HOMEWORK 1: SOLUTIONS - MATH 215
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Problem 1 *Make truth tables for the propositional forms* $(P \wedge Q) \vee (P \wedge R)$ *and* $(P \wedge Q) \vee (R \wedge \sim S)$.

Solution:

P	Q	R	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
F	F	F	F	F	F
F	F	T	F	F	F
F	T	F	F	F	F
F	T	T	F	F	F
T	F	F	F	F	F
T	F	T	F	T	T
T	T	F	T	F	T
T	T	T	T	T	T

P	Q	R	S	$\sim S$	$P \wedge Q$	$R \wedge \sim S$	$(P \wedge Q) \vee (R \wedge \sim S)$
F	F	F	F	T	F	F	F
F	F	F	T	F	F	F	F
F	F	T	F	T	F	T	T
F	F	T	T	F	F	F	F
F	T	F	F	T	F	F	F
F	T	F	T	F	F	F	F
F	T	T	F	T	F	T	T
F	T	T	T	F	F	F	F
T	F	F	F	T	F	F	F
T	F	F	T	F	F	F	F
T	F	T	F	T	F	T	T
T	F	T	T	F	F	F	F
T	T	F	F	T	T	F	T
T	T	F	T	F	T	F	T
T	T	T	F	T	T	T	T
T	T	T	T	F	T	F	T

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Problem 2 *Which of the following pairs of propositional forms are equivalent?*
(a) $\sim (P \wedge Q), \sim P \wedge \sim Q$ *(b)* $\sim (P \vee Q), (\sim P) \wedge (\sim Q)$.

Solution:

P	Q	$P \wedge Q$	$\sim P$	$\sim Q$	$\sim (P \wedge Q)$	$\sim P \wedge \sim Q$
F	F	F	T	T	T	T
F	T	F	T	F	T	F
T	F	F	F	T	T	F
T	T	T	F	F	F	F

The table shows that $\sim (P \wedge Q)$ and $\sim P \wedge \sim Q$ are not equivalent propositional forms.

P	Q	$P \vee Q$	$\sim P$	$\sim Q$	$\sim (P \vee Q)$	$\sim P \wedge \sim Q$
F	F	F	T	T	T	T
F	T	T	T	F	F	F
T	F	T	F	T	F	F
T	T	T	F	F	F	F

Hence $\sim (P \vee Q)$ and $(\sim P) \wedge (\sim Q)$ are indeed equivalent propositional forms. ■

Problem 3 *If P, Q and R are true while S and T are false, which of the following are true?*
(a) $((\sim P) \vee (\sim Q)) \vee ((\sim R) \vee (\sim S))$ *(b)* $((\sim T) \wedge P) \vee (T \wedge P)$.

Solution:

P	Q	R	S	$\sim P$	$\sim Q$	$\sim R$	$\sim S$	$\sim P \vee \sim Q$	$\sim R \vee \sim S$	$((\sim P) \vee (\sim Q)) \vee ((\sim R) \vee (\sim S))$
T	T	T	F	F	F	F	T	F	T	T

Thus $((\sim P) \vee (\sim Q)) \vee ((\sim R) \vee (\sim S))$ is true.

T	P	$\sim T$	$\sim T \wedge P$	$T \wedge P$	$((\sim T) \wedge P) \vee (T \wedge P)$
F	T	T	T	F	T

Thus $((\sim T) \wedge P) \vee (T \wedge P)$ is also true! ■

Problem 4 *Give a useful denial of the following statements:*
(a) We will win the first game or the second one.
(b) n is even and n is not a multiple of 5.

Solution: (a) We will win neither the first nor the second game.
(b) n is odd or n is a multiple of 5. ■

Problem 5 *P, Q and R are propositional forms, P is equivalent to Q and Q is equivalent to R. Prove that $\sim Q$ is equivalent to $\sim P$ and that $P \wedge Q$ is equivalent to $Q \wedge R$.*

Solution: We have to show that $\sim P$ and $\sim Q$ have the same truth value under the assumption that P and Q have the same truth value. We have

$$\begin{aligned} \sim P \text{ is true} \quad & \text{iff} \quad P \text{ is false} \\ & \text{iff} \quad Q \text{ is false} \\ & \text{iff} \quad \sim Q \text{ is true} \end{aligned}$$

Thus, $\sim P$ and $\sim Q$ are equivalent. Similarly, adding the assumption that Q and R are equivalent, we get

$$\begin{aligned} P \wedge Q \text{ is true} \quad & \text{iff} \quad P \text{ is true and } Q \text{ is true} \\ & \text{iff} \quad Q \text{ is true and } R \text{ is true} \\ & \text{iff} \quad Q \wedge R \text{ is true} \end{aligned}$$

whence, $P \wedge Q$ and $Q \wedge R$ are equivalent provided that P and Q are equivalent and Q and R are equivalent. ■

Problem 6 *Determine whether each of the following is a tautology, a contradiction or neither. Prove your answers.*
(a) $(P \wedge Q) \vee ((\sim P) \wedge (\sim Q))$ *(b)* $(Q \wedge \sim P) \wedge \sim (P \wedge R)$.

Solution: (a) $(P \wedge Q) \vee ((\sim P) \wedge (\sim Q))$ is neither a tautology nor a contradiction. To see this, note that the assignment of the value T to both P and Q results in $(P \wedge Q) \vee ((\sim P) \wedge (\sim Q))$ being T, whereas the assignment of the value T to P and of the value F to Q results in $(P \wedge Q) \vee ((\sim P) \wedge (\sim Q))$ being F.
(b) $(Q \wedge \sim P) \wedge \sim (P \wedge R)$ is neither a tautology nor a contradiction either. To see this, note that setting all three of P, Q and R true results in $(Q \wedge \sim P) \wedge \sim (P \wedge R)$ being false, whereas, setting P false but Q and R true results in $(Q \wedge \sim P) \wedge \sim (P \wedge R)$ being true. ■

Problem 7 *Which of the following conditional sentences are true?*
(a) If a hexagon has six sides, then the moon is made of cheese.
(b) If Euclid's birthday was April 2, then rectangles have 4 sides.
(c) $1 + 1 = 2$ *is sufficient for* $3 > 6$
(d) Horses have 4 legs whenever September 15 falls on a Saturday.

Solution: (a) T since $(T \Rightarrow F) = T$.
(b) T since $(- \Rightarrow T) = T$.
(c) F since $(T \Rightarrow F) = F$.
(d) T since $(- \Rightarrow T) = T$. ■

Problem 8 *Which of the following are true?*
(a) $7 + 5 = 12$ *iff* $1 + 1 = 2$
(b) A parallelogram has three sides iff 27 is a prime
(c) The Eiffel Tower is in Paris iff the chemical symbol for helium is H
(d) $\sqrt{10} + \sqrt{13} < \sqrt{11} + \sqrt{12}$ *iff* $\sqrt{13} - \sqrt{12} < \sqrt{11} - \sqrt{10}$.

Solution: (a) T since $(T \Leftrightarrow T) = T$.
(b) T since $(F \Leftrightarrow F) = T$.
(c) F since $(T \Leftrightarrow F) = F$.
(d) T since either both inequalities are true or both are false. ■

Problem 9 *Make truth tables for the propositional forms* $(P \wedge Q) \vee (Q \wedge R) \Rightarrow P \vee R$ *and* $[(Q \Rightarrow S) \wedge (Q \Rightarrow R)] \Rightarrow [(P \vee Q) \Rightarrow (S \vee R)]$.

Solution:

P	Q	R	$P \wedge Q$	$Q \wedge R$	$(P \wedge Q) \vee (Q \wedge R)$	$P \vee R$	$(P \wedge Q) \vee (Q \wedge R) \Rightarrow P \vee R$
F	F	F	F	F	F	F	T
F	F	T	F	F	F	T	T
F	T	F	F	F	F	F	T
F	T	T	F	T	T	T	T
T	F	F	F	F	F	T	T
T	F	T	F	F	F	T	T
T	T	F	T	F	T	T	T
T	T	T	T	T	T	T	T

Q	S	R	P	$Q \Rightarrow S$	$Q \Rightarrow R$	$P \vee Q$	$S \vee R$	$(Q \Rightarrow S) \wedge (Q \Rightarrow R)$	$(P \vee Q) \Rightarrow (S \vee R)$	
F	F	F	F	T	T	F	F	T	T	T
F	F	F	T	T	T	T	F	T	F	T
F	F	T	F	T	T	F	T	T	T	T
F	F	T	T	T	T	T	T	T	T	T
F	T	F	F	T	T	F	T	T	T	T
F	T	F	T	T	T	T	T	T	T	T
F	T	T	F	T	T	F	T	T	T	T
F	T	T	T	T	T	T	T	T	T	T
T	F	F	F	F	F	T	F	F	F	T
T	F	F	T	F	F	T	F	F	F	T
T	F	T	F	F	T	T	T	F	T	T
T	F	T	T	F	T	T	T	F	T	T
T	T	F	F	T	F	T	T	F	T	T
T	T	F	T	T	F	T	T	F	T	T
T	T	T	F	T	T	T	T	T	T	T
T	T	T	T	T	T	T	T	T	T	T

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Problem 10 *Show that the following pairs of statements are equivalent:*
(a) $P \Rightarrow (Q \vee R)$ *and* $(P \wedge \sim R) \Rightarrow Q$

(b) $(P \Rightarrow Q) \Rightarrow R$ *and* $(P \wedge \sim Q) \vee R$
(c) $P \Leftrightarrow Q$ *and* $(\sim P \vee Q) \wedge (\sim Q \vee P)$.

Solution: (a)

$$\begin{array}{rll} P \Rightarrow (Q \vee R) \text{ is false} & \text{iff} & P \text{ is true and } Q \vee R \text{ is false} \\ & \text{iff} & P \text{ is true and } Q, R \text{ are false} \\ & \text{iff} & P, \sim R \text{ are true and } Q \text{ is false} \\ & \text{iff} & P \wedge \sim R \text{ is true and } Q \text{ is false} \\ & \text{iff} & (P \wedge \sim R) \Rightarrow Q \text{ is false} \end{array}$$

(b)

$$\begin{array}{rll} (P \Rightarrow Q) \Rightarrow R \text{ is false} & \text{iff} & P \Rightarrow Q \text{ is true and } R \text{ is false} \\ & \text{iff} & (P \text{ is false or } Q \text{ is true) and } R \text{ is false} \\ & \text{iff} & (P \text{ is true and } Q \text{ is false) is false and } R \text{ is false} \\ & \text{iff} & P \wedge \sim Q \text{ is false and } R \text{ is false} \\ & \text{iff} & (P \wedge \sim Q) \vee R \text{ is false} \end{array}$$

(c)

$$\begin{array}{rll} P \Leftrightarrow Q \text{ is true} & \text{iff} & P \Rightarrow Q \text{ is true and } Q \Rightarrow P \text{ is true} \\ & \text{iff} & (P \text{ is false or } Q \text{ is true) and } (Q \text{ is false or } P \text{ is true}) \\ & \text{iff} & \sim P \vee Q \text{ is true and } \sim Q \vee P \text{ is true} \\ & \text{iff} & (\sim P \vee Q) \wedge (\sim Q \vee P) \text{ is true} \end{array}$$

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Problem 11 *Give, if possible, an example of a true conditional sentence for which (a) the converse is false (b) the contrapositive is true.*

Solution: (a) If $1 > 2$ then $1 + 1 = 2$.
(b) If $a > b$ then $a + c > b + c$. ■

Problem 12 *Give, if possible, an example of a false conditional sentence for which (a) the converse is true (b) the contrapositive is true.*

Solution: (a) If $1 + 1 = 2$, then $1 > 2$.
(b) This is not possible because the contrapositive is equivalent to the original sentence. ■

Problem 13 *Determine whether each of the following is a tautology, a contradiction or neither:*
(a) $(P \Leftrightarrow Q) \Leftrightarrow \sim (\sim P \vee Q) \vee (\sim P \wedge Q)$
(b) $(P \vee Q) \Rightarrow Q \Rightarrow P$
(c) $[P \Rightarrow (Q \wedge R)] \Rightarrow [R \Rightarrow (P \Rightarrow Q)]$.

Solution: (a) We show that the given formula is a contradiction by showing that $\sim(\sim P \vee Q) \vee (\sim P \wedge Q)$ is equivalent to $\sim(P \Leftrightarrow Q)$. We have

$$\begin{array}{rl} \sim(\sim P \vee Q) \vee (\sim P \wedge Q) & \text{iff} \quad \sim(\sim P \vee Q) \vee \sim(P \vee \sim Q) \\ & \text{iff} \quad \sim(P \Rightarrow Q) \vee \sim(Q \Rightarrow P) \\ & \text{iff} \quad \sim[(P \Rightarrow Q) \wedge (Q \Rightarrow P)] \\ & \text{iff} \quad \sim(P \Leftrightarrow Q) \end{array}$$

(b) This formula is neither a tautology nor a contradiction. Note that the assignment of true to both P and Q results in the formula being true, whereas the assignment of false to P and true to Q results in the formula being false.
(c) This formula is a tautology. To see this, we prove that no assignment results in its value being false. Suppose that such an assignment existed. Then under that assignment $R \Rightarrow (P \Rightarrow Q)$ must be false. Therefore R must be true and $P \Rightarrow Q$ must be false. But this implies that R and P must both be true and Q must be false. But, then, under this assignment, $P \Rightarrow (Q \wedge R)$ is false, whence the whole sentence is true, contrary to assumption. ■

Problem 14 *Translate the following English sentences into symbolic sentences with quantifiers. The universe for each is given in parentheses.*
(a) No right triangle is isosceles. (All triangles)
(b) All people are honest or no one is honest. (All people)
(c) For every positive real number x, there is a unique real number y such that $2^y = x$. (Real numbers)

Solution: (a) $\sim(\exists x)(x$ is right and$x$ is isosceles)
(b) $(\forall x)(x$ is honest$) \vee \sim(\exists x)(x$ is honest)
(c) $(\forall x)(x > 0 \Rightarrow (\exists! y)(2^y = x))$. ■

Problem 15 *Which of the following are true? The universe for each is given in parentheses.*
(a) $(\forall x)(x + x \geq x)$ *(Real numbers)*
(b) $(\exists x)(2x + 3 = 6x + 7)$ *(Natural numbers)*
(c) $(\exists x)(3(2 - x) = 5 + 8(1 - x))$ *(Real numbers)*
(d) $(\forall x)(x^2 + 6x + 5 \geq 0)$ *(Real numbers)*

Solution: (a) This is false. The real number $x = -1$ is, for instance, a counterexample.
(b) This is also false since the only real number that satisfies this equation is $x = -1$ which is not a natural number.
(c) This is true, since the real number $\frac{7}{5}$ satisfies the given equation.
(d) This is false. The real number $x = -3$ provides a counterexample. ■

Problem 16 *Give an English translation for each. The universe is given in parentheses.*
(a) $(\forall x)(x$ *is prime* $\wedge\, x \neq 2 \Rightarrow x$ *is odd*) *(Natural numbers)*

(b) $\sim (\exists x)(x^2 < 0)$ *(Real numbers)*
(c) $(\forall x)(x \text{ is odd} \Rightarrow x^2 \text{ is odd})$ *(Natural numbers)*

Solution: (a) Every prime natural number different from 2 is odd.
(b) The square of every real number is nonnegative.
(c) The square of every odd natural number is also odd. ■

Problem 17 *Which of the following are true for the universe of all real numbers?*
(a) $(\exists x)(\forall y)(x + y = 0)$
(b) $(\exists x)(\exists y)(x^2 + y^2 = -1)$
(c) $(\forall y)(\exists! x)(x = y^2)$
(d) $(\exists! x)(\exists! y)(\forall w)(w^2 > x - y)$

Solution: (a) This is not true, since, if we fix x, then only $y = -x$ satisfies $x + y = 0$.
(b) This is also false, since, for all reals x and y, $x^2 + y^2 \geq 0$.
(c) This is true since $f(y) = y^2$ is a function from the reals to the reals.
(d) This is false, since $(x, y) = (1, 2)$ and $(x, y) = (3, 4)$ both satisfy $(\forall w)(w^2 > x - y)$. ■

Problem 18 *Give a proof of* $(\exists! x)P(x) \Rightarrow (\exists x)P(x)$*. Then show that the converse of this conditional sentence is false.*

Solution: Suppose that $(\exists! x)P(x)$ is true. Then, there exists unique c in the universe, such that $P(c)$ is true. Hence $(\exists x)P(x)$ is also true, which proves the implication.

To see that the converse is true consider the universe of real numbers and the sentence $P(x) =$ "$x^2 - 4 = 0$". Then $(\exists x)P(x)$ is certainly true but $(\exists! x)P(x)$ is false, since both -2 and 2 satisfy the given equation. ■