:

We need to show that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ and $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

We first undertake $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. Let x be a particular but arbitrarily chosen element of $A \cup (B \cap C)$. Then, by the definition of union, $x \in A$ or $x \in B \cap C$.

1. If $x \in A$, then both ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$) are true statements. Hence, by the definition of union, both $x \in A \cup B$ and $x \in A \cup C$ are true. Therefore, by the definition of intersection, we get $x \in (A \cup B) \cap (A \cup C)$. Thus, in this case x is also in $(A \cup B) \cap (A \cup C)$.

2. If $x \in B \cap C$, then, by the definition of intersection, $x \in B$ and $x \in C$. Therefore, both ($x \in A$ or $x \in B$) and ($x \in A$ or ($x \in C$) are true statements. By the definition of the union, both $x \in A \cup B$ and $x \in A \cup C$ hold. Therefore, by the definition of intersection, $x \in (A \cup B) \cap (A \cup C)$. Thus, in this second case, we also showed that x must be in $(A \cup B) \cap (A \cup C)$.

Thus, in either case ($x \in A$ or $x \in B \cap C$) we have also that $x \in (A \cup B) \cap (A \cup C)$. Hence $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$, as was to be shown.

Next we need to show that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. So let x be a particular but arbitrarily chosen element of $(A \cup B) \cap (A \cup C)$. Then, by the definition of intersection, we have $x \in A \cup B$ and $x \in A \cup C$. We reason now by cases. We must have either $x \in A$ or $x \notin A$.

1. If $x \in A$, then $x \in A$ or $x \in B \cap C$ is a true statement and, therefore, by the definition of union, we get $x \in A \cup (B \cap C)$, which is what we wanted to show.

2. In the other case, if $x \notin A$, and since $x \in A \cup B$, we must have $x \in B$. Similarly, if $x \notin A$, and since $x \in A \cup C$, we must have $x \in C$. Hence x has to be in $B \cap C$. But then $x \in A$ or $x \in B \cap C$ is a true statement. Therefore, by the definition of the union, $x \in A \cup B \cap C)$.

We showed that in either case ($x \in A$ or $x \notin A$) x has to be in $A \cup (B \cap C)$. But then $x \in (A \cup B) \cap (A \cup C)$ implies that $x \in A \cup (B \cap C)$ which is what we wanted to show. ■

Exercise 121 *Show that for all sets* A, B *and* C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Example 107 *Show that, for all sets* A, B, $(A \cup B)^c = A^c \cap B^c$.

Solution:

We need to show that $(A \cup B)^c \subseteq A^c \cap B^c$ and that $A^c \cap B^c \subseteq (A \cup B)^c$.

We undertake first $(A \cup B)^c \subseteq A^c \cap B^c$. Let x be a particular but arbitrarily chosen element of $(A \cup B)^c$. Then, by the definition of complement, $x \notin A \cup B$. If $x \in A$ or $x \in B$ then $x \in A \cup B$, whence, since we know that $x \notin A \cup B$, we must have $x \notin A$ and $x \notin B$. Hence, by the definition of complement again, we have $x \in A^c$ and $x \in B^c$. But, then, by the definition of intersection, we get that $x \in A^c \cap B^c$. So we showed that every $x \in (A \cup B)^c$ is also an element of $A^c \cap B^c$, which gives $(A \cup B)^c \subseteq A^c \cap B^c$.

We next undertake the reverse inclusion $A^c \cap B^c \subseteq (A \cup B)^c$. So suppose that x is a particular but arbitrarily chosen element of $A^c \cap B^c$. By the definition of intersection, we then have $x \in A^c$ and $x \in B^c$. Then, by the definition of complement, we have that $x \notin A$ and $x \notin B$. If $x \notin A$ and $x \notin B$, then $x \notin A \cup B$, because in that case $x \in A$ or $x \in B$. Therefore, again by the definition of complement, we get $x \in (A \cup B)^c$. Thus, we showed that every element in $A^c \cap B^c$ is also an element in $(A \cup B)^c$, which shows that $A^c \cap B^c \subseteq (A \cup B)^c$, as required.

Having shown both $(A \cup B)^c \subseteq A^c \cap B^c$ and $A^c \cap B^c \subseteq (A \cup B)^c$ we may now conclude that $(A \cup B)^c = A^c \cap B^c$. ■

Exercise 122 *Show that, for all sets* A, B, $(A \cap B)^c = A^c \cup B^c$.

Example 108 *Show that, for all sets* A, B, *if* $A \subseteq B$, *then* $A \cap B = A$.

Solution:

Suppose that $A \subseteq B$. This is a global assumption and we are free to use it at any point in the proof that we need it. Then we have to show that $A \cap B \subseteq A$ and $A \subseteq A \cap B$.

We first undertake $A \cap B \subseteq A$. Let x be a particular but arbitrarily chosen element of $A \cap B$. We have then, by the definition of intersection, $x \in A$ and $x \in B$. In particular, $x \in A$ must be true. Thus, we showed that every element in $A \cap B$ is also an element in A. Therefore $A \cap B \subseteq A$.

Now we undertake to show that $A \subseteq A \cap B$. If $x \in A$, then, since $A \subseteq B$, we must also have $x \in B$. Therefore $x \in A$ and $x \in B$ are both true. Thus, by the definition of intersection, $x \in A \cap B$. Thus, we have shown that every element of A is also an element of $A \cap B$. This shows that $A \subseteq A \cap B$.

Having shown that $A \cap B \subseteq A$ and $A \subseteq A \cap B$ we may now conclude that, if $A \subseteq B$, then $A \cap B = A$. ■

Exercise 123 *Show that, for all sets* A, B, *if* $A \subseteq B$, *then* $A \cup B = B$.

Example 109 *Show that, for all sets* A, B, $B - A = B \cap A^c$.

Solution:

We need to show that $B - A \subseteq B \cap A^c$ and that $A \cap B^c \subseteq B - A$.

We first undertake $B - A \subseteq B \cap A^c$. So let x be a particular but arbitrarily chosen element of $B - A$. Then, by the definition of the relative complement, we have $x \in B$ and $x \notin A$. Thus, by the definition of complement, $x \in B$ and $x \in A^c$. Thus, by the definition of intersection, we get $x \in B = \cap A^c$. We have shown that every element of $B - A$ is also an element of $B \cap A^c$, i.e., that $B - A \subseteq B \cap A^c$.

We undertake, next, to show that $B \cap A^c \subseteq B - A$. So let x be a particular but arbitrarily chosen element of $B \cap A^c$. Then, by the definition of intersection, $x \in B$ and $x \in A^c$. Therefore, by the definition of complement, $x \in B$ and $x \notin A$. But, then, by the definition of the relative complement, we get $x \in B - A$. Thus, we showed that every element of $B \cap A^c$ is also an element of $B - A$, i.e., that $B \cap A^c \subseteq B - A$.

Having shown both $B - A \subseteq B \cap A^c$ and $A \cap B^c \subseteq B - A$, we may now conclude that $B - A = B \cap A^c$. ■

Exercise 124 *Show that, for all sets A, B, C,*

1. $(A - B) \cup (A \cap B) = A$.
2. $(A - B) - (B - C) = A - B$.
3. $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.
4. $(A - B) - C = A - (B \cup C)$.

Example 110 *Show that, for all sets A, B, $((A^c \cup B^c) - A)^c = A$.*

Solution:

We need to show that $((A^c \cup B^c) - A)^c \subseteq A$ and $A \subseteq ((A^c \cup B^c) - A)^c$.

We first undertake $((A^c \cup B^c) - A)^c \subseteq A$. So let x be a particular but arbitrarily chosen element of $((A^c \cup B^c) - A)^c$. By the definition of complement, $x \notin (A^c \cup B^c) - A$. This implies $x \notin A^c \cup B^c$ or $x \in A$.

1. If $x \notin A^c \cup B^c$, then $x \notin A^c$ and $x \notin B^c$. Therefore, by the definition of complement, $x \in A$ and $x \in B$. Thus, $x \in A$ is true, which is what we wanted to show.
2. If $x \in A$, we are done.

In either case, $x \in A$, which shows that every element of $((A^c \cup B^c) - A)^c$ is also an element of A, i.e., $((A^c \cup B^c) - A)^c \subseteq A$.

Next, we undertake to show that $A \subseteq ((A^c \cup B^c) - A)^c$. So let x be a particular but arbitrarily chosen element of A. Then $x \notin (A^c \cup B^c) - A$, since, otherwise, x would not be in A. Therefore, by the definition of complement, $x \in ((A^c \cup B^c) - A)^c$, as was to be shown.

Having shown that $((A^c \cup B^c) - A)^c \subseteq A$ and $A \subseteq ((A^c \cup B^c) - A)^c$, we may now conclude that $((A^c \cup B^c) - A)^c = A$. ■

Exercise 125 *Show that, for all sets A, B, $(B^c \cup (B^c - A))^c = B$.*

4.5 Finding Counterexamples and Detecting Mistakes

In this section a technique is introduced for showing that a given conjectured property between sets is not true for all sets. Some problems are also presented on how to detect fallacies in arguments showing that a set is a subset of another set or that two sets are equal.

Example 111 *Is the following property true?*

$$\text{For all sets } A, B, C,\ (A - B) \cup (B - C) = A - C.$$

Solution:
Draw a Venn diagram of the situation to realize where this property may fail. Then construct an example to show that the property is not true. Such an example is called a **counterexample** for the given property. We must

1. Construct a universe U.
2. Pick three sets A, B, C in the universe U for which the given property fails.
3. Verify the conjecture that the property fails for these three sets that we picked.

So let U be the set $U = \{a, b, c, d\}$. Then pick $A = \{a, b\}, B = \{b, c\}$ and $C = \{a, d\}$. Then we have

$$\begin{aligned}(A - B) \cup (B - C) &= (\{a, b\} - \{b, c\}) \cup (\{b, c\} - \{a, d\}) \\ &= \{a\} \cup \{b, c\} \\ &= \{a, b, c\}.\end{aligned}$$

On the other hand, $A - C = \{a, b\} - \{a, d\} = \{b\}$. Hence we see that $(A - B) \cup (B - C) \neq A - C$, for some sets A, B, C. ■

Example 112 *Is the following property true?*

$$\text{For all sets } A, B, C,\ A - (B - C) = (A - B) - C.$$

Solution:
Let $U = \{a, b, c, d\}, A = \{a, b\}, B = \{a, c\}$ and $C = \{a, d\}$. Then

$$\begin{aligned}A - (B - C) &= \{a, b\} - (\{a, c\} - \{a, d\}) \\ &= \{a, b\} - \{c\} \\ &= \{a, b\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(A - B) - C &= (\{a, b\} - \{a, c\}) - \{a, d\} \\ &= \{b\} - \{a, d\} \\ &= \{b\}.\end{aligned}$$

Therefore $A - (B - C) \neq (A - B) - C$ for some sets A, B and C. ■

Exercise 126 *Is the following property true?*

$$\text{For all sets } A, B, C,\ (A - B) \cap (C - B) = A - (B \cup C).$$

Exercise 127 *Is the following property true?*

$$\text{For all sets } A, B, C,\ \text{if } A \cap C = B \cap C,\ \text{then } A = B.$$

Exercise 128 *Is the following property true?*

$$\text{For all sets } A, B, C,\ \text{if } A \cup C = B \cup C,\ \text{then } A = B.$$

Exercise 129 *Is the following property true?*

$$\text{For all sets } A, B, C,\ \text{if } A \cap C \subseteq B \cap C \text{ and } A \cup C \subseteq B \cup C,\ \text{then } A = B.$$

Exercise 130 *Is the following property true?*

$$\text{For all sets } A, B, C,\ (A \cup B) \cap C = A \cup (B \cap C).$$

Exercise 131 *Is the following property true?*

$$\text{For all sets } A, B, C,\ \text{if } A \not\subseteq B \text{ and } B \not\subseteq C,\ \text{then } A \not\subseteq C..$$

Example 113 *Find the mistake in the following "proof" that, for all sets A, B and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$:*

"Proof: Suppose A, B and C are sets such that $A \subseteq B$ and $B \subseteq C$. Since $A \subseteq B$, there is an element x so that $x \in A$ and $x \in B$. Since $B \subseteq C$, there is an element x so that $x \in B$ and $x \in C$. Hence there is an element x so that $x \in A$ and $x \in C$ and so $A \subseteq C$."

Solution:

There are many errors. First, $A \subseteq B$ does not mean that there is an element x so that $x \in A$ and $x \in B$. Second, the x that is in A and in B may be a different element from the one (erroneously denoted by x as well) that is in B and C. ■

Exercise 132 *Find the mistake in the following "proof" of the statement that, for all sets A, B, $A^c \cup B^c \subseteq (A \cup B)^c$.*

"Proof: Suppose A and B are sets and suppose $x \in A^c \cup B^c$. Then $x \in A^c$ or $x \in B^c$ by definition of union. It follows that $x \notin A$ or $x \notin B$ by definition of complement. Therefore $x \notin A \cup B$ by definition of union. Thus $x \in (A \cup B)^c$ by definition of complement. Hence, $A^c \cup B^c \subseteq (A \cup B)^c$."

Exercise 133 *Find the mistake in the following "proof" of the statement that, for all sets A, B, $(A - B) \cup (A \cap B) \subseteq A$.*

"Proof: Suppose A and B are sets and suppose that $x \in (A - B) \cup (A \cap B)$. If $x \in A$, then $x \in A - B$. Then, by definition of difference, $x \in A$ and $x \notin B$. Hence $x \in A$ and so $(A - B) \cup (A \cap B) \subseteq A$ by the definition of subset."

4.6 Empty Set, Partitions and Power Sets

Chapter 5

Basic Combinatorics

5.1 Simple Counting

Example 114 *How many integers there are from 5 through 12?*

Solution:
We just count these integers:

integers	5	6	7	8	9	10	11	12
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
count	1	2	3	4	5	6	7	8

So there are 8 integers from 5 through 12. ■

Example 115 *Let m, n be two integers with $m \leq n$. How many integers are there from m through n?*

Solution:
Note that $m = m+0$ is element 1, $m+1$ is element number 2, etc. Setting up a counting list as above we get

integers	$m+0$	$m+1$	$m+2$	$\ldots$	$n = m + (n-m)$
	$\updownarrow$	$\updownarrow$	$\updownarrow$		$\updownarrow$
count	1	2	3	$\ldots$	$(n-m)+1$

Hence there are $n - m + 1$ elements in the given list. ■

Example 116 *How many three-digit integers (integers from 100 to 999 inclusive) are divisible by 5?*

Solution:

The integers that are divisible by 5 between 100 to 999 are the integers

$$100, 105, 110, 115, \ldots, 995.$$

These are the same as

$$5 \cdot 20, 5 \cdot 21, 5 \cdot 22, 5 \cdot 23 \ldots, 5 \cdot 199.$$

There are as many on this list as in the following list:

$$20, 21, 22, 23, \ldots, 199.$$

According to the counting principle above, this list contains $199 - 20 + 1 = 180$ elements. Thus, there are 180 three-digit integers that are divisible by 5. ■

Exercise 134 *How many positive two-digit integers are multiples of 3?*

Exercise 135 *How many positive three-digit integers are multiples of 6?*

Example 117 *Consider the list of numbers*

$$42, 43, 44, \ldots, 100.$$

What is the 27th element on this list?

Solution:
Suppose that n denotes the 27th element. We then have $n - 42 + 1 = 27$. Therefore $n = 27 + 42 - 1 = 68$. ■

Exercise 136 *Consider the list of numbers*

$$29, 30, 31, \ldots, 100.$$

What is the 27th element on this list?

Example 118 *If the largest of 56 consecutive integers is 279, what is the smallest?*

Solution:
Suppose that the smallest element in the list is n. Then, we have the list

$$n, n + 1, n + 2, \ldots, 279.$$

This list contains 56 integers. Thus $279 - n + 1 = 56$. Therefore $n = 279 - 56 + 1 = 224$. ■

Exercise 137 *If the largest of 87 consecutive integers is 326, what is the smallest?*

Example 119 *How many even integers are there between 1 and 1001?*

Solution:
The list of even integers between 1 and 1001 is

$$2, 4, 6, 8, \ldots, 1000.$$

These are the same as

$$2 \cdot 1, 2 \cdot 2, 2 \cdot 3, 2 \cdot 4, \ldots, 2 \cdot 500.$$

The number of these is the same as the number of the elements in the following list

$$1, 2, 3, 4, \ldots, 500.$$

here are $500 - 1 + 1 = 500$ elements in this list. Hence there are 500 even integers between 1 and 1001. ■

Exercise 138 *How many integers are there between 1 and 1001 that are multiples of 3?*

Example 120 *How many integers are there between 1 and 600 that are divisible by 7?*

Solution:
The list of integers between 1 and 600 that are divisible by 7 is

$$7, 14, 21, 28, \ldots, 595.$$

These are the same as

$$7 \cdot 1, 7 \cdot 2, 7 \cdot 3, 7 \cdot 4, \ldots, 7 \cdot 85.$$

There as many in this list as in the list

$$1, 2, 3, 4, \ldots, 85.$$

Hence there are $85 - 1 + 1 = 85$ integers between 1 and 600 that are divisible by 7. ■

Exercise 139 *How many integers between 100 and 800 are divisible by 4?*

5.2 The Multiplication Principle

A tree structure is very useful for keeping track of all possibilities in situations in which events happen in order.

Example 121 *Teams A and B are to play each other repeatedly until one wins two games in a row or a total of three games. One way this tournament can be played is for A to win the first game, B to win the second, and A to win the third and fourth games. Denote this by writing* $A-B-A-A$.

1. *How many ways can the tournament be played?*
2. *In how many of these ways 5 games are needed to determine the tournament winner?*

Solution:
Build a tree for the possible outcomes of the tournament.

1. The following are all the possibilities for complete tournaments:

$$A-A, A-B-B, A-B-A-A, A-B-A-B-A, A-B-A-B-B,$$

$$B-B, B-A-A, B-A-B-B, B-A-B-A-A, B-A-B-A-B.$$

   So there are 10 possible ways to play this game.

2. Only 4 of these ways require 5 games to determine the winner. ■

Exercise 140 *In baseball World Series the first team to win four games wins the series. Suppose that team A wins the first 3 games. How many ways can the series be completed?*

Exercise 141 *In a competition between players* X *and* Y *, the first player to win three games in a row or a total of four games wins. How many ways can the competition be played if* X *wins the first two games?*

Exercise 142 *Urn* $U1$ *contains two black balls* $B1$ *and* $B2$ *and one white ball. Urn* $U2$ *contains one black ball and two white balls* $W1$ *and* $W2$*. Suppose that an urn is chosen and then a ball is drawn from the chosen urn. Subsequently a second ball is also drawn without first replacing the first ball. How many possible outcomes are there in this experiment?*

Example 122 *Suppose that you want to travel from Ames, IA, through Chicago, IL, to Sault Sainte Marie, MI. You look at the map and find out that there are three possible routes* a, b, c *you could follow from Ames to Chicago and five possible routes* A, B, C, D, E *you can follow from Chicago to the Soo. How many different routes could you possibly follow from Ames to the Soo?*

Solution:

All the possibilities are given by an appropriate tree. It shows that the possible different routes are

$$aA, aB, aC, aD, aE, bA, bB, bC, bD, bE, cA, cB, cC, cD, cE.$$

Thus, there are $3 \times 5 = 15$ possible ways. ■

The Multiplication Principle:

If a task consists of k subtasks, the first of which can be performed in n_1 ways, the second in n_2 ways, and so on, and the k-th in n_k ways, then the entire task may be completed in $n_1 \cdot n_2 \cdot \ldots \cdot n_k$ different ways.

Exercise 143 *Suppose that you want to travel from Pittsburgh, PA, through Cleveland, OH, through Detroit, MI, to Sault Sainte Marie, MI. You look at the map and find out that there are two possible routes a, b you could follow from Pittsburgh to Cleveland, three possible routes A, B, C from Cleveland to Detroit and two possible routes x, y you can follow from Detroit to the Soo. How many different routes could you possibly follow from Pittsburgh to the Soo?*

Exercise 144 *You want to dress up for a formal Thanksgiving dinner. You discover that in you have available 3 different shirts, 4 different trousers and 3 pairs of socks and 2 pairs of shoes that are all matching in colors. How many different outfits could you make out of the available shirts, trousers, socks and shoes?*

Exercise 145 *A person buying a personal computer system is offered a choice of three models of the basic unit, two models of a keyboard, and two models of a printer. How many distinct systems can be purchased?*

Example 123 *A typical PIN is a sequence of four symbols chosen from the 26 letters of the alphabet and the ten digits, with repetition allowed. How many different PINs are possible?*

Solution:

Divide the task of setting up a PIN to the following four subtasks:

Subtask 1 Choose the first symbol.

Subtask 2 Choose the second symbol.

Subtask 3 Choose the third symbol.

Subtask 4 Choose the fourth symbol.

Subtask 1 may be performed in 36 ways, Subtask 2 in 36 ways, Subtask 3 in 36 ways and Subtask four in 36 ways. Thus, by the multiplication principle, the complete task may be performed in 36^4 ways. Thus many are the possible PINs. ■

Example 124 *Suppose that A, B and C are three sets with $6, 5$ and 9 elements, respectively. Show that the Cartesian product $A \times B \times C$ has 270 elements.*

Solution:

Recall the the Cartesian product $A \times B \times C$ is the set of all triples (a, b, c), where $a \in A, b \in B$ and $c \in C$. In how many ways is it possible to build such a triple? Subdivide the task into the following subtasks:

Subtask 1 Choose the first element of the triple.

Subtask 2 Choose the second element of the triple.

Subtask 3 Choose the third element of the triple.

Subtask 1 may be completed in 6 possible ways, Subtask 2 in 5 possible ways and Subtask 3 in 9 possible ways. Thus, by the multiplication principle, there are $6 \cdot 5 \cdot 9 = 270$ different ways to build a triple in $A \times B \times C$, i.e., there are 270 different triples available. ■

Example 125 *A four symbol PIN is to be formed as before, where each symbol is either a letter of the alphabet or a digit.*

1. *How many possible PINs are there with no repeated symbols?*
2. *How many PINs contain at least one repeated symbol?*

Solution:

1. Subdivide the task of forming a PIN to the following four subtasks:

Subtask 1 Choose the first symbol.

Subtask 2 Choose the second symbol.

Subtask 3 Choose the third symbol.

Subtask 4 Choose the fourth symbol.

Now the first subtasks may be carried out in 36 ways, the second in 35 ways, the third in 34 ways and the fourth in 33 ways. Thus, by the multiplication principle, there are $36 \cdot 35 \cdot 34 \cdot 33$ ways to perform the entire task. Hence, there are $36 \cdot 35 \cdot 34 \cdot 33$ different PINs with no repeated symbols.

2. The number of PINs with at least one repeated symbol may be computed by taking the total number of possible PINs and subtracting from it the number of these PINs with no repeated symbols. Thus, there are $36^4 - 36 \cdot 35 \cdot 34 \cdot 33$ PINs with at least one repeated symbol. ■

Exercise 146 *In a certain state automobile license plates each have three letters followed by three digits.*

1. *How many license plates are possible?*

2. *How many of these begin with A and end with a 0?*

3. *How many begin with PDQ?*

4. *How many are possible with all letters and digits distinct?*

Example 126 *Suppose that you want to construct the truth table for a sentence form that contains 3 different sentence variables. How many rows will that truth table have?*

Solution:

The task is to construct all sequences of length 3 out of the letters T and F. Subdivide this task to the subtasks

Subtask 1 Choose the first truth value.

Subtask 2 Choose the second truth value.

Subtask 3 Choose the third truth value.

Each of the subtasks may be performed in 2 ways. Therefore, by the multiplication principle, there are $2^3 = 8$ possible ways to carry out the complete task. Thus, there should be 8 rows in the truth table for the given sentence form. ■

Exercise 147 *A bit string is a sequence of 0'2 and 1's. How many bit strings have length 8? How many of these strings begin with a 1? How many begin and end with a 1?*

Exercise 148 *A coin is tossed four times. Each time the result H for heads or T for tails is recorded. An outcome of HHTT means that heads were obtained in the first two tosses and tails were obtained in the last two tosses. How many distinct out comes are possible? How many of these contain exactly two heads?*

5.3 Permutations

A **permutation** of a set of objects is an ordering of the objects in a row. For example the set of elements $\{a, b, c\}$ has six permutations

$$abc \quad acb \quad bac \quad bca \quad cab \quad cba$$

In general, if we are given a set of n elements, how many permutations does the set have? To calculate this number we use the multiplication principle. The task of forming a permutation may be subdivided into n subtasks:

Subtask 1 Choose the first element.

Subtask 2 Choose the second element.

$\vdots$

Subtask n Choose the n-th element.

Subtask 1 may be performed in n ways, Subtask 2 may be performed in $n-1$ ways, and so on, up to Subtask n which may be performed in just 1 way. Now, by the multiplication principle, there are $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$ ways to perform the complete task. In other words, there are

$$n! = n(n-1)(n-2)\cdots 2 \cdot 1$$

different permutations of a set with n elements.

Example 127 *1. In how many ways can the letters of the word "COMPUTER" be arranged in a row?*

2. In how many ways can the letters of the word "COMPUTER" be arranged in a row if the letters CO must remain next to each other (in order) as a unit?

Solution:

1. All letters in "COMPUTER" are distinct. So there are $8! = 8 \cdot 7 \cdot 6 \ldots \cdot 2 \cdot 1$ permutations of this set.

2. If CO must appear in the permutation together as a unit, then the number of possible arrangements is the same as the number of permutations of the seven different objects

$$CO, M, P, U, T, E, R.$$

Hence, there are $7! = 7 \cdot 6 \cdot \ldots \cdot 2 \cdot 1$ possible arrangements. ■

Exercise 149 *1. In how many ways can the letters of the word "ALGORITHM" be arranged in a row?*

2. *In how many ways can this be done if the letters A and L must remain together (in order) as a unit?*

3. *In how many ways can it be done if the letters GOR must remain together (in order) as a unit?*

4. *In how many ways can it be done if AL and GOR have to remain together (in order) as units?*

Exercise 150 *Six people go to the movies together.*

1. *In how many ways can they be seated in a row?*

2. *If one of them is a doctor who musty seat on the aisle in case she is paged, in how many ways can they be seated in six seats if exactly one of the seats is on the aisle?*

3. *If the six people consist of three couples and each couple wants to sit together with the boyfriend on the left, in how many ways can the six friends be seated in a row?*

Example 128 *At a meeting of diplomats, the six participants are to be seated around a circular table. Since the table has no ends to confer particular status, it does not matter who sits in which chair. But it does matter how the diplomats are seated relative to each other. In other words, two seatings are considered the same if one is a rotation of the other. In how many different ways can the diplomats be seated?*

Solution:

Fix one diplomat at a specific position. His position is irrelevant. Only the positions of the remaining 5 diplomats relative to him are important. Thus, there are $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ different possible ways to seat the diplomats at the round table. ■

Permutations like the one above are sometimes termed **cyclic permutations**. Thus, following the same argument as above, we may show that n different objects have $(n-1)!$ different cyclic permutations. I.e., they may be arranged in a cycle in $(n-1)!$ different ways.

Exercise 151 *Five people are to be seated around a circular table. Two seating are considered the same if one is a rotation of the other. In how many different ways can they be seated around the table?*

Given the set $\{a, b, c\}$, there are six ways to select two letters from the set and write them in order.

$$ab \quad ac \quad ba \quad bc \quad ca \quad cb$$

Each such ordering of two elements of $\{a, b, c\}$ is called a 2-**permutation** of $\{a, b, c\}$.

Exercise 152 1. *Write all the possible* 2*-permutations of* $\{W, X, Y, Z\}$.

2. *Write all possible* 3*-permutations of* $\{a, b, c, d\}$.

In general an r-**permutation** of a set of n elements is an ordered selection of r elements taken from the set of n elements. The number of r permutations of a set of n elements is denoted by $P(n, r)$.

Can we calculate how many r-permutations of a set of n elements there are? I.e., can we compute the number $P(n, r)$? We use the multiplication principle. Subdivide the task of forming an r-permutation into the following r subtasks:

Subtask 1 Choose the first element.

Subtask 2 Choose the second element.

$\vdots$

Subtask r Choose the r-th element.

Subtask 1 may be performed in n ways. Subtask 2 may be performed in $n-1$ ways. Subtask 3 may be performed in $n-2$ ways and so on until Subtask r, which may be performed in $n-r+1$ ways. Thus, by the multiplication principle, there are $n(n-1)(n-2)\dots(n-r+2)(n-r+1)$ possible ways to perform the complete task. I.e., there are $n(n-1)\dots(n-r+1)$ possible r-permutations of n elements. Thus, we obtain

$$P(n, r) = n(n-1)\dots(n-r+1) = \frac{n(n-1)\dots 1}{(n-r)(n-r-1)\dots 1} = \frac{n!}{(n-r)!}.$$

Example 129 1. *Evaluate* $P(5, 2)$.

2. *How many 4-permutations are there of a set of 7 objects?*

3. *How many 5-permutations are there of a set of 5 objects?*

Solution:

1. $P(5, 2) = \frac{5!}{(5-2)!} = \frac{5!}{3!} = 5 \cdot 4 = 20.$

2. $P(7, 4) = \frac{7!}{(7-4)!} = \frac{7!}{3!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840.$

3. $P(5, 5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = \frac{5!}{1} = 120.$ ■

Exercise 153 *Evaluate* $P(6, 4), P(6, 6), P(6, 2)$ *and* $P(6, 1)$.

Exercise 154 1. *How many different* 3*-permutations are there of a set of 5 objects?*

2. *How many different* 2*-permutations are there of a set of 7 objects?*

Example 130 *1. In how many different ways can three of the letters of the word "BYTES" be chosen and written in a row?*

2. In how many ways can this arrangement be performed if the first letters must be B?

Solution:

1. $P(5,3) = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5 \cdot 4 \cdot 3 = 60.$
2. The number is the same as the number of 2-permutations of the remaining 4 letters. Thus, it is the number $P(4,2) = \frac{4!}{(4-2)!} = \frac{4!}{2!} = 4 \cdot 3 = 12.$ ■

Exercise 155 *1. In how many different ways can three of the letters of the word "ALGORITHM" be selected and written in a row?*

2. In how many different ways can five of the letters of the word "ALGORITHM" be selected and written in a row?

3. In how many different ways can five of the letters of the word "ALGORITHM" be selected and written in a row if the first letter must be A? In how many different ways can five of the letters of the word "ALGORITHM" be selected and written in a row if the first two letters must be TH?

Example 131 *Prove that, for all integers $n \geq 2$,*

$$P(n,2) + P(n,1) = n^2.$$

Solution:

$$\begin{aligned} P(n,2) + P(n,1) &= \frac{n!}{(n-2)!} + \frac{n!}{(n-1)!} \\ &= n(n-1) + n \\ &= n^2 - n + n \\ &= n^2. \end{aligned}$$

■

Exercise 156 *1. Prove that, for all $n \geq 2$, $P(n+1,3) = n^3 - n$.*

2. Prove that, for all integers $n \geq 2$, $P(n+1,2) - P(n,2) = 2P(n,1)$.

3. Prove that, for all integers $n \geq 3$, $P(n+1,3) - P(n,3) = 3P(n,2)$.

4. Prove that, for all integers $n \geq 2$, $P(n,1) = P(n,n-1)$.

5.4 The Addition Rule

Suppose that a finite set A equals the union of k mutually disjoint subsets $A_1, A_2, \ldots, A_k$. Then, the **Addition Rule** states that

$$n(A) = n(A_1) + n(A_2) + \ldots + n(A_k).$$

Example 132 *A computer access code word consists of from one to three letters chosen from the 26 in the alphabet with repetitions allowed. How many different code words are possible?*

Solution:
The set of all code words A is the disjoint union of the set A_1 of code words of length 1, the set A_2 of code words of length 2 and the set A_3 of code words of length 3. The number of code words in each of these three sets may be determined by using the multiplication principle. We have

$$n(A_1) = 26, n(A_2) = 26^2, n(A_3) = 26^3.$$

Therefore, by the Addition rule,

$$n(A) = n(A_1) + n(A_2) + n(A_3) = 26 + 26^2 + 26^3.$$

■

Example 133 *How many three-digit integers (integers from 100 to 999 inclusive) are divisible by 5?*

Solution:
The set A of all three-digit integers that are divisible by 5 is the disjoint union of the set A_1 of all three-digit integers that end in 0 and of the set A_2 of all three-digit integers that end in 5. One may now use the multiplication principle to determine the number of elements in each of these sets

$$n(A_1) = 9 \cdot 10 \cdot 1 = 90, n(A_2) = 9 \cdot 10 \cdot 1 = 90.$$

Hence, by the Addition rule,

$$n(A) = n(A_1) + n(A_2) = 90 + 90 = 180.$$

■

If A is a finite set and B is a subset of A, then the **Difference Rule** states that

$$n(A - B) = n(A) - n(B).$$

Example 134 *Determine the number of 4-symbol PINs formed out of the 26 letters of the alphabet and the 10 digits that contain at least one repeated symbol.*

Solution:

Let A denote the set of all 4-symbol PINs and B be the set of all 4-symbol PINs that contain no repeated digits. We have, by the multiplication principle

$$n(A) = 36^4, n(B) = 36 \cdot 35 \cdot 34 \cdot 33.$$

Thus, by the Difference rule, we get

$$n(A - B) = n(A) - n(B) = 36^4 - 36 \cdot 35 \cdot 34 \cdot 33.$$

But $A - B$ is exactly the set of all 4-symbol PINs that contain at least one repeated symbol.
■

Example 135 *An experiment consists of rolling 3 dice.*

1. *How many possible outcomes are there in this experiment?*
2. *How many outcomes are there in which at least one 5 appears?*

Solution:

1. By the multiplication principle, the set A of all outcomes contains $n(A) = 6^3$ elements.
2. By the multiplication principle the set B of all outcomes in which no 5 appears has $n(B) = 5^3$ elements. Therefore the set $A - B$ of all outcomes in which at least one 5 appears has, by the Difference rule,

$$n(A - B) = n(A) - n(B) = 6^3 - 5^3$$

elements. ■

Exercise 157 *1. How many integers from 1 through 100,000 contain the digit 6 exactly once?*

2. *How many integers from 1 through 100,000 contain the digit 6 at least once?*
3. *How many integers from 1 through 100,000 contain the digit 6 at least twice?*

Exercise 158 *Six new employees, two of whom are married to each other, are two be assigned six desks that are lined up in a row. In how many ways may the desks be assigned so that the married couple will have two non-adjacent desks?*

Exercise 159 *Consider strings of length 10 over the alphabet $\{a, b, c, d\}$. How many of these contain at least one pair of consecutive characters that are the same?*

The Inclusion-Exclusion Principle for Two or Three Sets:
If A, B and C are any finite sets, then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

and

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C).$$

Example 136 1. *How many integers from 1 through 1,000 are multiples of 3 or multiples of 5?*

2. *How many integers from 1 through 1,000 are neither multiples of 3 not multiples of 5?*

Solution:

1. Denote by A the set of all integers from 1 to 1,000 that are multiples of 3 and by B the set of all those integers from 1 to 1,000 that are multiples of 5. Then $A \cup B$ is the set of all the integers from1 to 1,000 that are multiples of 3 or multiples of 5 and $A \cap B$ is the set of all integers from 1 to 1,000 that are multiples of 3 and multiples of 5, i.e., that are multiples of 15. We may use the techniques we saw in the first section of the chapter to calculate $n(A), n(B)$ and $n(A \cap B)$. We have for $n(A)$:

$$3, 6, 9, \ldots, 999$$

which is the same as

$$3 \cdot 1, 3 \cdot 2, 3 \cdot 3, \ldots, 3 \cdot 333$$

whence $n(A) = 333$. For $n(B)$,

$$5, 10, 15, \ldots, 1,000$$

which is the same as

$$5 \cdot 1, 5 \cdot 2, 5 \cdot 3, \ldots, 5 \cdot 200,$$

whence $n(B) = 200$. Finally, for $n(A \cap B)$,

$$15, 30, 45, \ldots, 990$$

which is the same as

$$15 \cdot 1, 15 \cdot 2, 15 \cdot 3, \ldots, 15 \cdot 66$$

i.e., $n(A \cap B) = 66$. Now apply the Inclusion-Exclusion Principle for two Sets to obtain

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 333 + 200 - 66 = 467.$$

2. The set of all numbers from 1 to 1,000 that are neither multiples of 3 nor multiples of 5 is the set $A^c \cap B^c$ which is the same as the set $(A \cup B)^c$. Then, by the difference formula,
$$n((A \cup B)^c) = 1000 - n(A \cup B) = 1000 - 467 = 533.$$

■

Exercise 160 *1. how many integers from 1 through a 1,000 are multiples of 4 or multiples of 7?*

2. How many integers from 1 through a 1,000 are neither multiples of 4 nor multiples of 7?

Exercise 161 *1. how many integers from 1 through a 1,000 are multiples of 2 or multiples of 9?*

2. How many integers from 1 through a 1,000 are neither multiples of 2 nor multiples of 9?

Example 137 *The food service at your school has conducted a survey for the tastes of the students using the service. Of the 100 students that responded to the survey, 60 like rice, 36 like spaghetti, 52 like pizza, 18 like both rice and spaghetti, 32 like rice and pizza, 16 like spaghetti and pizza and 94 like at least one of the three foods.*

1. *How many students do not like either of the three foods?*
2. *How many students like all three of the foods?*
3. *How many like rice and spaghetti but not pizza? How many rice but neither spaghetti nor pizza?*

Solution:

1. By the difference rule, the number of students that like neither of the three foods equals the number of respondents minus the number of all those that liked at least one of the foods:
$$100 - 94 = 6.$$

2. Let R, S, P denote, respectively, the sets of students that liked, rice, spaghetti and pizza. Then, by the Inclusion-Exclusion Principle for Three sets, we get
$$n(R \cup S \cup P) = n(R) + n(S) + n(P) - n(R \cap S) - n(R \cap P) - n(S \cap P) + n(R \cap S \cap P)$$
Thus
$$94 = 60 + 36 + 52 - 18 - 32 - 16 + n(R \cap S \cap P),$$
whence
$$n(R \cap S \cap P) = 12.$$
Thus 12 students liked all three foods.

3. To answer these two questions construct the Venn diagram for the three sets R, S and P. Then fill-in the regions with the appropriate numbers. The regions should contain the numbers given in the following table:

region	number
$R \cap S \cap P$	12
$R \cap S \cap P^c$	6
$R \cap S^c \cap P^c$	22
$R \cap S^c \cap P$	20
$R^c \cap S \cap P$	4
$R^c \cap S \cap P^c$	14
$R^c \cap S^c \cap P^c$	6
$R^c \cap S^c \cap P$	16

Then $n(R \cap S \cap P^c) = 6$ and $n(R \cap S^c \cap P^c) = 22$. ■

Exercise 162 *a market research project studied student readership of certain news magazines by asking students to place checks underneath the names of all news magazines they read occasionally. Out of 100 students, it was found that 28 checked* Time*, 26 checked* Newsweek*, 14 checked* U.S. News and World report*, 8 checked both* Time *and* Newsweek*, 4 checked both* time *and* U.S. News*, 3 checked both* Newsweek *and* U.S. News *and 2 checked all three.*

1. *How many students checked at least one of the magazines?*
2. *How many checked none of the magazines?*
3. *How many read* Time *and* Newsweek *but not* U.S. News*?*

Exercise 163 *A study was conducted to determine the efficacy of three drugs, A, B and C, in relieving headache pain. Over the period of the study, 40 subjects were given the chance to use all three drugs. The following results were obtained: 23 reported relief from A, 18 reported relief from B, 31 reported relief from C. 11 reported relief from both A and B, 19 relief from both A and C and 14 reported relief from both B and C. Finally, 37 reported relief from at least one of the drugs.*

1. *How many people got relief from none of the drugs?*
2. *How many people got relief from all three of the drugs?*
3. *How many got relief from drug A only?*

Exercise 164 *How many positive integers less than 1,000 have no common factors with 1,000?*

Exercise 165 *How many integers from 1 through 999,999 contain each of the digits 1,2 and 3 at least once?*

5.5 Combinations

Given a set S with n elements, how many subsets of size r can be chosen from S? This number is the same as the number of distinct subsets of size r that S has. Each of these subsets is called an r-**combination** of S. More precisely, let n, r be two distinct nonegative integers with $r \leq n$. An r-**combination** of a set of n elements is a subset containing r of the n elements. The symbol $\binom{n}{r}$, read "n choose r", denotes the number of subsets of size r (r-combinations) that can be chosen from a set of n elements.

Example 138 *Let $S = \{$Al, Bob, Chris, Dan$\}$. Each committe consisting of three of the four people in S is a 3-combination of S.*

1. *List all 3-combinations of S.*
2. *What is $\binom{4}{3}$?*

Solution:

1. Each 3-combination of S is a subset of S of size 3. Each of these subsets may be obtained by omitting one of the elements of S. Thus, the four 3-combinations are

$$\{\text{Bob, Chris, Dan}\}, \{\text{Al, Chris, Dan}\}, \{\text{Al, Bob, Dan}\}, \{\text{Al, Bob, Chris}\}.$$

2. The number $\binom{4}{3}$ is by definition the number of 3-combinations of a set of 4 elements. Therefore $\binom{4}{3} = 4$. ■

Example 139 *How many 2-element subsets does the set $\{0, 1, 2, 3\}$ have?*

Solution:
We list the subsets

$$\{0, 1\}, \{0, 2\}, \{0, 3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}.$$

Hence there are 6 2-element subsets of $\{0, 1, 2, 3\}$. This also shows that $\binom{4}{2} = 6$. ■

Exercise 166 *List all 3-combinations of the set $\{x_1, x_2, x_3, x_4, x_5\}$. Deduce the value of $\binom{5}{3}$.*

How can we compute the number $\binom{n}{r}$ of r-combinations of a set with n elements for arbitrary n and r? the secret is to relate the r-combinations to the r-permutations for the number of which we already have the formula $P(n, r) = \frac{n!}{(n-r)!}$. Here is a description on how this may be accomplished. Suppose we want to compute the number $P(n, r)$ of all r-permutations of a set S with n elements. We consider this as the main task and subdivide it into two subtasks:

Subtask 1 Choose r elements out of the set of n elements.

Subtask 2 Order the chosen r elements to form an r permutation of the set S.

Subtask 1 may be accomplished in $\binom{n}{r}$ by the definition of $\binom{n}{r}$. Subtask 2 may be accomplished in $P(r,r)$ ways by the definition of $P(r,r)$. The complete task may be accomplished in $P(n,r)$ ways by the definition of $P(n,r)$. Therefore, the multiplication principle gives:

$$P(n,r) = \binom{n}{r} P(r,r).$$

But we know that $P(n,r) = \frac{n!}{(n-r)!}$ and that $P(r,r) = \frac{r!}{(r-r)!}$. Therefore we have

$$\frac{n!}{(n-r)!} = \binom{n}{r} \frac{r!}{(r-r)!}$$

whence, since $(r-r)! = 0! = 1$, we get

$$\binom{n}{r} = \frac{\frac{n!}{(n-r)!}}{\frac{r!}{(r-r)!}} = \frac{n!}{r!(n-r)!}.$$

Exercise 167 *Write an equation relating $P(7,2)$ with $\binom{7}{2}$.*

Exercise 168 *Write an equation relating $P(8,5)$ with $\binom{8}{5}$.*

Example 140 *Compute $\binom{8}{5}$.*

Solution:
We have

$$\binom{8}{5} = \frac{8!}{5!(8-5)!} = \frac{8 \cdot 7 \cdot \ldots \cdot 1}{5 \cdot 4 \cdot \ldots \cdot 1 \cdot 3 \cdot 2 \cdot 1} = \frac{8 \cdot 7 \cdot 6}{6} = 8 \cdot 7 = 56.$$

■

Exercise 169 *Compute $\binom{5}{0}$, $\binom{5}{3}$, $\binom{5}{4}$ and $\binom{5}{5}$.*

Example 141 *Choose five members from a group of twelve to work as a team on a special project. How many distinct five person teams can be chosen?*

Solution:
The number of 5-member teams is the number of 5-combinations of a set of 12 elements. Thus, the number of different 5-member teams is $\binom{12}{5}$. ■

Exercise 170 *A student council consists of 15 students. In how many ways can a committee of six be selected from the membership of the council?*

Example 142 *Suppose that a team of five is to be formed out of a group of twelve people. Two members of the group insist on working as a pair, i.e., the team is going to contain either both or neither. How many five-person teams can be formed?*

Solution:

The set A of all teams of five members that can be formed is the disjoint union of the set A_1 of all five member teams that contain both persons and the set A_2 of all teams of five members that contain neither of the two persons. Thus, by the Addition Rule

$$n(A) = n(A_1) + n(A_2).$$

Now, the number of all five-member teams that contain both persons are as many as the 3-combinations of the set of the remaining people that has 10 elements (Just choose the remaining 3 members of the team from the remaining 10 people.). Thus $n(A_1) = \binom{10}{3}$. On the other hand, the number of all five-member teams that contain neither person are as many as the 5-combinations of the set of the remaining people that has 10 elements (Just choose all 5 members of the team from the set of the remaining 10 people.). Hence we have $n(A_2) = \binom{10}{5}$. Therefore, we have

$$n(A) = n(A_1) + n(A_2) = \binom{10}{3} + \binom{10}{5} = \frac{10!}{3! \cdot 7!} + \frac{10!}{5! \cdot 5!}.$$

■

Exercise 171

1. *A student council consists of 15 students. Two of the council members have the same major and they are not allowed to serve together on a committee. In how many ways can a committee of six be selected from the membership of the council?*

2. *Two council members always insist on serving on committees together. If they do not serve together then they will not serve at all. In how many ways can a committee of six be formed from the membership of the council?*

Example 143 *Suppose that a group of twelve people contains five men and seven women.*

1. *How many five person teams can be chosen that consist of three men and two women?*

2. *How many five-person teams contain at least one man?*

3. *How many five-person teams contain at most one man?*

Solution:

1. Divide the task of choosing a team of five persons three of which are men and two women in the following two subtasks:

Subtask 1 Choose the men.

Subtask 2 Choose the women.

The first subtask may be accomplished in $\binom{5}{3}$ ways and the second subtask in $\binom{7}{2}$ ways. Thus, by the multiplication principle, the whole task may be accomplished in $\binom{5}{3} \cdot \binom{7}{2}$ ways.

2. By the Difference rule, the number of teams that contain at leat one man are the total number of five-member teams minus the number of five-member teams that contains no man. The total number of five-member teams is $\binom{12}{5}$. The number of five-member teams that are entirely formed out of women is equal to $\binom{7}{5}$. Hence the number of teams that contain at least one man is $\binom{12}{5} - \binom{7}{5}$.

3. These are the teams that contain either no man at all ar exactly one man. The number of the teams that contain no man is $\binom{7}{5}$ and the number of the teams that contain exactly one man are $\binom{5}{1} \cdot \binom{7}{4}$. Hence the number of teams that contain at most one man is $\binom{7}{5} + \binom{5}{1} \cdot \binom{7}{4}$. ■

Exercise 172 *A student council consists of 15 students, eight men and seven women.*

1. *How many committees of six contain three men and three women?*

2. *How many committees of six contain at least one woman?*

3. *If the council consists of three freshmen, four sophomores, three juniors and five seniors. How many committees of eight contain two representatives from each class?*

Example 144 *How many eight-bit strings have exactly three 1's?*

Solution:

Think of the eight positions as eight spots, three out of which are to be chosen and filled with 1's. This the number of strings that contain exactly three 1's is the number of 3-combinations of a set of 8 elements. This is $\binom{8}{3}$. ■

Exercise 173 *An instructor gives an exam with fourteen questions. students are allowed to choose any ten to answer.*

1. *How many different choices of ten questions are there?*

2. *Suppose six questions require proof and eight do not.*

   (a) *How many groups of ten questions contain four that require proof and six that do not?*

   (b) *How many groups of ten questions contain at least one that requires proof?*

   (c) *How many groups of ten questions contain at most three that require proof?*

3. *Suppose the exam instructions specify that at most one of questions 1 and 2 maybe included among the ten. How many choices of ten questions are there?*

4. *Suppose the exam instructions specify that either both questions 1 and 2 are to be included among the ten or neither is to be included. How many different choices of ten questions are there?*

Exercise 174 *An all-male club is considering opening its membership to women. In a preliminary survey on the issue, 19 of the 30 members favored admitting women and 11 did not. A committee of six is to be chosen to give further study to the issue.*

1. *How many committees of six can be formed from the club membership?*

2. *ow many of the committees will contain at least three men who, in the preliminary survey, favored opening the membership to women?*

Exercise 175 *A coin is tossed ten times. In each case the outcome H or T is recorded.*

1. *How many possible outcomes are there?*

2. *In how many of these outcomes there are exactly five heads obtained?*

3. *In how many of the outcomes there are at least nine heads?*

4. *In how many outcomes there is at least one head?*

5. *In how many outcomes there is at most one head?*

Chapter 6

Basic Probability

Chapter 7

Graphs

Chapter 8

Finite State Automata

Chapter 9

Basic Number Theory

Chapter 10

The Principle of Mathematical Induction

Chapter 11

Number Systems: Complex Numbers

Chapter 12

Geometry