FINAL EXAM: SOLUTIONS - MATH 111
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Problem 1 *Find the point of intersection of the line that goes through the points* $(1,3)$ *and* $(4,9)$ *and of the line that is perpendicular to it and goes through* $(2, \frac{15}{2})$.

Solution:
The slope of the line passing through $(1,3)$ and $(4,9)$ is $m = \frac{9-3}{4-1} = 2$. Thus, its equation is given by $y - 3 = 2(x-1)$ i.e., $y = 2x+1$. The line that is perpendicular to it and passes through $(2, \frac{15}{2})$ has slope $-\frac{1}{2}$ and equation $y - \frac{15}{2} = -\frac{1}{2}(x-2)$, i.e., $y = -\frac{1}{2}x + \frac{17}{2}$. The point of intersection of these two lines is found by setting $2x - 1 = -\frac{1}{2} + \frac{17}{2}$. This gives $\frac{5}{2}x = \frac{15}{2}$, i.e., $x = 3$. Therefore $y = 2 \cdot 3 + 1 = 7$. The point of intersection is the point $(3,7)$. ■

Problem 2 *Find the domain of the function* $f(x) = \sqrt{\frac{-x+2}{x+5}}$.

Solution:
We have $(-x+2)(x+5) \geq 0$ if and only if $-(x-2)(x+5) \geq 0$ if and only if $(x-2)(x+5) \leq 0$. This holds if and only if $-5 \leq x \leq 2$. But $x+5$ appears in the denominator, whence we must also have $x \neq -5$. Therefore the domain of f is $D(f) = \{x : -5 < x \leq 2\}$. ■

Problem 3 *Find the vertex, the opening direction, the* x*- and* y*-intercepts and sketch the graph of* $f(x) = -3x^2 - 6x$.

Solution:
The vertex is at $V = (-\frac{b}{2a}, f(-\frac{b}{2a})) = (-1,3)$. The parabola opens down, the x-intercepts are $(0,0)$ and $(2,0)$ and the y-intercept is $(0,0)$. The graph would have been sketched here. ■

Problem 4 *Find the equation of the parabola that has vertex* $V = (4,1)$ *and goes through the point* $(0,2)$.

Solution:
The equation should be of the form $y = a(x-4)^2 + 1$, whence, since the parabola goes through $(0,2)$, we must have $2 = a(0-4)^2 + 1$ whence $2 = 16a + 1$ i.e., $16a = 1$, which gives $a = \frac{1}{16}$. Therefore, the parabola has equation $y = \frac{1}{16}(x-4)^2 + 1$. ■

Problem 5 *Solve the equations*

1. $5^{x^2-7} = 125^{2x}$.

2. $\log_2(x+1) - \log_2(x-5) = 2$.

Solution:

We have $5^{x^2-7} = 125^{2x}$ implies $5^{x^2-7} = (5^3)^{2x}$, whence $5^{x^2-7} = 5^{6x}$. Therefore $x^2 - 7 = 6x$, i.e., $x^2 - 6x - 7 = 0$. This yields $(x-7)(x+1) = 0$, i.e., $x = -1$ or $x = 7$.

For the second part $\log_2(x+1) - \log_2(x-5) = 2$ implies $\log_2 \frac{x+1}{x-5} = \log_2 4$. Therefore $\frac{x+1}{x-5} = 4$, whence $x + 1 = 4(x-5)$, which yields $x + 1 = 4x - 20$, i.e., $3x = 21$, or $x = 7$. This is an acceptable solution. ■

Problem 6 *Solve the following system* **by the Gauss-Jordan method**

$$\left\{ \begin{array}{rcrcrcr} x & + & y & + & z & = & 0 \\ -2x & - & y & + & z & = & 5 \\ x & + & 2y & - & 2z & = & -1 \end{array} \right\}.$$

Solution:

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ -2 & -1 & 1 & 5 \\ 1 & 2 & -2 & -1 \end{array} \right] \to \left[\begin{array}{rrr|r} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 5 \\ 0 & 1 & -3 & -1 \end{array} \right] \to \left[\begin{array}{rrr|r} 1 & 0 & -2 & -5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & -6 & -6 \end{array} \right] \to$$

$$\left[\begin{array}{rrr|r} 1 & 0 & -2 & -5 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 1 \end{array} \right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array} \right].$$

■

Problem 7 *An urn contains 3 red, 5 black, 2 white and 7 green marbles. Two marbles are drawn at random without replacement. Find the probability of*

1. *the first marble being white and the second being green,*
2. *one marble being black and one red.*

Solution:

$$P(1W \cap 2G) = P(1W)P(2G|1W) = \frac{2}{17}\frac{7}{16}.$$

$$\begin{array}{rcl} P(B \cap R) & = & P(1B \cap 2R) + P(1R \cap 2B) \\ & = & P(1B)P(2R|1B) + P(1R)P(2B|1R) \\ & = & \frac{5}{17}\frac{3}{16} + \frac{3}{17}\frac{5}{16}. \end{array}$$

■

Problem 8 *In a U.S. state, 20% of the population lives in inner cities, 35% in suburbs and 45% in rural areas. 20% of those living in inner cities receive poor medical care and the corresponding probabilities for those living in the suburbs and in rural areas are 5% and 10%, respectively. A person in the population selected at random receives satisfactory medical care. What is the probability that he came from the inner cities?*

Solution:

Let I, S, R and P denote inner cities, suburbs, rural areas and poor care, respectively. Then

$$\begin{aligned} P(I|P^c) &= \frac{P(P^c|I)P(I)}{P(P^c|I)P(I)+P(P^c|S)P(S)+P(P^c|R)P(R)} \\ &= \frac{0.8 \cdot 0.2}{0.8 \cdot 0.2+0.95 \cdot 0.35+0.9 \cdot 0.45}. \end{aligned}$$

■

Problem 9 *A committee of the United Nations consists of 6 Chinese, 5 Indian, 3 American, 2 Canadian and 4 European members.*

1. *A subcommittee of 5 is to be formed on Asian affairs. In how many ways can such a subcommittee be formed if it is to consist of 2 Chinese, 1 Indian and 2 non-Asian members?*
2. *A subcommittee of 8 is to be formed consisting of a Chairman, a Vice-Chairman, a Secretary and 5 members. In how many ways can such a subcommittee be formed?*

Solution:

1. $\binom{6}{2}\binom{5}{1}\binom{9}{2}$.
2. $20 \cdot 19 \cdot 18 \cdot \binom{17}{5}$.

■

Problem 10 *A pair of fair dice are rolled 11 times. Find the probabilities that*

1. *sum 7 appears at least once.*
2. *sum 5 appears at most twice.*

Solution:

1. Sum 7 has probability $\frac{1}{6}$ in a single roll. Thus

$$1 - \binom{11}{0}(\frac{1}{6})^0(\frac{5}{6})^{11}$$

is the probability of getting at least once a sum of 7.

2. Sum 5 has probability $\frac{1}{9}$, whence

$$\binom{11}{0}(\frac{1}{9})^0(\frac{8}{9})^{11} + \binom{11}{1}(\frac{1}{9})^1(\frac{8}{9})^{10} + \binom{11}{2}(\frac{1}{9})^2(\frac{8}{9})^9$$

is the probability of obtaining at most two sum 5's.

■