EXAM 1: SOLUTIONS - MATH 215
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Problem 1 *Let P and Q be propositional forms. Define the* new connective $\Box$ *by the following truth table*

P	Q	$P\Box Q$
F	F	F
F	T	T
T	F	T
T	T	F

1. *Prove that $P\Box Q$ is equivalent to the negation of $P \Leftrightarrow Q$.*

2. *Is $P\Box Q$ equivalent to $(P \vee Q) \wedge \sim (P \wedge Q)$?*

Solution:

1. We have

P	Q	$P \Leftrightarrow Q$	$P\Box Q$	$\sim (P \Leftrightarrow Q)$
F	F	T	F	F
F	T	F	T	T
T	F	F	T	T
T	T	T	F	F

Therefore $P\Box Q \equiv \sim (P \Leftrightarrow Q)$.

2. We have

P	Q	$P \vee Q$	$P \wedge Q$	$\sim (P \wedge Q)$	$P\Box Q$	$(P \vee Q) \wedge \sim (P \wedge Q)$
F	F	F	F	T	F	F
F	T	T	F	T	T	T
T	F	T	F	T	T	T
T	T	T	T	F	F	F

Therefore $P\Box Q \equiv (P \vee Q) \wedge \sim (P \wedge Q)$. ■

Problem 2 1. *Give the definition of the converse and of the contrapositive of an implication.*

2. *Prove that the contrapositive is equivalent to the original implication.*

Solution:

1. The converse of $P \Rightarrow Q$ is $Q \Rightarrow P$. The contrapositive of $P \Rightarrow Q$ is $\sim Q \Rightarrow \sim P$.

2. We have

P	Q	$\sim Q$	$\sim P$	$P \Rightarrow Q$	$\sim Q \Rightarrow \sim P$
F	F	T	T	T	T
F	T	F	T	T	T
T	F	T	F	F	F
T	T	F	F	T	T

Therefore $P \Rightarrow Q \equiv \sim Q \Rightarrow \sim P$. ■

Problem 3 *1. Is $((P \wedge Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \wedge (Q \Rightarrow R))$ a tautology? Explain!*

2. Prove that in $\mathbb{N}$,

$$(\exists x)(x + 3 = -x + 8) \Leftrightarrow (\forall x)(x^2 + 5 \textit{ is prime}).$$

Solution:

1. Look at the case where P, Q and R are assigned the truth values T, F and F, respectively. Then $(P \wedge Q) \Rightarrow R$ is assigned the value T whereas $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ is assigned the value F. Thus, under this assignment, the implication $((P \wedge Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \wedge (Q \Rightarrow R))$ is assigned the value F. This shows that $((P \wedge Q) \Rightarrow R) \Rightarrow ((P \Rightarrow R) \wedge (Q \Rightarrow R))$ is not a tautology!

2. The only solution of the equation on the left is $x = \frac{5}{2} \notin \mathbb{N}$. Therefore $(\exists x)(x + 3 = -x + 8)$ is false in $\mathbb{N}$!

   If you let $x = 1$, we get $x^2 + 5 = 6$ which is not a prime number. Therefore $(\forall x)(x^2 + 5$ is prime$)$ is also false in $\mathbb{N}$! But $F \Leftrightarrow F = T$, whence the given biconditional is true in $\mathbb{N}$! ■

Problem 4 *1. Suppose that a, b, c are positive integers. Prove that if a divides b and a divides $b + c$, then a divides c.*

2. Prove by contradiction that there is no odd integer that can be simultaneously expressed in the forms $4j - 1$ and $4k + 1$ for integers j and k.

Solution:

1. a divides b implies there exists $k \in \mathbb{N}$, such that $b = ka$. Similarly, a divides $b+c$ implies there exists $l \in \mathbb{N}$, such that $b + c = la$. Therefore $c = (b + c) - b = la - ka = (l - k)a$, whence a divides c, as was to be shown.

2. Suppose, for the sake of obtaining a contradiction, that there exists an odd integer x that may simultaneously be written in the forms $x = 4j - 1$ and $x = 4k + 1$ for $j, k \in \mathbf{Z}$. Then $4j - 1 = 4k + 1$, whence $4j = 4k + 2$ or $2j = 2k + 1$. But this last integer is both even and odd, which is a contradiction! Thus no odd integer may be written simultaneously in the forms $4j - 1$ and $4k + 1$ for integers j, k. ■

Problem 5 *1. Prove that if* $x < 1$ *or* $x > 3$*, then* $\frac{x-1}{x-3} > 0$.

2. Prove or disprove the following quantified statement: There is a unique three-digit number whose digits have sum 8 and product 10.

Solution:

1. This is a classic example of proof by cases! First case: If $x < 1$, then also $x < 3$, whence $x - 1 < 0$ and $x - 3 < 0$. Therefore $\frac{x-1}{x-3} > 0$. Second case: If $x > 3$, then also $x > 1$, whence $x - 1 > 0$ and $x - 3 > 0$. Therefore again $\frac{x-1}{x-3} > 0$. In every case, therefore, we have $\frac{x-1}{x-3} > 0$, as required.

2. Since, for example, both 125 and 152 are three-digit numbers, such that the sum of their three digits is 8 and the product is 10, the given statement is a false statement! ■