HOMEWORK 3 - MATH 215

DUE DATE: After Section 2.3 has been covered!
INSTRUCTOR: George Voutsadakis

Read each problem very carefully before starting to solve it. A few randomly selected problems will be graded for a total of 10 points. It is necessary to show your work.

GOOD LUCK!!

1. True or false?
   (a) $[2,5]=\{2,3,4,5\}$ (b) $(6,9]\subseteq[6,10)$ (c) $\{\{\emptyset\}\}\subseteq\{\emptyset,\{\emptyset\}\}$
   (d) $\{1,2\}\in\{\{1,2,3\},\{1,3\},1,2\}$ (e) $\{\{4\}\}\subseteq\{1,2,3,\{4\}\}$.

2. Give an example, if there is one, of sets A, B and C such that the following are true. If there is no example write "Not possible".
   (a) $A\subseteq B, B\subseteq C$ and $C\subseteq A$ (b) $A\subseteq B, B\not\subseteq C$ and $A\not\subseteq C$.

3. Write the power set $\mathcal{P}(X)$ for each of the sets (a) $X=\{S,\{S\}\}$ (b) $X=\{1,\{2,\{3\}\}\}$.

4. List all of the proper subsets for each of the following sets (a) $\{\emptyset,\{\emptyset\}\}$ (b) $\{0,\Delta,\Box\}$

5. Give an example, if there is one, of each of the following. If there is no example, write "Not possible".
   (a) A set A such that $\mathcal{P}(A)$ has 64 elements.
   (b) Sets A and B such that $A\subseteq B$ and $\mathcal{P}(B)\subseteq\mathcal{P}(A)$.
   (c) A set A such that $\mathcal{P}(A)=\emptyset$.
   (d) Sets A, B and C such that $A\subseteq B, B\subseteq C$ and $\mathcal{P}(A)\subseteq\mathcal{P}(C)$.

6. Prove that if $x\notin B$ and $A\subseteq B$, then $x\notin A$.

7. Let $X=\{x:P(x)\}$. Are the following statements true or false?
   (a) If $a\in X$, then $P(a)$. (b) If $P(a)$, then $a\in X$. (c) If $\sim P(a)$, then $a\notin X$.

8. Prove that $X=Y$, where $X=\{x:x\in\mathbb{R}$ and x is a solution to $x^2-7x+12=0\}$ and $Y=\{3,4\}$.

9. Prove that $X=Y$, where $X=\{x\in\mathbb{N}:x^2<30\}$ and $Y=\{1,2,3,4,5\}$.

10. Let the universe be all real numbers. Let $A=[3,8), B=[2,6], C=(1,4)$ and $D=(5,\infty)$. Find $B\cup C, A\cap B, D-A, \tilde{D}$ and $(A\cup C)-(B\cap D)$.

11. Let $U=\{1,2,3\}$ be the universe for the sets $A=\{1,2\}$ and $B=\{2,3\}$. Find $\mathcal{P}(A)\cap\mathcal{P}(B)$ and $\mathcal{P}(A)-\mathcal{P}(B)$.

12. Let A, B, C be sets.
    (a) Prove that $(A-B)-C=(A-C)-(B-C)$.
    (b) Prove that if $A\subseteq C$ and $B\subseteq C$, then $A\cup B\subseteq C$.

13. Let A, B, C, D be sets. Prove that if $A\cup B\subseteq C\cup D, A\cap B=\emptyset$ and $C\subseteq A$, then $B\subseteq D$.

14. Let A, B be sets. Prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

15. Provide counterexamples for each of the following:
(a) If $A \cap C \subseteq B \cap C$, then $A \subseteq B$.
(b) $A - (B - C) = (A - B) - C$.

16. Define the symmetric difference operation $\triangle$ on sets by $A \triangle B = (A - B) \cup (B - A)$. prove that (a) $A \triangle B = (A \cup B) - (A \cap B)$ (b) $A \triangle \emptyset = A$.

17. Find the union and intersection of each of the following families or indexed collections.
(a) Let $\mathbb{R}^+ = (0, \infty)$. For $r \in \mathbb{R}^+$, let $A_r = [-\pi, r)$, and let $\mathcal{A} = \{A_r : r \in \mathbb{R}^+\}$.
(b) For each natural number $n \geq 3$, let $A_n = [\frac{1}{n}, 2 + \frac{1}{n})$ and $\mathcal{A} = \{A_n : n \geq 3\}$.
(c) For each $n \in \mathbb{N}$, let $D_n = (-n, \frac{1}{n})$ and $\mathcal{D} = \{D_n : n \in \mathbb{N}\}$.

18. Let $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ be a family of sets and let B be a set. Prove that $B \cup \bigcap_{\alpha \in \Delta} A_\alpha = \bigcap_{\alpha \in \Delta} (B \cup A_\alpha)$.

19. Let $\mathcal{A}$ be a family of sets, and suppose $\emptyset \in \mathcal{A}$. Prove that $\bigcap_{A \in \mathcal{A}} A = \emptyset$.

20. If $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ is a family of sets and if $\Gamma \subseteq \Delta$, prove that $\bigcap_{\alpha \in \Delta} A_\alpha \subseteq \bigcap_{\alpha \in \Gamma} A_\alpha$.

21. Give an example of an indexed collection of sets $\{A_\alpha : \alpha \in \Delta\}$ such that each $A_\alpha \subseteq (0, 1)$, and for all α and $\beta \in \Delta$, $A_\alpha \cap A_\beta \neq \emptyset$ but $\bigcap_{\alpha \in \Delta} A_\alpha = \emptyset$.

22. Let $\mathcal{A}$ and $\mathcal{B}$ be two pairwise disjoint families of sets. Let $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$ and $\mathcal{D} = \mathcal{A} \cup \mathcal{B}$.
(a) Prove that $\mathcal{C}$ is a family of pairwise disjoint sets.
(b) Give an example to show that $\mathcal{D}$ need not be pairwise disjoint.
(c) Prove that if $\bigcup_{A \in \mathcal{A}} A$ and $\bigcup_{B \in \mathcal{B}} B$ are disjoint, then $\mathcal{D}$ is pairwise disjoint.