EXAM 1: SOLUTIONS - MATH 110
INSTRUCTOR: George Voutsadakis

Problem 1 *What is the smallest natural number that has three different factors? (Of course* ± 1 *are not counted as valid factors.)*

Solution:

First, let's look at the factors of the first few numbers to understand the meaning of the problem.

number	its factors
1	1
2	$1, 2$
3	$1, 3$
4	$1, 2, 4$
5	$1, 5$
6	$1, 2, 3, 6$
$\vdots$	

Thus, if 1 had not been excluded from our considerations, then 4 would be the smallest natural number with three different factors; the factors $1, 2$ and 4. Since 1 is not allowed, the first such number in our list is 6 who has three different factors except for 1, namely the factors $2, 3$ and 6. ■

Problem 2 *Take a line segment of length* 1*. Divide it into three equal pieces and take out the middle third. Then divide the remaining two pieces into 3 equal pieces each and take out the middle thirds. Now you have 4 pieces remaining. Divide each to three pieces and take out the middle thirds. Continue in this way indefinitely.*

1. *What would be the total length of the remaining pieces after* $1, 2, 3, 4, 5$ *iterations of this process?*
2. *Do you see the pattern? What would be the total length of the remaining pieces after* n *iterations?*
3. *What would be the final total length of the remaining pieces if this process is repeated infinitely many times?*

Solution:

First draw some pictures to get a feel for the process. Note that after 1 step there are 2 pieces each of length $\frac{1}{3}$ so that the total length remaining is $\frac{2}{3}$. After 2 steps there are 4 pieces remaining, each of length $\frac{1}{9}$. So the total length is $\frac{4}{9}$. After the 3rd step there are 8 pieces remaining each of length $\frac{1}{27}$, whence the total length is $\frac{8}{27}$. The first few steps are summarized below

step	number of segments remaining	length of each segment	total length
1	2	$\frac{1}{3}$	$\frac{2}{3}$
2	2^2	$(\frac{1}{3})^2$	$(\frac{2}{3})^2$
3	2^3	$(\frac{1}{3})^3$	$(\frac{2}{3})^3$
4	2^4	$(\frac{1}{3})^4$	$(\frac{2}{3})^4$
5	2^8	$(\frac{1}{3})^5$	$(\frac{2}{3})^5$
$\vdots$			

Now the pattern is clear: After n steps the total length of the remaining pieces adds up to $(\frac{2}{3})^n$. This number for large n goes to 0 and, therefore, if the process is repeated infinitely many times, then there will be no part of the original segment remaining. ■