EXAM 3: SOLUTIONS - MATH 111
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Problem 1 *Solve the logarithmic equations*

1. $\log_2 (\log_2 (\log_2 x)) = 2$
2. $\log (x^{27}) = (\log x)^4$.

Solution:

1. $\log_2 (\log_2 (\log_2 x)) = 2$ implies $\log_2 \log_2 x = 2^2$, whence $\log_2 x = 2^{2^2} = 2^4 = 16$. Thus $x = 2^{16}$.
2. $\log (x^{27}) = (\log x)^4$ gives $27 \log x = (\log x)^4$, whence $(\log x)^4 - 27 \log x = 0$, and, therefore $\log x((\log x)^3 - 27) = 0$. This gives $\log x = 0$ or $\log x = 3$, whence $x = 10^0$ or $x = 10^3$, i.e., $x = 1$ or $x = 1000$.

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Problem 2 *1. Find the domain of* $f(x) = \ln \frac{x-2}{x+5}$.

2. Solve the exponential equation $2^{x^2-28} = (\frac{1}{8})^{-x}$.

Solution:

1. The domain of a logarithm imposes the expression inside the logarithm to be strictly positive, i.e., we must have $\frac{x-2}{x+5} > 0$. Now a sign table gives $x < -5$ or $x > 2$.
2. We have $2^{x^2-28} = (\frac{1}{8})^{-x}$ implies $2^{x^2-28} = (2^{-3})^{-x}$, whence $2^{x^2-28} = 2^{3x}$, i.e., $x^2 - 28 = 3x$ or $x^2 - 3x - 28 = 0$. This has the factorization $(x - 7)(x + 4) = 0$, whence $x = -4$ or $x = 7$.

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Problem 3 *A company has agreed to pay \$3 million in 5 years to settle a lawsuit. How much must they invest now in an account paying 10% compounded quarterly to have the amount available when it is due?*

Solution:
$A = P(1 + \frac{r}{m})^{mt}$ implies $P = \frac{A}{(1+\frac{r}{m})^{mt}}$, whence $P = \frac{3{,}000{,}000}{(1+\frac{0.1}{4})^{4 \cdot 5}} = \frac{3{,}000{,}000}{1.025^{20}}$. ■

Problem 4 *Your uncle, who is 50 years old, is trying to figure out how much money he'll have available for retirement. You are college educated, whereas he has only his high school degree. He comes to you for help. He says that he is going to put \$3,000 in a retirement account at the end of each semester until he reaches the age of 60 and then he will make no further deposits. If the account pays 5% interest compounded semiannually, how much will his account have when he retires at the age of 65?*

Solution:

The annuity will yield after 10 years the amount $S = 3,000\frac{(1+0.025)^{20}-1}{0.025}$. This amount will receive compounded interest for another 5 years. So the final amount will be $A = S(1+0.025)^{10} = 3,000\frac{1.025^{20}-1}{0.025}1.025^{10}$. ■

Problem 5 *Solve the following system by the* **augmented matrix** *(which is also called the* **Gauss-Jordan***)* **method***:*

$$\left\{ \begin{array}{rcrcrcr} x & - & y & + & z & = & 8 \\ 2x & + & y & - & z & = & -2 \\ -x & + & 2y & - & 3z & = & -19 \end{array} \right\}.$$

Solution:

$$\left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 2 & 1 & -1 & -6 \\ -1 & 2 & -3 & -19 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 0 & 3 & -3 & -18 \\ 0 & 1 & -2 & -11 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & -1 & 1 & 8 \\ 0 & 1 & -1 & -6 \\ 0 & 1 & -2 & -11 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & -6 \\ 0 & 0 & -1 & -5 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & -1 & -6 \\ 0 & 0 & 1 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Thus, we have the solution $(x, y, z) = (2, -1, 5)$. ■

Problem 6 *Let* $A = \left[\begin{array}{rr} 1 & 2 \\ -2 & 6 \end{array}\right], B = \left[\begin{array}{rr} -1 & 2 \\ 0 & -3 \\ 1 & 5 \end{array}\right]$.

1. *Compute* $A \cdot B$ *and* $B \cdot A$.
2. *Is* A *invertible? If yes, find* A^{-1}.

Solution:

AB cannot be computed because the dimensions do not match. $BA = \left[\begin{array}{rr} -5 & 10 \\ 6 & -18 \\ -9 & 32 \end{array}\right]$.

Finally, we have $\det(A) = 6 + 4 = 10 \neq 0$, whence A is invertible with inverse

$$A^{-1} = \frac{1}{\det(A)} \left[\begin{array}{rr} 6 & -2 \\ 2 & 1 \end{array}\right] = \left[\begin{array}{rr} 0.6 & -0.2 \\ 0.2 & 0.1 \end{array}\right]$$

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