HOMEWORK 4 - MATH 215
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Problem 1 *Prove or find a counterexample for each:*
(a) If A and B are inductive, then $A \cup B$ is inductive.
(b) If A and B are inductive, then $A \cap B$ is inductive.

Solution: (a) Suppose that A is inductive and B is inductive. Let $n \in A \cup B$. Then $n \in A$ or $n \in B$. If $n \in A$, then, since A is inductive, $n+1 \in A$ and, therefore $n+1 \in A \cup B$. If $n \in B$, then, since B is inductive, $n+1 \in B$, whence $n+1 \in A \cup B$. Thus, in every case $n+1 \in A \cup B$ and $A \cup B$ is inductive.
(b) Suppose that $n \in A \cap B$. Then $n \in A$ and $n \in B$, whence, since both A and B are inductive, $n+1 \in A$ and $n+1 \in B$. Therefore $n+1 \in A \cap B$ and $A \cap B$ is inductive as well. ■

Problem 2 *Give an inductive definition for each:*
(a) A set formed as an arithmetic progression $\{a, a+d, a+2d, \ldots\}$.
(b) A set formed as a geometric progression $\{a, ar, ar^2, \ldots\}$.
(c) $\bigcup_{i=1}^n A_i$, for some indexed family $\{A_i : i \in \mathbb{N}\}$.
(d) The product $\prod_{i=1}^n x_i = x_1 \cdot x_2 \cdot \ldots \cdot x_n$ of n real numbers.

Solution: (a) A is the smallest set, such that: $a \in A$ and, if $x \in A$, then $x + d \in A$.
(b) A is the smallest set such that: $a \in A$ and, if $x \in A$, then $xr \in A$.
(c) $\bigcup_{i=1}^1 A_i = A_1$, and, for all $n \in \mathbb{N}$, $\bigcup_{i=1}^{n+1} A_i = \bigcup_{i=1}^n A \cup A_{n+1}$.
(d) $\prod_{i=1}^1 x_i = x_1$, and, for all $n \in \mathbb{N}$, $\prod_{i=1}^{n+1} x_i = \prod_{i=1}^n x_i \cdot x_{n+1}$. ■

Problem 3 *Use the PMI to prove the following for all natural numbers n.*
(a) $1 + 4 + 7 + \ldots + (3n-2) = \frac{1}{2}n(3n-1)$.
(b) $n^3 + 5n + 6$ is divisible by 3.
(c) $10^n + 3 \cdot 4^{n+2} + 5$ is divisible by 9.
(d) $\prod_{i=1}^n (2i-1) = \frac{(2n)!}{n!2^n}$.
(e) Using the differentiation formulas $\frac{d}{dx}(x) = 1$ and $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$, prove that for all $n \in \mathbb{N}$, $\frac{d}{dx}(x^n) = nx^{n-1}$.

Solution: (a) For $n = 1$, $1 = \frac{1}{2}1 \cdot (3 \cdot 1 - 1)$ which is true. Suppose that the given equality holds for $n = k$, i.e., that $1 + 4 + 7 + \ldots + (3k-2) = \frac{1}{2}k(3k-1)$. We now prove it for $n = k+1$:

$$\begin{aligned} 1 + 4 + 7 + \ldots + (3k-2) + (3(k+1)-2) &= \tfrac{1}{2}k(3k-1) + (3(k+1)-2) \\ &= \tfrac{3k^2 - k + 6k + 6 - 4}{2} \\ &= \tfrac{3k^2 + 5k + 2}{2} \\ &= \tfrac{(k+1)(3k+2)}{2} \\ &= \tfrac{1}{2}(k+1)(3(k+1)-1). \end{aligned}$$

(b) For $n = 1$, we have $n^3 + 5n + 6 = 12$ which is divisible by 3. Suppose that $k^3 + 5k + 6$ is divisible by 3. Then, for $n = k+1$, we have

$$\begin{aligned} (k+1)^3 + 5(k+1) + 6 &= k^3 + 3k^2 + 3k + 1 + 5k + 5 + 6 \\ &= (k^3 + 5k + 6) + 3k^2 + 3k + 6 \\ &= (k^3 + 5k + 6) + 3(k^2 + k + 2). \end{aligned}$$

The first summand is divisible by 3 by our inductive hypothesis and the second summand is obviously divisible by 3, whence the sum is also divisible by 3.
(c) For $n = 1$, $10^n + 3 \cdot 4^{n+2} + 5 = 207$, which is divisible by 9. Suppose that $10^k + 3 \cdot 4^{k+2} + 5$ is divisible by 9. Then we have

$$\begin{array}{rcl} 10^{k+1} + 3 \cdot 4^{k+1+2} + 5 & = & 10 \cdot 10^k + 3 \cdot 4 \cdot 4^{k+2} + 5 \\ & = & 10^k + 3 \cdot 4^{k+2} + 5 + 9 \cdot 10^k + 3 \cdot 3 \cdot 4^{k+2} \\ & = & 10^k + 3 \cdot 4^{k+2} + 5 + 9(10^k + 4^{k+2}), \end{array}$$

whence, since the first summand is divisible by 9 by the inductive hypothesis and the second summand is obviously divisible by 9, we get that the sum is also divisible by 9.
(d) For $n = 1$, we get $1 = \frac{2!}{1!2^1}$, which is true. Suppose that $\prod_{i=1}^{k}(2i-1) = \frac{(2k)!}{k!2^k}$. For $n = k+1$, we get

$$\begin{array}{rcl} \prod_{i=1}^{k+1}(2i-1) & = & \prod_{i=1}^{k}(2i-1) \cdot (2(k+1)-1) \\ & = & \frac{(2k)!}{k!2^k} \cdot (2k+1) \\ & = & \frac{(2k)!(2k+1)(2k+2)}{k!2^k(2k+2)} \\ & = & \frac{(2k+2)!}{k!2^k 2(k+1)} \\ & = & \frac{(2(k+1))!}{(k+1)!2^{k+1}}. \end{array}$$

(e) For $n = 1$, we get $\frac{d}{dx}(x) = 1 \cdot x^0 = 1$. Suppose that $\frac{d}{dx}(x^k) = kx^{k-1}$. Then we have

$$\begin{array}{rcl} \frac{d}{dx}(x^{k+1}) & = & \frac{d}{dx}(x^k \cdot x) \\ & = & x^k \frac{d}{dx}(x) + \frac{d}{dx}(x^k) \cdot x \\ & = & x^k + kx^{k-1}x \\ & = & x^k + kx^k \\ & = & (k+1)x^{(k+1)-1}. \end{array}$$

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Problem 4 *Use the generalized PMI to prove the following:*
(a) $(n+1)! > 2^{n+3}$ *for* $n \geq 5$.
(b) For all $n > 2$, *the sum of the angle measures of the interior angles of a convex polygon of* n *sides is* $(n-2) \cdot 180^o$.

Solution: (a) For $n = 5$, we have $(5+1)! = 720 > 256 = 2^{5+3}$. Suppose that $(k+1)! > 2^{k+3}$. Then

$$\begin{array}{rcl} ((k+1)+1)! & = & (k+2)! \\ & = & (k+1)!(k+2) \\ & > & 2^{k+3}(k+2) \\ & > & 2^{k+3}2 \\ & = & 2^{k+4} \\ & = & 2^{(k+1)+3}. \end{array}$$

(b) It is well-known theorem of Euclidean geometry that the sum of the angle measures of a triangle is $180^o = (3-2)180^o$. This settles the inductive base. Now suppose that the sum of the angle measures of the interior angles of a convex polygon of k sides is $(k-2) \cdot 180^o$. Consider a convex polygon of $k+1$ sides $A_0A_1A_2A_3 \ldots A_kA_0$. The sum of the measures of its interior angles is the sum of the measures of the interior angles of $A_0A_1A_2$ plus the sum of the measures of the interior angles of $A_0A_2A_3\dots A_kA_0$. The first sum is equal to 180^o and the second sum is equal to $(k-2)180^o$ by the inductive hypothesis. Thus the sum of the angle measures of the interior angles of a convex polygon of $k+1$ sides is $180^o + (k-2)180^o = ((k+1)-2)180^o$. ■

Problem 5 *Suppose that a statement $P(n)$ satisfies:*
(a) $P(1)$ is true.
(b) if $P(n)$ is true, then $P(n+2)$ is true.
Is $P(n)$ true for all $n \in \mathbb{N}$? Explain.

Solution: The conclusion is not true. Let, for instance, $P(n)$ be the statement "n is odd". Then $P(1)$ is true and, if $P(n)$ is true, then $P(n+2)$ is also true. But obviously, $P(n)$ is not true for all $n \in \mathbb{N}$. ■

Problem 6 *Let $a_1 = 2, a_2 = 4$ and $a_{n+2} = 5a_{n+1} - 6a_n$ for all $n \geq 3$. Prove that $a_n = 2^n$ for all natural numbers n.*

Solution: We have $a_1 = 2^1$ and $a_2 = 2^2$. Suppose that $a_k = 2^k$, for $n = k, k+1$. Then we have, for $n = k+2$,

$$\begin{aligned} a_{k+2} &= 5a_{k+1} - 6a_k \\ &= 5 \cdot 2^{k+1} - 6 \cdot 2^k \\ &= 5 \cdot 2 \cdot 2^k - 6 \cdot 2^k \\ &= 10 \cdot 2^k - 6 \cdot 2^k \\ &= 4 \cdot 2^k \\ &= 2^2 2^k \\ &= 2^{k+2}. \end{aligned}$$

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Problem 7 *Let the "Foobar-nacci" numbers g_n be defined as follows: $g_1 = 2, g_2 = 2$ and $g_{n+2} = g_{n+1}g_n$, for all $n \geq 1$.*
(a) Calculate the first five "Foorbar-nacci" numbers.
(b) Show that $g_n = 2^{f_n}$.

Solution: (a) $g_1 = 2, g_2 = 2, g_3 = 4, g_4 = 8, g_5 = 32$.
(b) For $n = 1$, we have $g_1 = 2 = 2^1 = 2^{f_1}$. For $n = 2$, $g_2 = 2 = 2^1 = 2^{f_2}$. Suppose that, for all $n < k$, $g_n = 2^{f_n}$. Then

$$\begin{aligned} g_k &= g_{k-1}g_{k-2} \\ &= 2^{f_{k-1}}2^{f_{k-2}} \\ &= 2^{f_{k-1}+f_{k-2}} \\ &= 2^{f_k}. \end{aligned}$$

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Problem 8 *Find (a) $\#\{n \in \mathbf{Z} : n^2 < 41\}$ (b) $\#\{n \in \mathbb{N} : n+1 = 4n-10\}$*

Solution: (a) $\#\{n \in \mathbf{Z} : n^2 < 41\} = \#\{-6, -5, \dots, 5, 6\} = 13$.
(b) $\#\{n \in \mathbb{N} : n+1 = 4n-10\} = \#\emptyset = 0$. ■

Problem 9 *Of the four teams in a softball league, one team has four pitchers and the other teams have three each. Give the counting rules that apply to determine each of the following.*
(a) the number of possible selections of pitchers for an all-star team, if exactly four pitchers are to be chosen.
(b) The number of possible selections if one pitcher is to be chosen from each team.
(c) The number of possible selections of four pitchers, if exactly two of the five left-handed pitchers in the league must be selected.
(d) The number of possible orders in which the four pitchers, once they are selected, can appear (one at a time) in the all-star game.

Solution: (a) $\binom{13}{4}$ possible choices.
(b) $4 \cdot 3 \cdot 3 \cdot 3$.
(c) $\binom{5}{2}\binom{13-5}{2}$.
(d) $4!$. ■

Problem 10 *Among the 40 first-time campers at Camp Forlorn one week, 14 fell into the lake during the week, 13 suffered from poison ivy, and 16 got lost trying to find the dining hall. Three of these campers had poison ivy rash and fell into the lake, 5 fell into the lake and got lost, 8 had poison ivy and got lost and 2 experienced all three misfortunes. How many first-time campers got through the week without any of these mishaps?*

Solution: Draw a Venn diagram and fill the appropriate regions. You should conclude that 11 campers got through the week without any of these mishaps. ■

Problem 11 *Find the number of ways seven school children can line up to board a school bus.*

Solution: $7!$. ■

Problem 12 *Suppose the seven children of the previous exercise are three girls and four boys. Find the number of ways they could line up subject to these conditions.*
(a) The three girls are first in line.
(b) The three girls are together in line.
(c) The four boys are together in line.
(d) No two boys are together.

Solution: (a) $3!4!$ (b) $3!5!$ (c) $4!4!$ (d) $4!3!$ ■

Problem 13 *Among ten lottery finalists, four will be selected to win individual amounts of* $\$1,000$, $\$2,000, \$5,000$ *and* $\$10,000$. *In how many ways may the money be distributed?*

Solution: $P(10,4) = 10 \cdot 9 \cdot 8 \cdot 7$. ■

Problem 14 *From a second-grade class of 11 boys and 8 girls, 3 are selected for flag duty.*
(a) How many selections are possible?
(b) How many of these selections have exactly 2 boys?
(c) Exactly 1 boy?

Solution: (a) $\binom{19}{3}$. (b) $\binom{11}{2}\binom{8}{1}$. (c) $\binom{11}{1}\binom{8}{2}$. ∎

Problem 15 *Prove combinatorially that if n is odd, then the number of ways to select an even number of objects from n is equal to the number of ways to select an odd number of objects.*

Solution: The number of ways to select a group of included objects equals the number of ways to select the group of objects to be left out. Thus, if n is odd, selection of an even number of objects (to be included) amounts to selection of an odd number of objects (to be left out). So the number of ways to perform these two choices is the same. ∎