HOMEWORK 2 SOLUTIONS - MATH 215
INSTRUCTOR: George Voutsadakis

Problem 1 *Let x and y be integers. Prove that*
(a) if x and y are even, then $x + y$ is even.
(b) if x and y are even, then xy is divisible by 4
(c) if x is even and y is odd, then xy is even.

Solution: (a) Suppose that x and y are even. Then there exist $m, n \in \mathbf{Z}$ such that $x = 2m$ and $y = 2n$. Hence $x + y = 2m + 2n = 2(m + n)$, which is also even, since $m + n$ is an integer.
(b) Suppose that x and y are even. Then, there exist $m, n \in \mathbf{Z}$, such that $x = 2m$ and $y = 2n$. Hence $xy = 2m \cdot 2n = 4(mn)$ which is an integer multiple of 4. Therefore xy is divisible by 4.
(c) Suppose that x is even and y is odd. Then, there exist $m, n \in \mathbf{Z}$, such that $x = 2m$ and $y = 2n + 1$. Hence $xy = 2m(2n + 1)$. Since $m(2n + 1)$ is an integer, xy is also an even number. ■

Problem 2 *Let a and b be real numbers. Prove that*
(a) $|a - b| = |b - a|$ (b) $|a| \leq b$ iff $-b \leq a \leq b$.

Solution: (a) We give a proof by cases. If $a \geq b$, then $a - b \geq 0$ and $b - a \leq 0$, whence $|a - b| = a - b$, whereas $|b - a| = -(b - a)$. Therefore we have

$$|a - b| = a - b = -(b - a) = |b - a|.$$

If, on the other hand, $a < b$, then $a - b < 0$ and $b - a > 0$, whence $|a - b| = -(a - b)$ and $|b - a| = b - a$. Therefore

$$|a - b| = -(a - b) = b - a = |b - a|.$$

(b) We prove first the implication from left to write, i.e., we show that if $|a| \leq b$ then $-b \leq a \leq b$. We employ again the method of proof by cases: If $a \geq 0$, then $|a| \leq b$ implies $a \leq b$, whence $a, b \geq 0$ and, therefore $-b \leq 0 \leq a \leq b$. If, on the other hand, $a < 0$, then $|a| \leq b$ implies $-a \leq b$, whence $-b \leq a$. But then $-b \leq a < 0 \leq |a| \leq b$, which again verifies the required inequalities.

Now we prove the reverse implication, i.e., that if $-b \leq a \leq b$ then $|a| \leq b$. We employ once more the method of proof by cases. If $a \geq 0$, then $a \leq b$ implies $|a| \leq b$. On the other hand, if $a < 0$, then $-b \leq a$ implies $-a \leq b$, whence $|a| \leq b$, which proves the required inequality. ■

Problem 3 *Suppose a, b, c and d are positive integers. Prove that*
(a) if a is odd, then $a + 1$ is even (b) if a divides b, then a divides bc.

Solution: (a) If a is odd, then there exists $n \in \mathbb{N}$, such that $a = 2n + 1$. hence $a + 1 = 2n + 1 + 1 = 2n + 2 = 2(n + 1)$, which shows that $a + 1$ is even.
(b) Suppose that a divides b. Then, there exists $n \in \mathbb{N}$, such that $b = an$. Hence $bc = (an)c = a(nc)$, whence a divides bc. ■

Problem 4 *Prove by cases that if n is a natural number, $n^2 + n + 3$ is odd.*

Solution: First, suppose that n is even. Then, there exists $k \in \mathbb{N}$, such that $n = 2k$. Therefore

$$n^2 + n + 3 = 4k^2 + 2k + 3 = 2(2k^2 + k + 1) + 1,$$

whence n^2+n+3 is odd. Next, suppose that n is odd, i.e., there exists $k \in \mathbb{N}$, such that $n = 2k+1$. Then

$$n^2+n+3 = (2k+1)^2+2k+1+3 = 4k^2+4k+1+2k+4 = 4k^2+6k+4+1 = 2(2k^2+3k+2)+1,$$

whence $n^2 + n + 3$ is odd again, as was to be shown. ■

Problem 5 *Use the technique of working backward from the desired conclusion to prove that*
(a) if $x^3 + 2x^2 < 0$, *then* $2x + 5 < 11$
(b) if an isosceles triangle has sides of length a, b *and* c, *where* $c = \sqrt{2ab}$, *then it is a right triangle.*

Solution: (a)

$$\begin{aligned} 2x + 5 < 11 \quad &\text{iff} \quad 2x < 6 \\ &\text{iff} \quad x < 3 \end{aligned}$$

So, it suffices to show that if $x^3 + 2x^2 < 0$, then $x < 3$.

Suppose $x^3 + 2x^2 < 0$. Then $x^2(x + 2) < 0$, whence $x + 2 < 0$, i.e., $x < -2$. But, in this case, obviously, $x < 3$.

(b)

$$\begin{aligned} \text{It is a right triangle} \quad &\text{iff} \quad c^2 = a^2 + b^2 \\ &\text{iff} \quad c^2 = (a-b)^2 + 2ab \\ &\text{iff} \quad c^2 = 2ab \\ &\text{iff} \quad c = \sqrt{2ab} \end{aligned}$$

■

Problem 6 *Let* x, y *and* z *be integers. Write a proof by contraposition to show that*
(a) if x *is odd then* $x + 2$ *is odd*
(b) if xy *is even, then either* x *or* y *is even*
(c) if 8 does not divide $x^2 - 1$, *then* x *is even*
(d) if x *does not divide* yz, *then* x *does not divide* z

Solution: (a) Suppose that $x + 2$ is not odd. Then $x + 2$ is even, i.e., there exists $k \in \mathbf{Z}$, such that $x + 2 = 2k$. Thus $x = 2k - 2$, whence $x = 2(k - 1)$, and, therefore x is even. Thus x is not odd. This shows that, if x is odd then $x + 2$ is also odd.

(b) Suppose that both x and y are odd. Then, there exist $k, l \in \mathbf{Z}$, such that $x = 2k + 1$ and $y = 2l + 1$. Therefore $xy = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1$, whence xy is not even.

(c) If x is odd, then there exists $k \in \mathbf{Z}$, such that $x = 2k + 1$. Therefore $x^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4k^2 + 4k = 8\frac{k(k+1)}{2}$. Note that $\frac{k(k+1)}{2}$ is always an integer, since either k or $k+1$ is even. Therefore 8 divides $x^2 - 1$.

(d) If x divides z, then there exists $k \in \mathbf{Z}$, such that $z = kx$. But then $yz = ykx = (yk)x$, whence x divides yz. ■

Problem 7 *Write a proof by contraposition to show that for any real number x, if $x^3 + x > 0$, then $x > 0$.*

Solution: Suppose $x \leq 0$. Then $x^3 \leq 0$, whence $x^3 + x \leq 0$. ■

Problem 8 *A circle has center $(2, 4)$.*
(a) Prove that $(-1, 5)$ and $(5, 1)$ are not both on the circle.
(b) Prove that if the radius is less than 5, then the circle does not intersect the line $y = x - 6$.

Solution: (a) It suffices to show that the distance between $(2, 4)$ and $(-1, 5)$ is not equal to the distance between $(2, 4)$ and $(5, 1)$. We have

$$d((2, 4), (-1, 5)) = \sqrt{(-1 - 2)^2 + (5 - 4)^2} = \sqrt{10}$$

whereas

$$d((2, 4), (5, 1)) = \sqrt{(5 - 2)^2 + (1 - 4)^2} = \sqrt{18}.$$

(b) Suppose that the radius is $r < 5$. Then the equation of the circle is $(x - 2)^2 + (y - 4)^2 = r^2$. Suppose that (s, t) is a point both on the circle and on the given line. Then $t = s - 6$, whence $(s - 2)^2 + (s - 6 - 4)^2 = r^2$, i.e., $s^2 - 4s + 4 + s^2 - 20s + 100 = r^2$, whence $2s^2 - 24s + 104 = r^2$, and, therefore $2s^2 - 24s + 104 - r^2 = 0$. The discriminant of this quadratic is

$$D = 24^2 - 4 \cdot 2 \cdot (104 - r^2) = 576 - 832 + 8r^2 = -256 + 8r^2 = 8(-32 + r^2).$$

For this to be nonnegative, we must have $-32 + r^2 \geq 0$, whence $r^2 \geq 32$, and, therefore $r > 5$, contrary to hypothesis. ■

Problem 9 *Suppose a and b are positive integers. Write a proof by contradiction to show that*
(a) if a is odd, then $a + 1$ is even
(b) if $a - b$ is odd, then $a + b$ is odd

Solution: (a) Suppose that a is odd and $a + 1$ is odd. Then, there exists $k \in \mathbb{N}$, such that $a = 2k + 1$, whence $a + 1 = 2k + 2 = 2(k + 1)$. Therefore $a + 1$ is both odd and even, which is a contradiction.
(b) Suppose that $a-b$ is odd and $a+b$ is even. Then, there exist $k, l \in \mathbb{N}$, such that $a-b = 2k+1$ and $a+b = 2l$. Therefore, by adding the two equations, we get $2a = 2k+2l+1$, whence $2a = 2(k+l)+1$. But this is a contradiction, since the same number cannot be both even and odd! ■

Problem 10 *Suppose a, b, c are positive integers. Write a proof of each biconditional statement.*
(a) ac divides bc if and only if a divides b.
(b) $a + 1$ divides b and b divides $b + 3$ if and only if $a = 2$ and $b = 3$.

Solution: (a) If ac divides bc, then a divides b: Suppose ac divides bc. Then, there exists $k \in \mathbb{N}$, such that $bc = kac$, whence $b = ka$, i.e., a divides b.

If a divides b, then ac divides bc: Suppose that a divides b. Then, there exists $k \in \mathbb{N}$, such that $b = ka$. Thus, $bc = kac$, whence ac divides bc.
(b) If $a = 2$ and $b = 3$, then $a + 1 = 3$ and $b + 3 = 6$, whence, obviously, $a + 1$ divides b and b divides $b + 3$.

Suppose conversely, that $a+1$ divides b and b divides $b+3$. Then, there exists $k, l \in \mathbb{N}$, such that $b = k(a+1)$ and $b+3 = lb$. Then $(l-1)b = 3$, which shows that $l = 4$ and $b = 1$, or $l = 2$ and $b = 3$. The first case yields $1 = k(a+1)$, which is not possible, since $k, a \in \mathbb{N}$, whence $b = 3$. Therefore $3 = k(a+1)$. But then $k = 1$ and $a = 2$. ■

Problem 11 *Prove by contradiction that if n is a natural number, then $\frac{n}{n+1} > \frac{n}{n+2}$.*

Solution: Suppose that $\frac{n}{n+1} \leq \frac{n}{n+2}$. Then $n(n+2) \leq n(n+1)$, whence $n^2 + 2n \leq n^2 + n$, i.e., $n \leq 0$, which contradicts $n \in \mathbb{N}$. ■

Problem 12 *Prove that*
(a) there exist integers m and n such that $15m + 12n = 3$.
(b) there do not exist integers m and n such that $12m + 15n = 1$.
(c) if m and n are odd integers and $mn = 4k - 1$ for some integer k, then m or n is of the form $4j - 1$ for some integer j.

Solution: (a) Take $m = 1$ and $n = -1$.
(b) If such integers existed, then 3 would divide $12m + 15n$, whence 3 would also divide 1, a contradiction.
(c) Suppose that m, n are odd. Then there exist $p, q \in \mathbf{Z}$, such that $m = 2p + 1$ and $n = 2q + 1$. Therefore $mn = (2p+1)(2q+1) = 4pq+2p+2q+1 = 4k-1$. This shows that $2(p+q) = -4pq+4k-2$, i.e., $p + q = -2pq + 2k - 1$. Therefore, either p or q must be odd. If p is odd, then $p = 2s + 1$, for some s, whence $m = 2p + 1 = 2(2s + 1) + 1 = 4s + 3 = 4(s + 1) - 1$. If q is odd, then $q = 2t + 1$, for some t, whence $n = 2q + 1 = 2(2t + 1) + 1 = 4t + 3 = 4(t + 1) - 1$. In both cases either m or n is in fact of the form $4j - 1$ for some integer j. ■

Problem 13 *Prove that, for all integers a, b, c and d, if a divides b and a divides c, then for all integers x, y, a divides $bx + cy$.*

Solution: Suppose a divides b and a divides c. Then, there exist $m, n \in \mathbf{Z}$, such that $b = na$ and $c = ma$. Therefore $bx + cy = nax + may = (nx + my)a$ and, hence a divides $bx + cy$. ■

Problem 14 *Prove that if every even natural number greater than 2 is the sum of two primes, then every odd natural number greater than 5 is the sum of three primes.*

Solution: Assume that every even natural number greater than 2 is the sum of two primes and suppose that n is an odd natural number greater than 5. Then, there exists $k > 2$, such that $n = 2k + 1 = 2k - 2 + 3 = 2(k - 1) + 3$. But $2(k - 1)$ is an even natural number greater than 2, whence it can be written as the sum of two primes $2(k - 1) = p + q$ by our hypothesis. Therefore $n = 2(k - 1) + 3 = p + q + 3$ is the sum of the three primes $p, q, 3$. ■

Problem 15 *Provide either a proof or a counterexample of each of these statements:*
(a) $(\forall x)(\exists y)(x + y = 0)$ (Universe of all reals)
(b) $(\forall x)(\forall y)(x > 1 \wedge y > 0 \Rightarrow y^x > x)$ (Universe of all reals)
(c) For all positive real numbers x, $x^2 - x > 0$.

Solution: (a) Given x, there exists $y = -x$, such that $x + y = 0$. So this is a true statement.
(b) This statement is false: $x = 2$ and $y = 1$ provide a counterexample.
(c) This is also a false statement. $x = \frac{1}{2}$ provides a counterexample. ■

Problem 16 *Prove that*
(a) there is a natural number M, such that for every natural number n, $\frac{1}{n} < M$.
(b) there is no largest natural number.

Solution: (a) Take $M = 2$. Then, for every $n \in \mathbb{N}$, $\frac{1}{n} \leq 1 < 2$.
(b) Suppose M is a largest natural number. Then $M < M+1$ and $M+1$ is also a natural number larger than M which contradicts the choice of M. ■

Problem 17 *Prove that*
(a) for all integers n, $5n^2 + 3n + 1$ is odd
(b) the sum of 5 consecutive integers is always divisible by 5.

Solution: (a) We use the method of proof by cases: If n is even, then there exists $k \in \mathbf{Z}$, such that $n = 2k$. Thus

$$5n^2 + 3n + 1 = 5(2k)^2 + 32k + 1 = 20k^2 + 6k + 1 = 2(10k^2 + 3k) + 1$$

which shows that $5n^2 + 3n + 1$ is odd. On the other hand, if n is odd, then there exists $k \in \mathbf{Z}$, such that $n = 2k + 1$. Therefore

$$\begin{aligned} 5n^2 + 3n + 1 &= 5(2k+1)^2 + 3(2k+1) + 1 \\ &= 20k^2 + 20k + 5 + 6k + 3 + 1 \\ &= 20k^2 + 26k + 8 + 1 \\ &= 2(10k^2 + 13k + 4) + 1, \end{aligned}$$

which again proves that $5n^2 + 3n + 1$ is odd.
(b) Let $k, k+1, k+2, k+3, k+4$ be the consecutive 5 integers. Then, we have

$$k + (k+1) + (k+2) + (k+3) + (k+4) = 5k + 10 = 5(k+2)$$

which shows that this sum is divisible by 5. ■

Problem 18 *Let l be the line $2x + ky = 3k$. prove that*
(a) if $k \neq -6$, then l does not have slope $\frac{1}{3}$.
(b) for every real number k, l is not parallel to the x-axis.
(c) there is a unique real number k, such that l passes through $(1, 4)$.

Solution: (a) We prove the contrapositive. If the slope is $\frac{1}{3}$, then we must have $-\frac{2}{k} = \frac{1}{3}$, which gives $k = -6$.
(b) By contradiction: Suppose that such a k exists. Then $-\frac{2}{k} = 0$, which is impossible.
(c) We have $2 + k = 3k$ implies $2k = 2$, i.e., $k = 1$. For this value of k, $2x + ky = 3k$ goes through the point $(1, 4)$. ■

Problem 19 *Prove that*
(a) every point on the line $y = 6 - x$ is outside the circle with radius 4 and center $(-3, 1)$.
(b) there exists a three-digit natural number less than 400 with distinct digits such that the sum of the digits is 17 and the product of the digits is 108.

Solution: (a) If the line had a point on or inside the given circle, then there would be a point of intersection of the line with the given circle. We show that this is not possible by showing the the system of equations, consisting of the equations $(x+3)^2+(y-1)^2 = 16$ and $y = 6-x$ does not have a solution. Substituting into the first equation, we get $(x+3)^2+(6-x-1)^2 = 16$, which after algebraic manipulations yields $x^2+2x+9 = 0$. It is easy to see that this quadratic has discriminant $D = -32 < 0$, whence the quadratic does not have any real solutions.
(b) There are at least two such numbers: 269 and 296. Both are less than 400 with distinct digits, whose sum of the digits is 17 and whose product is 108. ∎

Problem 20 *Prove that for all nonnegative real numbers x, $\frac{|2x-1|}{x+1} \leq 2$.*

Solution: We employ proof by cases: If $2x - 1 \geq 0$, i.e., if $x \geq \frac{1}{2}$, we have $|2x - 1| = 2x - 1$, whence

$$\begin{aligned} \tfrac{|2x-1|}{x+1} \leq 2 \quad &\text{iff} \quad \tfrac{2x-1}{x+1} \leq 2 \\ &\text{iff} \quad 2x-1 \leq 2x+2 \\ &\text{iff} \quad 0 \leq 3. \end{aligned}$$

Next suppose that $2x - 1 < 0$, i.e., $0 \leq x < \frac{1}{2}$. Then we have $|2x - 1| = -2x + 1$, whence

$$\begin{aligned} \tfrac{|2x-1|}{x+1} \leq 2 \quad &\text{iff} \quad \tfrac{-2x+1}{x+1} \leq 2 \\ &\text{iff} \quad -2x+1 \leq 2x+2 \\ &\text{iff} \quad -1 \leq 4x \\ &\text{iff} \quad -\tfrac{1}{4} \leq x. \end{aligned}$$

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Problem 21 *Let a, b, c and n be natural numbers and $\mathrm{LCM}(a, b) = m$. Prove that*
(a) if a divides n and b divides n, then $m \leq n$.
(b) for all natural numbers n, $\mathrm{LCM}(an, bn) = n \cdot \mathrm{LCM}(a, b)$.

Solution: (a) a divides n and b divides n imply that n is a common multiple of a, b. Therefore, since m is their least common multiple, m must divide n. Thus $m \leq n$.
(b) We have to show that $n \cdot \mathrm{LCM}(a, b)$ is a common multiple of an and bn and that it divides every other common multiple of an and bn.

Since $\mathrm{LCM}(a, b)$ is a common multiple of a, b, there exist $k, l \in \mathbb{N}$, such that $\mathrm{LCM}(a, b) = ka$ and $\mathrm{LCM}(a, b) = lb$. Therefore $n\mathrm{LCM}(a, b) = kna$ and $n\mathrm{LCM}(a, b) = lnb$, whence $n\mathrm{LCM}(a, b)$ is a common multiple of na and nb.

Now suppose that m is a common multiple of na and nb. Then, there exist $k, l \in \mathbb{N}$, such that $m = kna$ and $m = lnb$. Thus, $\frac{m}{n} = ka$ and $\frac{m}{n} = lb$. Therefore $\frac{m}{n}$ is a common multiple of a and b and, therefore, it is a multiple of $\mathrm{LCM}(a, b)$. I.e., there exists $j \in \mathbb{N}$, such that $\frac{m}{n} = j\mathrm{LCM}(a, b)$, whence $m = jn\mathrm{LCM}(a, b)$ and, therefore, $n\mathrm{LCM}(a, b)$ divides m. ∎