HOMEWORK 8: SOLUTIONS - MATH 111
INSTRUCTOR: George Voutsadakis

Problem 1 *Solve the system of equations using the* **Gauss-Jordan** *method* $\left\{\begin{array}{rcl} y & = & x-1 \\ y & = & 8+z \\ z & = & -3-x \end{array}\right\}$

Solution:
The given system can be rewritten as

$$\left\{\begin{array}{rrrrrrr} x & - & y & & & = & 1 \\ & & y & - & z & = & 8 \\ x & & & + & z & = & -3 \end{array}\right\}$$

Thus, using the augmented matrix and the Gauss-Jordan gives

$$\left[\begin{array}{rrr|r} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 8 \\ 1 & 0 & 1 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 8 \\ 0 & 1 & 1 & -4 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 9 \\ 0 & 1 & -1 & 8 \\ 0 & 0 & 2 & -12 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 9 \\ 0 & 1 & -1 & 8 \\ 0 & 0 & 1 & -6 \end{array}\right] \longrightarrow \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -6 \end{array}\right].$$

Thus $(x, y, z) = (3, 2, -6)$. ■

Problem 2 *Let* $A = \left[\begin{array}{rrr} -1 & 3 & -5 \\ 2 & 6 & -1 \end{array}\right]$ *and* $B = \left[\begin{array}{rr} 1 & 0 \\ 3 & 2 \\ -1 & 7 \end{array}\right]$. *Compute* $A \cdot B$ *and* $B \cdot A$.

Solution:

$$A \cdot B = \left[\begin{array}{rrr} -1 & 3 & -5 \\ 2 & 6 & -1 \end{array}\right] \cdot \left[\begin{array}{rr} 1 & 0 \\ 3 & 2 \\ -1 & 7 \end{array}\right] = \left[\begin{array}{rr} 13 & -29 \\ 21 & 5 \end{array}\right].$$

$$B \cdot A = \left[\begin{array}{rr} 1 & 0 \\ 3 & 2 \\ -1 & 7 \end{array}\right] \cdot \left[\begin{array}{rrr} -1 & 3 & -5 \\ 2 & 6 & -1 \end{array}\right] = \left[\begin{array}{rrr} -1 & 3 & -5 \\ 1 & 21 & -17 \\ 15 & 39 & -2 \end{array}\right].$$

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Problem 3 *Let* $X = \left[\begin{array}{cc} x & 0 \\ 0 & y \end{array}\right]$. *Solve the matrix equation* $X^2 + \left[\begin{array}{rr} -14 & 0 \\ 0 & 4 \end{array}\right] = -5X$.

Solution:

$X^2 + \begin{bmatrix} -14 & 0 \\ 0 & 4 \end{bmatrix} = -5X$ implies $\begin{bmatrix} x^2-14 & 0 \\ 0 & y^2+4 \end{bmatrix} = \begin{bmatrix} -5x & 0 \\ 0 & -5y \end{bmatrix}$. This, in turn, yields the system of equations $\left\{ \begin{array}{rcl} x^2-14 & = & -5x \\ y^2+4 & = & -5y \end{array} \right\}$, i.e., $\left\{ \begin{array}{rcl} x^2+5x-14 & = & 0 \\ y^2+5y+4 & = & 0 \end{array} \right\}$, whence

$$\left\{ \begin{array}{rcl} (x+7)(x-2) & = & 0 \\ (y+4)(y+1) & = & 0 \end{array} \right\},$$

which gives ($x=-7$ or $x=2$) and ($y=-4$ or $y=-1$). Thus, the four solutions for X are $X = \begin{bmatrix} -7 & 0 \\ 0 & -4 \end{bmatrix}$ or $X = \begin{bmatrix} -7 & 0 \\ 0 & -1 \end{bmatrix}$ or $X = \begin{bmatrix} 2 & 0 \\ 0 & -4 \end{bmatrix}$ or $X = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$. ■

Problem 4 *Compute the inverses of the matrices* $A = \begin{bmatrix} 3 & -2 \\ 1 & -1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & -1 \\ -6 & -3 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{rr|rr} 3 & -2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rr|rr} 1 & -1 & 0 & 1 \\ 3 & -2 & 1 & 0 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{rr|rr} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{rr|rr} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & -3 \end{array}\right].$$

Thus $A^{-1} = \begin{bmatrix} 1 & -2 \\ 1 & -3 \end{bmatrix}$. Similarly,

$$\left[\begin{array}{rr|rr} 2 & -1 & 1 & 0 \\ -6 & -3 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rr|rr} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ -6 & -3 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rr|rr} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -6 & 3 & 1 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{rr|rr} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{6} \end{array}\right] \longrightarrow \left[\begin{array}{rr|rr} 1 & 0 & \frac{1}{4} & -\frac{1}{12} \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{6} \end{array}\right].$$

Thus $B^{-1} = \begin{bmatrix} \frac{1}{4} & -\frac{1}{12} \\ -\frac{1}{2} & -\frac{1}{6} \end{bmatrix}$. ■

Problem 5 *Find the inverse of the matrix* $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & 1 \end{bmatrix}$.

Solution:

Working as in the previous problem we get $A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & 0 & 0 \\ -\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$. ■

Problem 6 *Let A, B be sets in a universe U. Suppose that $n(A) = 10, n(B) = 15, n(A \cap B) = 3$ and $n(U) = 35$. Find is $n((A' \cap B)')$?*

Solution:

We have

region	number
$A \cap B$	3
$A \cap B'$	7
$A' \cap B$	12
$A' \cap B'$	13

Thus $n((A' \cap B)') = 35 - n(A' \cap B) = 35 - 12 = 23$. ■

Problem 7 *Suppose that* A, B *and* C *are sets in a universe* U*. If* $n(A) = 11, n(B) = 17, n(C) = 21, n(A \cap B) = 5, n(A \cap C) = 7, n(B \cap C) = 6, n(A \cap B \cap C) = 2$ *and* $n(U) = 35$*, fill in the number of elements of each region in the appropriate Venn diagram.*

Solution:

region	number
$A \cap B \cap C$	2
$A \cap B \cap C'$	3
$A \cap B' \cap C'$	1
$A \cap B' \cap C$	5
$A' \cap B \cap C$	4
$A' \cap B \cap C'$	8
$A' \cap B' \cap C'$	2
$A' \cap B' \cap C$	10

■

Problem 8 *You are a teacher of a class with 40 students. 17 of your students are doing their homework regularly. 19 of your students are getting SI help regularly and 21 of your students do well in the exams. If 8 of the students are doing homework and getting SI help, 12 of them are doing their homework and doing well in the exams and 13 are getting help and doing well in the exams, whereas only 5 of them are doing homework, getting SI help and doing well in the exams at the same time, how many of your students are not doing homework, nor are they getting SI help nor do they do well in the exams?*

Solution:

Build the Venn diagram as in the previous problem and conclude that 11 students were not doing homework, nor were they getting SI help nor did they do well in the exams. ■