# HOMEWORK 3 - MATH 215
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**Problem 1** *True or false?*
*(a)* $[2, 5] = \{2, 3, 4, 5\}$ *(b)* $(6, 9] \subseteq [6, 10)$ *(c)* $\{\{\emptyset\}\} \subseteq \{\emptyset, \{\emptyset\}\}$
*(d)* $\{1, 2\} \in \{\{1, 2, 3\}, \{1, 3\}, 1, 2\}$ *(e)* $\{\{4\}\} \subseteq \{1, 2, 3, \{4\}\}$.

**Solution:** (a) False, $2.5 \in [2, 5]$ but $2.5 \notin \{2, 3, 4, 5\}$.
(b) This is true, since if $6 < x \leq 9$, then $6 \leq x < 10$.
(c) This is true since $\{\emptyset\} \in \{\emptyset, \{\emptyset\}\}$.
(d) False, since the set $\{1, 2\}$ is not an element of the set $\{\{1, 2, 3\}, \{1, 3\}, 1, 2\}$.
(e) This is true, since $\{4\} \in \{1, 2, 3, \{4\}\}$. ■

**Problem 2** *Give an example, if there is one, of sets $A, B$ and $C$ such that the following are true. If there is no example write "Not possible".*
*(a)* $A \subseteq B, B \subseteq C$ *and* $C \subseteq A$ *(b)* $A \subseteq B, B \not\subseteq C$ *and* $A \not\subseteq C$.

**Solution:** (a) Let $A = B = C = \emptyset$.
(b) Let $A = B = \{\emptyset\}$ and $C = \emptyset$. ■

**Problem 3** *Write the power set $\mathcal{P}(X)$ for each of the sets (a) $X = \{S, \{S\}\}$ (b) $X = \{1, \{2, \{3\}\}\}$.*

**Solution:** (a) $\mathcal{P}(X) = \{\emptyset, \{S\}, \{\{S\}\}, \{S, \{S\}\}\}$.
(b) $\mathcal{P}(X) = \{\emptyset, \{1\}, \{\{2, \{3\}\}\}, \{1, \{2, \{3\}\}\}\}$. ■

**Problem 4** *List all of the proper subsets for each of the following sets (a) $\{\emptyset, \{\emptyset\}\}$ (b) $\{0, \Delta, \square\}$*

**Solution:** (a) $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}$.
(b) $\emptyset, \{0\}, \{\Delta\}, \{\square\}, \{0, \Delta\}, \{0, \square\}, \{\Delta, \square\}$. ■

**Problem 5** *Give an example, if there is one, of each of the following. If there is no example, write "Not possible".*
*(a) A set $A$ such that $\mathcal{P}(A)$ has 64 elements.*
*(b) Sets $A$ and $B$ such that $A \subseteq B$ and $\mathcal{P}(B) \subseteq \mathcal{P}(A)$.*
*(c) A set $A$ such that $\mathcal{P}(A) = \emptyset$.*
*(d) Sets $A, B$ and $C$ such that $A \subseteq B, B \subseteq C$ and $\mathcal{P}(A) \subseteq \mathcal{P}(C)$.*

**Solution:** (a) $A = \{0, 1, 2, 3, 4, 5\}$.
(b) $A = B = \emptyset$.
(c) No such example exists because $\emptyset \in \mathcal{P}(A)$ for every set $A$.
(d) Set $A = B = C = \emptyset$. ■

**Problem 6** *Prove that if $x \notin B$ and $A \subseteq B$, then $x \notin A$.*

**Solution:** Suppose to the contrary that $x \notin B, A \subseteq B$ and $x \in A$. Since $x \in A$ and $A \subseteq B$, we get $x \in B$. But this contradicts the assumption $x \notin B$. ■

**Problem 7** *Let $X = \{x : P(x)\}$. Are the following statements true or false?*
*(a) If $a \in X$, then $P(a)$. (b) If $P(a)$, then $a \in X$. (c) If $\sim P(a)$, then $a \notin X$.*

**Solution:** (a) This is true, by the definition of $X$.
(b) This is also true by the definition of $X$.
(c) This is also true since it is the contrapositive of part (a). ■

**Problem 8** *Prove that $X = Y$, where $X = \{x : x \in \mathbb{R}$ and $x$ is a solution to $x^2 - 7x + 12 = 0\}$ and $Y = \{3, 4\}$.*

**Solution:** We first show that $Y \subseteq X$. Suppose that $x \in Y$. Then $x = 3$ or $x = 4$. Note that $3^2 - 7 \cdot 3 + 12 = 0$ and $4^2 - 7 \cdot 4 + 12 = 0$, whence $x \in X$ in both cases, and, therefore $Y \subseteq X$.

We now show that $X \subseteq Y$. Suppose that $x \in X$. Therefore $x \in \mathbb{R}$ and $x^2 - 7x + 12 = 0$, whence $(x - 3)(x - 4) = 0$. Therefore $x = 3$ or $x = 4$ and, therefore $x \in Y$. Thus, $X \subseteq Y$. ■

**Problem 9** *Prove that $X = Y$, where $X = \{x \in \mathbb{N} : x^2 < 30\}$ and $Y = \{1, 2, 3, 4, 5\}$.*

**Solution:** First, we show that $X \subseteq Y$. Suppose $x \in X$. Then $x \in \mathbb{N}$ and $x^2 < 30$. Thus $x = 1, 2, 3, 4$ or 5. Therefore $x \in Y$ and $X \subseteq Y$.

Next, we show that, conversely, $Y \subseteq X$. Suppose $x \in Y$. Then $x = 1, 2, 3, 4$ or 5. Note that $1, 2, 3, 4, 5 \in \mathbb{N}$ and $1, 4, 9, 16, 25 < 30$, whence $x \in X$ and $Y \subseteq X$. ■

**Problem 10** *Let the universe be all real numbers. Let $A = [3, 8), B = [2, 6], C = (1, 4)$ and $D = (5, \infty)$. Find $B \cup C, A \cap B, D - A, \tilde{D}$ and $(A \cup C) - (B \cap D)$.*

**Solution:** $B \cup C = (1, 6], A \cap B = [3, 6], D - A = [8, \infty), \tilde{D} = (-\infty, 5]$ and

$$(A \cup C) - (B \cap D) = (1, 8) - (5, 6] = (1, 5] \cup (6, 8).$$

■

**Problem 11** *Let $U = \{1, 2, 3\}$ be the universe for the sets $A = \{1, 2\}$ and $B = \{2, 3\}$. Find $\mathcal{P}(A) \cap \mathcal{P}(B)$ and $\mathcal{P}(A) - \mathcal{P}(B)$.*

**Solution:** we have $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$ and $\mathcal{P}(B) = \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$. So

$$\mathcal{P}(A) \cap \mathcal{P}(B) = \{\emptyset, \{2\}\} \text{ and } \mathcal{P}(A) - \mathcal{P}(B) = \{\{1\}, \{1, 2\}\}.$$

■

**Problem 12** *Let $A, B, C$ be sets.*
*(a) Prove that $(A - B) - C = (A - C) - (B - C)$.*
*(b) Prove that if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.*

**Solution:** (a) We first show that $(A - B) - C \subseteq (A - C) - (B - C)$. Suppose that $x \in (A - B) - C$. Then $x \in A - B$ and $x \notin C$. Thus $x \in A$ and $x \notin B$ and $x \notin C$. Therefore $x \in A$ and $x \notin C$ and, at the same time $x \notin B - C \subseteq B$. Therefore $x \in (A - C) - (B - C)$ and $(A - B) - C \subseteq (A - C) - (B - C)$.

Next, we show that $(A-C)-(B-C) \subseteq (A-B)-C$. Suppose that $x \in (A-C)-(B-C)$. Then $x \in A-C$ but $x \notin B-C$. Therefore $x \in A$ and $x \notin C$ and ($x \notin B$ or $x \in C$). Since $x \notin C$, this last disjunction yields $x \notin B$. Therefore $x \in (A-B)-C$ and $(A-C)-(B-C) \subseteq (A-B)-C$.
(b) Suppose that $A \subseteq C$ and $B \subseteq C$ and let $x \in A \cup B$. Then $x \in A$ or $x \in B$. If $x \in A$, then, since $A \subseteq C$, $x \in C$. If $x \in B$, then, since $B \subseteq C$, we also get $x \in C$. Therefore in each case, $x \in C$, which yields that $A \cup B \subseteq C$. ■

**Problem 13** *Let $A, B, C, D$ be sets. Prove that if $A \cup B \subseteq C \cup D, A \cap B = \emptyset$ and $C \subseteq A$, then $B \subseteq D$.*

**Solution:** Suppose $A \cup B \subseteq C \cup D, A \cap B = \emptyset$ and $C \subseteq A$ and let $b \in B$. Then $b \in A \cup B$, whence, since $A \cup B \subseteq C \cup D$, $b \in C \cup D$. Thus, either $b \in C$ or $b \in D$. If $b \in C$, then, since $C \subseteq A$, $b \in A$, which together with $b \in B$, contradicts $A \cap B = \emptyset$. Therefore $b \notin C$ and, hence $b \in D$. Thus $B \subseteq D$. ■

**Problem 14** *Let $A, B$ be sets. Prove that $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.*

**Solution:** Suppose that $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$. Then $X \in \mathcal{P}(A)$ or $X \in \mathcal{P}(B)$. If $X \in \mathcal{P}(A)$, then $X \subseteq A \subseteq A \cup B$. Thus $X \subseteq A \cup B$ and $X \in \mathcal{P}(A \cup B)$. If, on the other hand, $X \in \mathcal{P}(B)$, then $X \subseteq B \subseteq A \cup B$, whence $X \in \mathcal{P}(A \cup B)$ as well. Thus $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. ■

**Problem 15** *Provide counterexamples for each of the following:*
*(a) If $A \cap C \subseteq B \cap C$, then $A \subseteq B$.*
*(b) $A - (B - C) = (A - B) - C$.*

**Solution:** (a) Let $A = \{0\}$ and $B = C = \emptyset$. Then $A \cap C = \emptyset \subseteq \emptyset = B \cap C$, but, obviously, $A = \{0\} \not\subseteq \emptyset = B$.
(b) Set $A = B = C = \{0\}$. Then $A - (B - C) = \{0\}$, whereas $(A - B) - C = \emptyset$. ■

**Problem 16** *Define the symmetric difference operation $\triangle$ on sets by $A \triangle B = (A - B) \cup (B - A)$. prove that (a) $A \triangle B = (A \cup B) - (A \cap B)$ (b) $A \triangle \emptyset = A$.*

**Solution:** (a) We first show that $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$. So suppose $x \in (A - B) \cup (B - A)$. Then $x \in A - B$ or $x \in B - A$. If $x \in A - B$, Then $x \in A$ and $x \notin B$. Thus $x \in A \cup B$ and $x \notin A \cap B$. Therefore $x \in (A \cup B) - (A \cap B)$. On the other hand, if $x \in B - A$, then $x \in B$ and $x \notin A$, whence $x \in A \cup B$ and $x \notin A \cap B$. Therefore $x \in (A \cup B) - (A \cap B)$ and, therefore $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$.
Next, we show that $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$. Suppose that $x \in (A \cup B) - (A \cap B)$. Therefore $x \in A \cup B$ and $x \notin A \cap B$. Thus, $x \in A$ and $x \notin A \cap B$ or $x \in B$ and $A \notin A \cap B$. In the first case $x \in A$ and $x \notin B$, i.e., $x \in A - B$, whereas in the second case $x \in B$ and $x \notin A$, whence $x \in B - A$. Thus, in both cases $x \in (A - B) \cup (B - A)$. Therefore $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$.
(b) $A \triangle \emptyset = (A - \emptyset) \cup (\emptyset - A) = A \cup \emptyset = A$. ■

**Problem 17** *Find the union and intersection of each of the following families or indexed collections.*
*(a) Let $\mathbb{R}^+ = (0, \infty)$. For $r \in \mathbb{R}^+$, let $A_r = [-\pi, r)$, and let $\mathcal{A} = \{A_r : r \in \mathbb{R}^+\}$.*
*(b) For each natural number $n \geq 3$, let $A_n = [\frac{1}{n}, 2 + \frac{1}{n})$ and $\mathcal{A} = \{A_n : n \geq 3\}$.*
*(c) For each $n \in \mathbb{N}$, let $D_n = (-n, \frac{1}{n})$ and $\mathcal{D} = \{D_n : n \in \mathbb{N}\}$.*

**Solution:** (a) $\bigcup_{r\in\mathbb{R}^+}[-\pi, r) = [-\pi, \infty), \bigcap_{r\in\mathbb{R}^+}[-\pi, r) = [-\pi, 0]$.
(b) $\bigcup_{n\geq 3}[\frac{1}{n}, 2+\frac{1}{n}) = (0, \frac{7}{3}), \bigcap_{n\geq 3}[\frac{1}{n}, 2+\frac{1}{n}) = [\frac{1}{3}, 2]$.
(c) $\bigcup_{n\in\mathbb{N}}(-n, \frac{1}{n}) = (-\infty, 1), \bigcap_{n\in\mathbb{N}}(-n, \frac{1}{n}) = (-1, 0]$. ■

**Problem 18** *Let $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ be a family of sets and let $B$ be a set. Prove that $B \cup \bigcap_{\alpha\in\Delta} A_\alpha = \bigcap_{\alpha\in\Delta}(B \cup A_\alpha)$.*

**Solution:** We have

$$\begin{aligned} x \in B \cup \textstyle\bigcap_{\alpha\in\Delta} A_\alpha \quad & \text{iff} \quad x \in B \text{ or } x \in \textstyle\bigcap_{\alpha\in\Delta} A_\alpha \\ & \text{iff} \quad x \in B \text{ or } x \in A_\alpha, \text{ for all } \alpha \in \Delta \\ & \text{iff} \quad (x \in B \text{ or } x \in A_\alpha), \text{ for all } \alpha \in \Delta \\ & \text{iff} \quad x \in B \cup A_\alpha, \text{ for all } \alpha \in \Delta \\ & \text{iff} \quad x \in \textstyle\bigcap_{\alpha\in\Delta}(B \cup A_\alpha) \end{aligned}$$

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**Problem 19** *Let $\mathcal{A}$ be a family of sets, and suppose $\emptyset \in \mathcal{A}$. Prove that $\bigcap_{A\in\mathcal{A}} A = \emptyset$.*

**Solution:** We show the contrapositive. Suppose that $\bigcap_{A\in\mathcal{A}} A \neq \emptyset$. Then, there exists $x \in \bigcap_{A\in\mathcal{A}} A$. Therefore $x \in A$, for all $A \in \mathcal{A}$. Hence $A \neq \emptyset$, for all $A \in \mathcal{A}$. Thus, $\emptyset \notin \mathcal{A}$. ■

**Problem 20** *If $\mathcal{A} = \{A_\alpha : \alpha \in \Delta\}$ is a family of sets and if $\Gamma \subseteq \Delta$, prove that $\bigcap_{\alpha\in\Delta} A_\alpha \subseteq \bigcap_{\alpha\in\Gamma} A_\alpha$.*

**Solution:** Suppose that $\Gamma \subseteq \Delta$ and let $x \in \bigcap_{\alpha\in\Delta} A_\alpha$. Then $x \in A_\alpha$, for all $\alpha \in \Delta$. Thus, since $\Gamma \subseteq \Delta$, $x \in A_\alpha$, for all $\alpha \in \Gamma$, whence $x \in \bigcap_{\alpha\in\Gamma} A_\alpha$. Therefore $\bigcap_{\alpha\in\Delta} A_\alpha \subseteq \bigcap_{\alpha\in\Gamma} A_\alpha$. ■

**Problem 21** *Give an example of an indexed collection of sets $\{A_\alpha : \alpha \in \Delta\}$ such that each $A_\alpha \subseteq (0, 1)$, and for all $\alpha$ and $\beta \in \Delta$, $A_\alpha \cap A_\beta \neq \emptyset$ but $\bigcap_{\alpha\in\Delta} A_\alpha = \emptyset$.*

**Solution:** Suppose that $\Delta = \mathbb{N}$ and set $A_n = (0, \frac{1}{n})$, for all $n \in \mathbb{N}$. Then, for all $m, n \in \mathbb{N}$, $(0, \frac{1}{n}) \cap (0, \frac{1}{m}) = (0, \frac{1}{\max(m,n)}) \neq \emptyset$, whereas $\bigcap_{n\in\mathbb{N}}(0, \frac{1}{n}) = \emptyset$. ■

**Problem 22** *Let $\mathcal{A}$ and $\mathcal{B}$ be two pairwise disjoint families of sets. Let $\mathcal{C} = \mathcal{A} \cap \mathcal{B}$ and $\mathcal{D} = \mathcal{A} \cup \mathcal{B}$.*
*(a) Prove that $\mathcal{C}$ is a family of pairwise disjoint sets.*
*(b) Give an example to show that $\mathcal{D}$ need not be pairwise disjoint.*
*(c) Prove that if $\bigcup_{A\in\mathcal{A}} A$ and $\bigcup_{B\in\mathcal{B}} B$ are disjoint, then $\mathcal{D}$ is pairwise disjoint.*

**Solution:** (a) Let $C_1, C_2 \in \mathcal{C}$. then $C_1, C_2 \in \mathcal{A}$ and $C_1, C_2 \in \mathcal{B}$. Thus, since $\mathcal{A}$ is pairwise disjoint, $C_1 \cap C_2 = \emptyset$, whence $\mathcal{C}$ is also pairwise disjoint.
(b) Let $\mathcal{A} = \{\{0\}\}, \mathcal{B} = \{\{0, 1\}\}$. Obviously, both $\mathcal{A}$ and $\mathcal{B}$ are pairwise disjoint since they each contain only one set. But $\mathcal{D} = \mathcal{A} \cup \mathcal{B} = \{\{0\}, \{0, 1\}\}$, which is not pairwise disjoint.
(c) We show the contrapositive. Suppose that $\mathcal{A} \cup \mathcal{B}$ is not pairwise disjoint. Therefore, there exist $A \in \mathcal{A}$ and $B \in \mathcal{B}$, such that $A \cap B \neq \emptyset$. Thus, there exist $x$, such that $x \in A \in \mathcal{A}$ and $x \in B \in \mathcal{B}$. But then $x \in \bigcup_{A\in\mathcal{A}} A$ and $x \in \bigcup_{B\in\mathcal{B}} B$, whence $\bigcup_{A\in\mathcal{A}} A$ and $\bigcup_{B\in\mathcal{B}} B$ are not disjoint. ■