HOMEWORK 5 - MATH 215
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Problem 1 *Prove that for any sets* A, B, C, D, $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Solution: We first show that $(A \times B) \cap (C \times D) \subseteq (A \cap C) \times (B \cap D)$. Suppose $(x, y) \in (A \times B) \cap (C \times D)$. Then $(x, y) \in A \times B$ and $(x, y) \in C \times D$. Therefore $x \in A$ and $y \in B$ and $x \in C$ and $y \in D$. Therefore $x \in A \cap C$ and $y \in B \cap D$. Hence $(x, y) \in (A \cap C) \times (B \cap D)$.

We show next that $(A \cap C) \times (B \cap D) \subseteq (A \times B) \cap (C \times D)$. Suppose that $(x, y) \in (A \cap C) \times (B \cap D)$. Then $x \in A \cap C$ and $y \in B \cap D$. Hence $x \in A$ and $x \in C$ and $y \in B$ and $y \in D$. Therefore $(x, y) \in A \times B$ and $(x, y) \in C \times D$, which shows that $(x, y) \in (A \times B) \cap (C \times D)$. ∎

Problem 2 *Give an example of sets* A, B *and* C *such that*
(a) $(A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D)$. *(b)* $(C \times C) - (A \times B) \neq (C - A) \times (C - B)$.

Solution: (a) Let $A = B = \{0\}, C = D = \{1\}$. Then $(A \times B) \cup (C \times D) = \{(0, 0)\} \cup \{(1, 1)\} = \{(0, 0), (1, 1)\}$ whereas $(A \cup C) \times (B \cup D) = \{0, 1\} \times \{0, 1\} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.
(b) Let $C = A = \{0\}$ and $B = \emptyset$. Then $(C \times C) - (A \times B) = \{(0, 0)\}$ and $(C - A) \times (C - B) = \emptyset \times \{0\} = \emptyset$. ∎

Problem 3 *Let* T *be the relation* $\{(3, 1), (2, 3), (3, 5), (2, 2), (1, 6), (2, 6), (1, 2)\}$. *Find*
(a) $\mathrm{Dom}(T)$ *(b)* $\mathrm{Rng}(T)$ *(c)* T^{-1} *(d)* $(T^{-1})^{-1}$.

Solution: (a) $\mathrm{Dom}(T) = \{1, 2, 3\}$.
(b) $\mathrm{Rng}(T) = \{1, 2, 3, 5, 6\}$.
(c) $T^{-1} = \{(1, 3), (3, 2), (5, 3), (2, 2), (6, 1), (6, 2), (2, 1)\}$.
(d) $(T^{-1})^{-1} = T$. ∎

Problem 4 *The inverse of* $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = 2x + 1\}$ *may be expressed in the form* $R^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \frac{(x-1)}{2}$, *the set of all pairs* (x, y), *subject to some condition. Use this form to give the inverses of the following relations. In (c)* P *is the set of all people.*
(a) $R_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = -5x + 2\}$.
(b) $R_2 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y < x + 1\}$.
(c) $R_3 = \{(x, y) \in P \times P : y \text{ loves } x\}$.

Solution: (a) $R_1^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = \frac{2-x}{5}\}$.
(b) $R_2^{-1} = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y > x - 1\}$.
(c) $R_3^{-1} = \{(x, y) \in P \times P : y \text{ is loved by } x\}$. ∎

Problem 5 *Let* $R = \{(1, 5), (2, 2), (3, 4), (5, 2)\}, S = \{(2, 4), (3, 4), (3, 1), (5, 5)\}$ *and* $T = \{(1, 4), (3, 5), (4, 1)\}$. *Find*
(a) $R \circ S$ *(b)* $T \circ T$ *(c)* $R \circ (S \circ T)$.

Solution: (a) $R \circ S = \{(3, 5), (5, 2)\}$.
(b) $T \circ T = \{(1, 1), (4, 4)\}$.
(c) $R \circ (S \circ T) = R \circ \{(3, 5)\} = \{(3, 2)\}$. ∎

Problem 6 *Let $S = \{(1,3),(2,1)\}$ be a relation on $\{1,2,3\}$. Give the digraphs for the following relations on the set $\{1,2,3\}$.*
(a) S (b) S^{-1} (c) $S \circ S$.

Solution:

$\blacksquare$

Problem 7 *Let $A = \{a,b,c,d\}$. Give an example of relations R and S on A such that*
(a) $R \circ S \neq S \circ R$ (b) $(S \circ R)^{-1} \neq S^{-1} \circ R^{-1}$.

Solution: (a) Let $R = \{(a,b)\}, S = \{(b,a)\}$. Then $R \circ S = \{(b,b)\}$, whereas $S \circ S = \{(a,a)\}$.
(b) Let $R = \{(a,b)\}$ and $S = \{(b,a)\}$. Then $(S \circ R)^{-1} = \{(a,a)\}^{-1} = \{(a,a)\}$, whereas $S^{-1} \circ R^{-1} = \{(a,b)\} \circ \{(b,a)\} = \{(b,b)\}$. $\blacksquare$

Problem 8 *Prove that if A has m elements and B has n elements, then $A \times B$ has mn elements.*

Solution: By induction on m. It is easy to see that, if $m = 1$, then $A \times B$ has $n = m \times n$ elements. Suppose that, if $m = k$, then $A \times B$ has $k \times n$ elements. Now let A be a set with $m = k+1$ elements. If $a \in A$, then

$$\begin{aligned} \#(A \times B) &= \#(\{a\} \times B) + \#((A - \{a\}) \times B) \\ &= n + kn \\ &= (k+1)n \\ &= mn. \end{aligned}$$

$\blacksquare$

Problem 9 *Indicate which of the following relations on the given sets are reflexive, which are symmetric and which are transitive.*
(a) "divides" on $\mathbb{N}$.
(b) $\perp = \{(l,m) : l$ and m are lines and l is perpendicular to $m\}$.
(c) R, where $(x,y)R(z,w)$ iff $x + z \leq y + w$, on the set $\mathbb{R} \times \mathbb{R}$.

Solution: (a) "divides" on $\mathbb{N}$ is reflexive and transitive but not symmetric.
(b) $\perp$ is symmetric but it is neither reflexive nor transitive.
(c) R is symmetric but it is neither reflexive nor transitive. $\blacksquare$

Problem 10 *Let A be the set $\{1,2,3\}$. List the ordered pairs in a relation on A which is*
(a) reflexive, not symmetric and transitive (b) not reflexive, symmetric and transitive.

Solution: (a) $R = \{(1,1),(1,2),(2,2),(3,3)\}$.
(b) $R = \{(1,1),(1,2),(2,1),(2,2)\}$. $\blacksquare$

Problem 11 *For each of the following verify that the relation is an equivalence relation. then give information about the equivalence classes as specified.*
(a) the relation R on $\mathbf{Z}$ *given by xRy iff $x^2 = y^2$. Give the equivalence class of 0; of 4; of -72.*
(b) The relation V on $\mathbb{R}$ given by xVy iff $x = y$ or $xy = 1$. Give the equivalence class of 3; of $-\frac{2}{3}$; of 0.
(c) The relation R on the set of all ordered triples from the set $\{1, 2, 3, 4\}$ given by $(x, y, z)R(a, b, c)$ iff $y = b$. List five elements of $(4, 2, 1)/R$. How many elements are in the equivalence class of $(1, 1, 1)$?

Solution: (a) $x^2 = x^2$, for all $x \in \mathbf{Z}$, whence xRx, for all $x \in \mathbf{Z}$ and R is reflexive. xRy means $x^2 = y^2$, whence $y^2 = x^2$ and hence yRx. Therefore R is symmetric. Finally, if xRy and yRz, then $x^2 = y^2$ and $y^2 = z^2$, whence $x^2 = z^2$ and xRz, i.e., R is also transitive. Thus R is an equivalence relation on $\mathbf{Z}$.

We have $0/R = \{0\}, 4/R = \{-4, 4\}$ and $-72/R = \{-72, 72\}$.

(b) xVx, for all $x \in \mathbb{R}$, by the definition of R, whence R is reflexive. If xVy, then $x = y$ or $xy = 1$, whence $y = x$ or $yx = 1$, which gives yVx and V is symmetric. Finally suppose that xVy and yVz. Then $x = y$ or $xy = 1$ and $y = z$ or $yz = 1$. So there are four cases to consider: If $x = y$ and $y = z$, then $x = z$ and xVz. If $x = y$ and $yz = 1$, then $xz = 1$, whence xVz. If $xy = 1$ and $y = z$, then $xz = 1$, whence xVz. Finally, if $xy = 1$ and $yz = 1$, then $x = z = \frac{1}{y}$, whence xVz. Therefore V is also transitive and, thus, an equivalence relation on $\mathbb{R}$.

$3/V = \{\frac{1}{3}, 3\}, -\frac{2}{3}/V = \{-\frac{2}{3}, -\frac{3}{2}\}, 0/V = \{0\}$.

(c) $(x, y, z)R(x, y, z)$, since $y = y$. So R is reflexive. $(x, y, z)R(u, v, w)$ implies $y = v$, whence $v = y$ and, therefore $(u, v, w)R(x, y, z)$. Thus R is symmetric. Finally, if $(x, y, z)R(u, v, w)$ and $u, v, w)R(a, b, c)$, then $y = v$ and $v = b$, whence $y = b$ and, therefore $(x, y, z)R(a, b, c)$ and R is also transitive, i.e., an equivalence relation on $\{1, 2, 3, 4\}^3$.

$(4, 2, 1), (1, 2, 1), (2, 2, 1), (3, 2, 1), (5, 2, 1), (5, 2, 2)$ are five elements of $(4, 2, 1)/R$. There are 16 elements in total in the equivalence class of $(1, 1, 1)$. ■

Problem 12 *Which of the following digraphs represent relations that are (i) reflexive (ii) symmetric (iii) transitive?*

Solution:

■

Problem 13 *For the equivalence relation $\equiv_m$, prove that*
(a) if $x \equiv_m y$, then $\overline{x} = \overline{y}$

(b) if $\overline{x} = \overline{y}$, then $x \equiv_m y$.
(c) if $\overline{x} \cap \overline{y} \neq \emptyset$, then $\overline{x} = \overline{y}$.

Solution: (a) Suppose that $x \equiv_m y$ and let $z \in \overline{x}$. Then $z \equiv_m x$, whence $z \equiv_m y$, and therefore $z \in \overline{y}$. Thus $\overline{x} \subseteq \overline{y}$. The reverse inclusion follows similarly.
(b) Suppose $\overline{x} = \overline{y}$. then $x \in \overline{y}$, whence $x \equiv_m y$.
(c) Suppose that $\overline{x} \cap \overline{y} \neq \emptyset$. Let $z \in \overline{x} \cap \overline{y}$. We show that $\overline{x} \subseteq \overline{y}$. The reverse inclusion can be proved then similarly. Let $w \in \overline{x}$. Then $w \equiv_m x$. But $x \equiv_m z$, whence $w \equiv_m z$. Now $z \equiv_m y$ yields $w \equiv_m y$, whence $w \in \overline{y}$ and $\overline{x} \subseteq \overline{y}$. ■

Problem 14 *Consider the relations R and S on $\mathbb{N}$ defined by xRy iff 2 divides $x + y$ and xSy iff 3 divides $x + y$.*
(a) Prove that R is an equivalence relation.
(b) Prove that S is not an equivalence relation.

Solution: (a) We show that R is an equivalence on $\mathbb{N}$. First $x + x = 2x$ is divisible by 2, for all $x \in \mathbb{N}$, whence $(x, x) \in \mathbb{N}$, for all $x \in \mathbb{N}$ and R is reflexive. Suppose xRy. Then $x + y$ is divisible by 2, whence $y + x = x + y$ is divisible by 2 and $(y, x) \in R$. Thus R is also symmetric. Finally, suppose $(x, y) \in R$ and $(y, z) \in R$. Then 2 divides $x + y$ and 2 divides $y + z$, whence 2 divides $(x + z) + 2y$, which implies that 2 divides $x + z$. Therefore $(x, z) \in R$ and R is also transitive.
(b) For instance $(2, 2) \notin S$, whence S is not reflexive. ■

Problem 15 *The* **complement** *of a digraph has the same vertex set as the original digraph and an edge from x to y exactly when the original digraph does not have an edge from x to y. Call a digraph symmetric or transitive iff its relation is symmetric or transitive, respectively.*
(a) Show that the complement $\tilde{D}$ of a symmetric digraph D is symmetric.
(b) Show by example that the complement of a transitive digraph need not be transitive.

Solution: (a) Suppose $xy \in \tilde{D}$. Then $xy \notin D$, whence, since D is symmetric $yx \notin D$. Hence $yx \in \tilde{D}$ and, therefore $\tilde{D}$ is in fact symmetric.
(b) Let D have two vertices x, y and the edges (x, x) and (y, y). This D is transitive. Its complement $\tilde{D}$ has vertices x, y and the edges $(x, y), (y, x)$. It is easy to see that $\tilde{D}$ is not transitive. ■

Problem 16 *Describe the partition for the following equivalence relation: for $x, y \in \mathbb{R}, xRy$ iff $x - y \in \mathbf{Z}$.*

Solution: The partition is in one-to-one correspondence with the subset $[0, 1)$ of the real numbers. The set in the partition corresponding to $r \in [0, 1)$ consists of all reals x that have the same fractional part as r does. ■

Problem 17 *Describe the equivalence relation on each of the following sets with the given partition:*
(a) $\mathbb{R}, \{(-\infty, 0), \{0\}, (0, \infty)\}$.
(b) $\mathbf{Z}, \{A, B\}$, where $A = \{x \in \mathbf{Z} : x < 3\}$ and $B = \mathbf{Z} - A$.

Solution: (a) R on $\mathbb{R}$ is defined by

$$xRy \quad \text{iff} \quad x = y = 0 \text{ or } xy > 0.$$

(b) R on $\mathbf{Z}$ may be defined by

$$xRy \quad \text{iff} \quad (x - \frac{5}{2})(y - \frac{5}{2}) > 0.$$

■

Problem 18 *For each $a \in \mathbb{R}$, let $A_a = \{(x,y) \in \mathbb{R} \times \mathbb{R} : y = a - x^2\}$.*
(a) Sketch a graph of the set A_a for $a = -2, -1, 0, 1, 2$.
(b) Prove that $\{A_a : a \in \mathbb{R}\}$ is a partition of $R \times \mathbb{R}$.
(c) Describe the equivalence relation associated with this partition.

Solution: (a)

(b) First, we show that every part $A_a, a \in \mathbb{R}$, is nonempty. This is true, since, for all $a \in \mathbb{R}$, $(0, a) \in A_a$.

Next we show that, if $a \neq b$, then $A_a \cap A_b = \emptyset$. Suppose $(x, y) \in A_a \cap A_b$. Then $y = a - x^2$ and $y = b - x^2$. Therefore $y + x^2 = a = b$.

Finally, we show that the union of all parts is the whole space $\mathbb{R} \times \mathbb{R}$. Given $(x, y) \in \mathbb{R} \times \mathbb{R}$, $y = (y + x^2) - x^2$, whence $(x, y) \in A_{y+x^2}$. So $\bigcup_{a \in \mathbb{R}} A_a = \mathbb{R} \times \mathbb{R}$.
(c) $(x, y)R(z, w)$ if and only if $y + x^2 = w + z^2$. ■

Problem 19 *List the ordered pairs in the equivalence relation on $A = \{1, 2, 3, 4, 5\}$ associated with the partition $\{\{1\}, \{2\}, \{3, 4\}, \{5\}\}$.*

Solution: $R = \{(1, 1), (2, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5)\}$. ■

Problem 20 *Let R be a relation on a set A that is reflexive and symmetric but not transitive. Let $R(x) = \{y : xRy\}$. [Note that $R(x)$ is the same as x/R except that R is not an equivalence relation in this exercise.] Does the set $\mathcal{A} = \{R(x) : x \in A\}$ always form a partition of A? Prove that your answer is correct.*

Solution: $\mathcal{A}$ is not always a partition on A. Let for instance $A = \{1, 2, 3\}, R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1)\}$. R is a reflexive and symmetric relation on A but it is not transitive. We have $1/R = \{1, 2\}, 2/R = \{1, 2, 3\}, 3/R = \{2, 3\}$. Obviously, these do not form a partition on A. ■