HOMEWORK 7: SOLUTIONS - MATH 110
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Problem 1 *Is the matrix* $A = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix}$ *invertible? If yes, can you find its inverse* A^{-1} *and verify that* $AA^{-1} = A^{-1}A = I$*?*

Solution:
We have $\begin{vmatrix} 1 & -2 \\ 3 & 5 \end{vmatrix} = 1 \cdot 5 - (-2) \cdot 3 = 5 + 6 = 11 \neq 0$. Hence A is invertible and

$$A^{-1} = \frac{1}{11} \begin{bmatrix} 5 & 2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ -\frac{3}{11} & \frac{1}{11} \end{bmatrix}$$

To verify that A^{-1} is the matrix computed above, we multiply A on the left and on the right by A^{-1} and check whether we get $\mathbf{I}_2$ as the result of both multiplications.

$$A \cdot A^{-1} = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ -\frac{3}{11} & \frac{1}{11} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2.$$

Similarly,

$$A^{-1} \cdot A = \begin{bmatrix} \frac{5}{11} & \frac{2}{11} \\ -\frac{3}{11} & \frac{1}{11} \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2.$$

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Problem 2 *Is the matrix* $A = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}$ *invertible? If yes, find its inverse.*

Solution:
We have $\begin{vmatrix} 2 & -1 \\ 5 & 3 \end{vmatrix} = 2 \cdot 3 - (-1) \cdot 5 = 6 + 5 = 11 \neq 0$. Hence A is invertible and

$$A^{-1} = \frac{1}{11} \begin{bmatrix} 3 & 1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{11} & \frac{1}{11} \\ -\frac{5}{11} & \frac{2}{11} \end{bmatrix}.$$

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Problem 3 *Use the* **Gauss-Jordan method** *to solve the system of equations*

$$\left\{ \begin{array}{rcl} 2x + y + z & = & 6 \\ -x + 5y - z & = & -10 \\ -2x - 3y + z & = & 2 \end{array} \right\}.$$

Solution:

We take the augmented matrix of the system and perform row operations to find the solutions:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 6 \\ -1 & 5 & -1 & -10 \\ -2 & -3 & 1 & 2 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|c} -1 & 5 & -1 & -10 \\ 2 & 1 & 1 & 6 \\ -2 & -3 & 1 & 2 \end{array}\right] \xrightarrow{r_1 \leftarrow (-1) r_1} \left[\begin{array}{ccc|c} 1 & -5 & 1 & 10 \\ 2 & 1 & 1 & 6 \\ -2 & -3 & 1 & 2 \end{array}\right] \xrightarrow{r_2 \leftarrow r_2 - 2r_1}$$

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & 10 \\ 0 & 11 & -1 & -14 \\ -2 & -3 & 1 & 2 \end{array}\right] \xrightarrow{r_3 \leftarrow r_3 + 2r_1} \left[\begin{array}{ccc|c} 1 & -5 & 1 & 10 \\ 0 & 11 & -1 & -14 \\ 0 & -13 & 3 & 22 \end{array}\right] \xrightarrow{r_2 \leftarrow \frac{1}{11} r_2} \left[\begin{array}{ccc|c} 1 & -5 & 1 & 10 \\ 0 & 1 & -\frac{1}{11} & -\frac{14}{11} \\ 0 & -13 & 3 & 22 \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 + 5r_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{6}{11} & \frac{40}{11} \\ 0 & 1 & -\frac{1}{11} & -\frac{14}{11} \\ 0 & -13 & 3 & 22 \end{array}\right] \xrightarrow{r_3 \leftarrow r_3 + 13r_2} \left[\begin{array}{ccc|c} 1 & 0 & \frac{6}{11} & \frac{40}{11} \\ 0 & 1 & -\frac{1}{11} & -\frac{14}{11} \\ 0 & 0 & \frac{20}{11} & \frac{60}{11} \end{array}\right] \xrightarrow{r_3 \leftarrow \frac{11}{20} r_3} \left[\begin{array}{ccc|c} 1 & 0 & \frac{6}{11} & \frac{40}{11} \\ 0 & 1 & -\frac{1}{11} & -\frac{14}{11} \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{r_1 \leftarrow r_1 - \frac{6}{11} r_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -\frac{1}{11} & -\frac{14}{11} \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow{r_2 \leftarrow r_2 + \frac{1}{11} r_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Hence $(x, y, z) = (2, -1, 3)$. ■

Problem 4 *Use the* **inverse matrix method** *to compute the solution of the system*

$$\left\{ \begin{array}{rcl} 2x + y & = & 5 \\ -3x + 2y & = & 3 \end{array} \right\}.$$

Solution:

We have

$$\left| \begin{array}{cc} 2 & 1 \\ -3 & 2 \end{array} \right| = 4 + 3 = 7 \neq 0.$$

Thus, the matrix $A = \left[\begin{array}{cc} 2 & 1 \\ -3 & 2 \end{array}\right]$ is invertible with inverse $A^{-1} = \frac{1}{7}\left[\begin{array}{cc} 2 & -1 \\ 3 & 2 \end{array}\right] = \left[\begin{array}{cc} \frac{2}{7} & -\frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{array}\right]$. Therefore

$$\left[\begin{array}{c} x \\ y \end{array}\right] = \left[\begin{array}{cc} \frac{2}{7} & -\frac{1}{7} \\ \frac{3}{7} & \frac{2}{7} \end{array}\right] \cdot \left[\begin{array}{c} 5 \\ 3 \end{array}\right] = \left[\begin{array}{c} 1 \\ 3 \end{array}\right],$$

i.e., $(x, y) = (1, 3)$. ■

Problem 5 *Let* $\rightarrow$ *be the connective that is defined by the following truth table*

P	Q	$P \rightarrow Q$
F	F	T
F	T	T
T	F	F
T	T	T

Use the truth table method to show that $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$ *and* $P \rightarrow Q \equiv \neg Q \rightarrow \neg P$.

Solution:
We construct the truth tables

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg(P \vee Q)$	$\neg P \wedge \neg Q$
F	F	T	T	F	T	T
F	T	T	F	T	F	F
T	F	F	T	T	F	F
T	T	F	F	T	F	F

Since the last two columns agree on all rows, we have $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$. We proceed similarly with the second logical equivalence

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$\neg Q \to \neg P$
F	F	T	T	T	T
F	T	T	F	T	T
T	F	F	T	F	F
T	T	F	F	T	T

Since the last two columns agree on all rows, we have $P \to Q \equiv \neg Q \to \neg P$. ■

Problem 6 *The logical connective* $|$, *called* **Sheffer stroke** *is defined by the following truth table*

P	Q	$P\|Q$
F	F	T
F	T	T
T	F	T
T	T	F

Create the truth table for $P|P$ *and for* $(P|Q)|(P|Q)$. *What do you observe?*

Solution:
We have

P	$P\|P$
F	T
T	F

P	Q	$P\|Q$	$(P\|Q)\|(P\|Q)$
F	F	T	F
F	T	T	F
T	F	T	F
T	T	F	T

We observe that $P|P \equiv \neg P$ and that $(P|Q)|(P|Q) \equiv P \wedge Q$. ■

Problem 7 *Use truth tables to determine whether the following argument form is valid*

$$\frac{P \to Q, Q \to P}{P \vee Q}.$$

Solution:
We have

P	Q	$P \to Q$	$Q \to P$	$P \vee Q$
F	F	T	T	F
F	T	T	F	T
T	F	F	T	T
T	T	T	T	T

Since in the first row both $P \to Q$ and $Q \to P$ are true but $P \vee Q$ is false, one may not conclude $P \vee Q$ from the assumptions $P \to Q$ and $Q \to P$. Hence, the given argument form is not valid. ■