HOMEWORK 9: SOLUTIONS - MATH 110
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Problem 1 *Let* $U = \mathbf{N}, A = \{n \in \mathbf{N} : n = 2^p 3^q$ *for some nonnegative integers* $p, q\}$ *and* $B = \{m \in \mathbf{N} : m = 6^r$ *for some nonnegative integer* $r\}$. *Is* $A = B$*? Rigorously justify your answer.*

Solution:

Note that $2 \in A$, since $2 = 2^1 \cdot 3^0$. But $2 \notin B$, since B only contains powers of 6 and 2 is not a power of 6. Hence $A \neq B$. ■

Problem 2 *Let* $U = \{a, b, c, d, e, f, g\}, A = \{a, c, e, g\}$ *and* $B = \{d, e, f, g\}$. *Find* $A \cup B, A \cap B, A - B$ *and* B^c.

Solution:

We have $A \cup B = \{a, c, d, e, f, g\}$. $A \cap B = \{e, g\}$. Also $A - B = \{a, c\}$ and $B^c = \{a, b, c\}$. ■

Problem 3 *Let* U *be the set of all car owners in the U.S. Let* F *be the set of all persons in* U *that own a Ford and* C *the set of all those that own a Chevy. Describe the members of* $F \cap C, F^c \cap C^c, C - F$ *and* F^c.

Solution:

$F \cap C$ consists of all car owners that own both a Ford and a Chevy, $F^c \cap C^c$ consists of those car owners that own neither a Ford nor a Chevy. $C - F$ is the set of all those car owners that own a Ford but do not own a Chevy. Finally F^c consists of all those car owners that do not own a Ford. ■

Problem 4 *Let* $U = \mathbf{R}, A = \{x \in \mathbf{R} : -3 \leq x < 1\}$ *and* $B = \{x \in \mathbf{R} : -3 < x < 3\}$. *Find* $A \cap B, A \cup B$ *and* $B - A$.

Solution:

We have

$$A \cap B = \{x \in \mathbf{R} : -3 < x < 1\}.$$

$$A \cup B = \{x \in \mathbf{R} : -3 \leq x < 3\}.$$

$$B - A = \{x \in \mathbf{R} : 1 \leq x < 3\}.$$

■

Problem 5 *Let* $A = \{a, b\}$ *and* $B = \{0, 1, 2\}$. *Write the sets* $A \times B, B \times (A \times B)$ *and* $(B \times B) \times A$.

Solution:

$$A \times B = \{(a,0),(a,1),(a,2),(b,0),(b,1),(b,2)\}.$$

$$\begin{aligned} B \times (A \times B) &= \{(0,(a,0)),(0,(a,1)),(0,(a,2)),(0,(b,0)),(0,(b,1)),(0,(b,2)),\\ &\quad (1,(a,0)),(1,(a,1)),(1,(a,2)),(1,(b,0)),(1,(b,1)),(1,(b,2)),\\ &\quad (2,(a,0)),(2,(a,1)),(2,(a,2)),(2,(b,0)),(2,(b,1)),(2,(b,2))\} \end{aligned}$$

$$B \times B = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(3,1),(3,2),(3,3)\}.$$

Therefore

$$\begin{aligned} (B \times B) \times A &= \{((0,0),a),((0,1),a),((0,2),a),((1,0),a),((1,1),a),\\ &\quad ((1,2),a),((2,0),a),((2,1),a),((2,2),a),\\ &\quad ((0,0),b),((0,1),b),((0,2),b),((1,0),b),((1,1),b),\\ &\quad ((1,2),b),((2,0),b),((2,1),b),((2,2),b)\} \end{aligned}$$

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Problem 6 *Do you think that for all sets A, B, C it is true that $A \cap (B - C) = (A \cap B) - (A \cap C)$? If yes, prove it. If no, find a universe U and three sets A, B, C in U, such that the above identity fails (i.e., give a* **counterexample** *to the statement).*

Solution:

The given statement is true, for all sets A, B and C. To show this we need to show both $A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$ and $(A \cap B) - (A \cap C) \subseteq A \cap (B - C)$.

First, we show that $A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$. Let x be a particular but arbitrary element of $A \cap (B - C)$. Then, by the definition of intersection, $x \in A$ and $x \in B - C$. Hence, by the definition of difference, $x \in A$ and $x \in B$ and $x \notin C$ are all true. Since $x \in A$ and $x \in B$ are both true, by the definition of intersection, $x \in A \cap B$. Also, since $x \notin C$, we must have $x \notin A \cap C$, since $A \cap C \subseteq C$. Therefore, both $x \in A \cap B$ and $x \notin A \cap C$ are true. Hence, by the definition of difference, $x \in (A \cap B) - (A \cap C)$. So every $x \in A \cap (B - C)$ is also an element in $(A \cap B) - (A \cap C)$. I.e., $A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$.

Next, we show that $(A \cap B) - (A \cap C) \subseteq A \cap (B - C)$. Let x be a particular but arbitrary element of $(A \cap B) - (A \cap C)$. Then, by the definition of difference, $x \in A \cap B$ and $A \notin A \cap C$. Thus, by the definition of intersection, $x \in A$ and $x \in B$ and at the same time $x \notin A$ or $x \notin C$. Therefore, since $x \in A$ and $x \in B$, for the or statement to be true, we must have $x \notin C$. In other words, $x \in A, x \in B$ and $x \notin C$ are all true. Thus, by the definition of difference $x \in A$ and $x \in B - C$. Thus, by the definition of intersection, $x \in A \cap (B - C)$. Thus, every element of $(A \cap B) - (A \cap C)$ is also an element of $A \cap (B - C)$. This shows that $(A \cap B) - (A \cap C) \subseteq A \cap (B - C)$.

Having shown that both $A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$ and $(A \cap B) - (A \cap C) \subseteq A \cap (B - C)$ are true, we may now conclude that $A \cap (B - C) = (A \cap B) - (A \cap C)$. ■