HOMEWORK 2: SOLUTIONS - MATH 110
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Problem 1 *Suppose that you were able to take a large piece of paper of ordinary thickness and fold it in half 50 times. What is the height of the folded paper? (Look on the web for a reasonable estimate of the "ordinary" thickness.)*

Solution:

Suppose that you found that the thickness of the paper is a inches. If you fold the paper 1 time you will have 2 sheets on top of each other. If you repeat, i.e., fold it 2 times, you will have 4 sheets on top of each other. For 3 folds, you will have 8 sheets on top of each other. For 4 folds, you get a 16 sheets thick bunch. The pattern now should be becoming clear. If we fold the paper n times, we'll have a 2^n sheets thick bunch. Thus the number we were asked to compute should be

$$a \times 2^{50}.$$

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Problem 2 *Suppose that 370 people are attending a party? Are there two people in the party having the same birthday?*

Solution:

Think of the different possible birthdays as boxes: $1/1, 1/2, 1/3, \ldots, 2/1, \ldots, 12/31$. Then place each of the 370 people in the box that is labelled by his/her birthday. By the pigeon-hole principle, at least one box will have to contain at least two people. Thus, there will surely be at least two people with the same birthday at the party. ■

Problem 3 *What proportion of the first 1,000 numbers have a 3 somewhere in them? What proportion of the first 10,000 numbers have a 3? Explain why almost all million-digit numbers contain a 3.*

Solution:

It is much easier to compute the number of numbers that do not contain a 3. In the first 10 numbers 9 do not contain a 3. In the first 100 numbers 81 do not contain a 3. (This is because these numbers are formed by selecting a digit out of $\{0, 1, 2, 4, 5, 6, 7, 8, 9\}$ for the first position and another digit out of the same set for the last position and this may be done in $9 \times 9 = 81$ ways. In the first 1000 numbers, reasoning as above, there are 9^3 numbers that do not contain a 3. Now the pattern should be becoming obvious. In the first 10^n numbers there are 9^n that do not contain a 3. This shows that the number of those numbers in the first 10^n that do contain a 3 is $10^n - 9^n$. Thus, the ratio of those numbers that do contain a 3 over the total is

$$\frac{10^n - 9^n}{10^n} = \frac{10^n}{10^n} - \frac{9^n}{10^n} = 1 - (\frac{9}{10})^n.$$

Verify that this goes to 1 as n gets very large. Therefore almost all million digit numbers do contain a 3. ■

Problem 4 *By experimenting with numerous examples in search of a pattern, determine a simple formula for* $(F_{n+1})^2 + (F_n)^2$*, that is a formula for the sum of the squares of two consecutive Fibonacci numbers.*

Solution:
We have

n	0	1	2	3	4	5	6	7	8	9	10
F_n	1	1	2	3	5	8	13	21	34	55	89
F_n^2	1	1	4	9	25	64	169	441	1156	3025	7921
F_{n+1}^2	1	4	9	25	64	169	441	1156	3025	7921	
$F_n^2 + F_{n+1}^2$	2	5	13	34	89	233	610	1597	4181	10946	

From the table above, it is clear that $F_0^2 + F_1^2 = F_2, F_1^2 + F_2^2 = F_4$, $F_2^2 + F_3^2 = F_6, F_3^2 + F_4^2 = F_8^2, \ldots$. Thus, the pattern that emerges is

$$F_n^2 + F_{n+1}^2 = F_{2(n+1)}.$$

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Problem 5 *By experimenting with numerous examples in search of a pattern, determine a formula for* $F_n + L_n$*, that is a formula for the sum of a Fibonacci number and the corresponding Lucas number.*

Solution:
We have

n	0	1	2	3	4	5	6	7	8	9	10
F_n	1	1	2	3	5	8	13	21	34	55	89
L_n	2	1	3	4	7	11	18	29	47	76	123
$F_n + L_n$	3	2	5	7	12	19	31	50	81	131	212
$2F_n + F_{n-2}$			5	7	12	19	31	50	81	131	212

From the table above, it is clear that the pattern that emerges is

$$F_n + L_n = 2F_n + F_{n-2}, \quad \text{for } n \geq 2.$$

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Problem 6 *Let's start with the numbers 0,0,1 and generate future numbers in our sequence by adding up the the previous three numbers. Write out the first 15 terms in this sequence, starting with the first 1. Evaluate the values of the quotients of consecutive terms (dividing the smaller term into the larger one). Do the quotients seem to be approaching a fixed number?*

Solution:
We have

n	0	1	2	3	4	5	6	7	8
S_n	0	0	1	1	2	4	7	13	24
S_{n+1}	0	1	1	2	4	7	13	24	
$\frac{S_n}{S_{n+1}}$			1	0.5	0.5	.571428	.538461	.541666	.545454

n	9	10	11	12	13	14	15	16	17
S_n	44	81	149	274	504	927	1705	3136	
S_{n+1}	44	81	149	274	504	927	1705	3136	5768
$\frac{S_n}{S_{n+1}}$	.543209	.543624	.543795	.543650	.543689	.543695	.543686	543689	

We observe that the numbers are indeed approaching a specific number. ■