HOMEWORK 2 - MATH 215

DUE DATE: When Chapter 1 has been covered!
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Read each problem very carefully before starting to solve it. A few randomly selected problems will be graded for a total of 10 points. It is necessary to show your work.

GOOD LUCK!!

1. Let x and y be integers. Prove that
   (a) if x and y are even, then $x+y$ is even.
   (b) if x and y are even, then xy is divisible by 4
   (c) if x is even and y is odd, then xy is even.

2. Let a and b be real numbers. Prove that
   (a) $|a-b| = |b-a|$ (b) $|a| \leq b$ iff $-b \leq a \leq b$.

3. Suppose a, b, c and d are positive integers. Prove that
   (a) if a is odd, then $a+1$ is even (b) if a divides b, then a divides bc.

4. Prove by cases that if n is a natural number, n^2+n+3 is odd.

5. Use the technique of working backward from the desired conclusion to prove that
   (a) if $x^3+2x^2<0$, then $2x+5<11$
   (b) if an isosceles triangle has sides of length a, b and c, where $c=\sqrt{2ab}$, then it is a right triangle.

6. Let x, y and z be integers. Write a proof by contraposition to show that
   (a) if x is odd then $x+2$ is odd
   (b) if xy is even, then either x or y is even
   (c) if 8 does not divide x^2-1, then x is even
   (d) if x does not divide yz, then x does not divide z

7. Write a proof by contraposition to show that for any real number x, if $x^3+x>0$, then $x>0$.

8. A circle has center $(2,4)$.
   (a) Prove that $(-1,5)$ and $(5,1)$ are not both on the circle.
   (b) Prove that if the radius is less than 5, then the circle does not intersect the line $y=x-6$.

9. Suppose a and b are positive integers. Write a proof by contradiction to show that
   (a) if a is odd, then $a+1$ is even
   (b) if $a-b$ is odd, then $a+b$ is odd

10. Suppose a, b, c and d are positive integers. Write a proof of each biconditional statement.
    (a) ac divides bc if and only if a divides b.
    (b) $a+1$ divides b and b divides $b+3$ if and only if $a=2$ and $b=3$.

11. Prove by contradiction that if n is a natural number, then $\frac{n}{n+1} > \frac{n}{n+2}$.

12. Prove that

    (a) there exist integers m and n such that $15m + 12n = 3$.
    (b) there do not exist integers m and n such that $12m + 15n = 1$.
    (c) if m and n are odd integers and $mn = 4k - 1$ for some integer k, then m or n is of the form $4j - 1$ for some integer j.

13. Prove that, for all integers a, b, c and d, if a divides b and a divides c, then for all integers x, y, a divides $bx + cy$.

14. Prove that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes.

15. Provide either a proof or a counterexample of each of these statements:

    (a) $(\forall x)(\exists y)(x + y = 0)$ (Universe of all reals)
    (b) $(\forall x)(\forall y)(x > 1 \wedge y > 0 \Rightarrow y^x > x)$ (Universe of all reals)
    (c) For all positive real numbers x, $x^2 - x > 0$.

16. Prove that

    (a) there is a natural number M, such that for every natural number n, $\frac{1}{n} < M$.
    (b) there is no largest natural number.

17. Prove that

    (a) for all integers n, $5n^2 + 3n + 1$ is odd
    (b) the sum of 5 consecutive integers is always divisible by 5.

18. Let l be the line $2x + ky = 3k$. prove that

    (a) if $k \neq -6$, then l does not have slope $\frac{1}{3}$.
    (b) for every real number k, l is not parallel to the x-axis.
    (c) there is a unique real number k, such that l passes through $(1, 4)$.

19. Prove that

    (a) every point on the line $y = 6 - x$ is outside the circle with radius 4 and center $(-3, 1)$.
    (b) there exists a three-digit natural number less than 400 with distinct digits such that the sum of the digits is 17 and the product of the digits is 108.

20. Prove that for all nonnegative real numbers x, $\frac{|2x-1|}{x+1} \leq 2$.

21. Let a, b, c and n be natural numbers and $\text{LCM}(a, b) = m$. Prove that

    (a) if a divides n and b divides n, then $m \leq n$.
    (b) for all natural numbers n, $\text{LCM}(an, bn) = n \cdot \text{LCM}(a, b)$.