HOMEWORK 5 - MATH 215

DUE DATE: After Section 3.3 has been covered!
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Read each problem very carefully before starting to solve it. A few randomly selected problems will be graded for a total of 10 points. It is necessary to show your work.

GOOD LUCK!!

1. Prove that for any sets A, B, C, D, $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

2. Give an example of sets A, B and C such that
   (a) $(A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D)$. (b) $(C \times C) - (A \times B) \neq (C - A) \times (C - B)$.

3. Let T be the relation $\{(3,1),(2,3),(3,5),(2,2),(1,6),(2,6),(1,2)\}$. Find
   (a) $\mathrm{Dom}(T)$ (b) $\mathrm{Rng}(T)$ (c) T^{-1} (d) $(T^{-1})^{-1}$.

4. The inverse of $R = \{(x,y) \in \mathbb{R} \times \mathbb{R} : y = 2x + 1\}$ may be expressed in the form $R^{-1} = \{(x,y) \in \mathbb{R} \times \mathbb{R} : y = \frac{(x-1)}{2}$, the set of all pairs (x,y), subject to some condition. Use this form to give the inverses of the following relations. In (c) P is the set of all people.
   (a) $R_1 = \{(x,y) \in \mathbb{R} \times \mathbb{R} : y = -5x + 2\}$.
   (b) $R_2 = \{(x,y) \in \mathbb{R} \times \mathbb{R} : y < x + 1\}$.
   (c) $R_3 = \{(x,y) \in P \times P : y \text{ loves } x\}$.

5. Let $R = \{(1,5),(2,2),(3,4),(5,2)\}, S = \{(2,4),(3,4),(3,1),(5,5)\}$ and $T = \{(1,4),(3,5),(4,1)\}$. Find
   (a) $R \circ S$ (b) $T \circ T$ (c) $R \circ (S \circ T)$.

6. Let $S = \{(1,3),(2,1)\}$ be a relation on $\{1,2,3\}$. Give the digraphs for the following relations on the set $\{1,2,3\}$.
   (a) S (b) S^{-1} (c) $S \circ S$.

7. Let $A = \{a,b,c,d\}$. Give an example of relations R and S on A such that
   (a) $R \circ S \neq S \circ R$ (b) $(S \circ R)^{-1} \neq S^{-1} \circ R^{-1}$.

8. Prove that if A has m elements and B has n elements, then $A \times B$ has mn elements.

9. Indicate which of the following relations on the given sets are reflexive, which are symmetric and which are transitive.
   (a) "divides" on $\mathbb{N}$.
   (b) $\perp = \{(l,m) : l \text{ and } m \text{ are lines and } l \text{ is perpendicular to } m\}$.
   (c) R, where $(x,y)R(z,w)$ iff $x + z \leq y + w$, on the set $\mathbb{R} \times \mathbb{R}$.

10. Let A be the set $\{1,2,3\}$. List the ordered pairs in a relation on A which is
    (a) reflexive, not symmetric and transitive (b) not reflexive, symmetric and transitive.

11. For each of the following verify that the relation is an equivalence relation. then give information about the equivalence classes as specified.

(a) the relation R on $\mathbf{Z}$ given by xRy iff $x^2 = y^2$. Give the equivalence class of 0; of 4; of -72.
(b) The relation V on $\mathbb{R}$ given by xVy iff $x = y$ or $xy = 1$. Give the equivalence class of 3; of $-\frac{2}{3}$; of 0.
(c) The relation R on the set of all ordered triples from the set $\{1, 2, 3, 4\}$ given by $(x, y, z)R(a, b, c)$ iff $y = b$. List five elements of $(4, 2, 1)/R$. How many elements are in the equivalence class of $(1, 1, 1)$?

12. Which of the following digraphs represent relations that are (i) reflexive (ii) symmetric (iii) transitive?

13. For the equivalence relation $\equiv_m$, prove that
(a) if $x \equiv_m y$, then $\overline{x} = \overline{y}$
(b) if $\overline{x} = \overline{y}$, then $x \equiv_m y$.
(c) if $\overline{x} \cap \overline{y} \neq \emptyset$, then $\overline{x} = \overline{y}$.

14. Consider the relations R and S on $\mathbb{N}$ defined by xRy iff 2 divides $x + y$ and xSy iff 3 divides $x + y$.
(a) Prove that R is an equivalence relation.
(b) Prove that S is not an equivalence relation.

15. The **complement** of a digraph has the same vertex set as the original digraph and an edge from x to y exactly when the original digraph does not have an edge from x to y. Call a digraph symmetric or transitive iff its relation is symmetric or transitive, respectively.
(a) Show that the complement $\tilde{D}$ of a symmetric digraph D is symmetric.
(b) Show by example that the complement of a transitive digraph need not be transitive.

16. Describe the partition for the following equivalence relation: for $x, y \in \mathbb{R}, xRy$ iff $x - y \in \mathbf{Z}$.

17. Describe the equivalence relation on each of the following sets with the given partition:
(a) $\mathbb{R}, \{(-\infty, 0), \{0\}, (0, \infty)\}$.
(b) $\mathbf{Z}, \{A, B\}$, where $A = \{x \in \mathbf{Z} : x < 3\}$ and $B = \mathbf{Z} - A$.

18. For each $a \in \mathbb{R}$, let $A_a = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = a - x^2\}$.
(a) Sketch a graph of the set A_a for $a = -2, -1, 0, 1, 2$.
(b) Prove that $\{A_a : a \in \mathbb{R}\}$ is a partition of $R \times \mathbb{R}$.
(c) Describe the equivalence relation associated with this partition.

19. List the ordered pairs in the equivalence relation on $A = \{1, 2, 3, 4, 5\}$ associated with the partition $\{\{1\}, \{2\}, \{3, 4\}, \{5\}\}$.

20. Let R be a relation on a set A that is reflexive and symmetric but not transitive. Let $R(x) = \{y : xRy\}$. [Note that $R(x)$ is the same as x/R except that R is not an equivalence relation in this exercise.] Does the set $\mathcal{A} = \{R(x) : x \in A\}$ always form a partition of A? Prove that your answer is correct.