HOMEWORK 6 - MATH 215
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Problem 1 *Which of the relations on the given sets are antisymmetric?*
(a) $A = \{1, 2, 3, 4, 5\}, R = \{(1, 3), (1, 1), (2, 4), (3, 2), (5, 4), (4, 2)\}$
(b) $\mathbb{R}, xRy$ *iff* $x^2 = y^2$.
(c) $A = \{1, 2, 3, 4\}$ *and* R *is given by the following digraph.*

Solution: (a) R is not antisymmetric, since $(2, 4) \in R$ and $(4, 2) \in R$ but $2 \neq 4$.
(b) This R is not antisymmetric either: We have, for instance $-1R1$ and $1R-1$, but $-1 \neq 1$.
(c) This relation is antisymmetric, since for no x, y is it the case that xRy and yRx. ■

Problem 2 *Show that if* R *is antisymmetric, then* xRy *and* $x \neq y$ *implies* $y \not R x$.

Solution: Suppose that R is antisymmetric, xRy and $x \neq y$. If yRx, then, by antisymmetry, we have $x = y$, contrary to the hypothesis. ■

Problem 3 *Give an example of a relation* R *on a set* A *which is antisymmetric and such that* xRx *for some, but not all,* $x \in A$.

Solution: Let $A = \{1, 2\}$ and $R = \{(1, 1)\}$. Clearly, xRx is true for some but not for all $x \in A$. Also R is antisymmetric, since the only pair (x, y) for which xRy and yRx is the pair $(1, 1)$ and $1 = 1$. ■

Problem 4 *Give an example of a relation* S *on a set* $A = \{a, b, c, d\}$ *such that* S *is transitive, antisymmetric and irreflexive (that is,* xSx *is false for all* $x \in A$*).*

Solution: Let $S = \{(a, b)\}$. ■

Problem 5 *Define the relation on* $\mathbb{R} \times \mathbb{R}$ *by* $(a, b)R(x, y)$ *iff* $a \leq x$ *and* $b \leq y$*. Prove that* R *is a partial ordering for* $\mathbb{R} \times \mathbb{R}$.

Solution: Clearly, $x \leq x$ and $y \leq y$, for all $x, y \in \mathbb{R}$, whence $(x, y)R(x, y)$, for all $(x, y) \in \mathbb{R}^2$, and R is reflexive. Suppose $(x, y)R(a, b)$ and $(a, b)R(x, y)$. Then $x \leq a$ and $y \leq b$ and $a \leq x$ and $b \leq y$. therefore $x = a$ and $y = b$, which shows that $(x, y) = (a, b)$ and R is antisymmetric. Finally, suppose $(x, y)R(a, b)$ and $(a, b)R(z, w)$. Then $x \leq a, y \leq b, a \leq z$ and $b \leq w$. Therefore $x \leq z$ and $y \leq w$, which yields $(x, y)R(z, w)$ and R is also transitive. Thus, R is a partial ordering on $\mathbb{R}^2$. ■

Problem 6 *Let* $\mathbf{C}$ *be the complex numbers. Define* $(a + bi)R(c + di)$ *iff* $a^2 + b^2 \leq c^2 + d^2$*. Is* R *a partial order for* $\mathbf{C}$*? Justify your answer.*

Solution: R is not a partial ordering on $\mathbf{C}$. The reason is that R is not antisymmetric. For instance $-1R1$ and $1R-1$, but $-1 \neq 1$. ■

Problem 7 *Use your own judgement about which tasks should precede others to draw a Hasse diagram for the partial order among the tasks for the following complex job:*
To back a car out of the garage, Kim must perform 11 tasks:
t_1*: put the key in the ignition*
t_2*:step on the gas*
t_3*: check to see if the driveway is clear*
t_4*:start the car*
t_5*:adjust the mirror*
t_6*:open the garage door* t_7*:fasten the seat belt*
t_8*:adjust the position of the driver's seat*
t_9*:get in the car*
t_{10}*:put the car in reverse gear*
t_{11}*: step on the break.*

Solution:

■

Problem 8 *Let A be a set. Consider the partial order $\subseteq$ on $\mathcal{P}(A)$.*
(a) Let C and D be subsets of A. Prove that the least upper bound of $\{C, D\}$ is $C \cup D$ and the greatest lower bound of $\{C, D\}$ is $C \cap D$.
(b) Let $\mathcal{B}$ be a family of subsets of A. Prove that the least upper bound of $\mathcal{B}$ is $\bigcup_{B\in\mathcal{B}} B$ and the greatest lower bound of $\mathcal{B}$ is $\bigcap_{B\in\mathcal{B}} B$.

Solution: (a) We first show that $C \cup D$ is the least upper bound of $\{C, D\}$. Clearly, $C \subseteq C \cup D$ and $D \subseteq C \cup D$. Thus $C \cup D$ is an upper bound of $\{C, D\}$. Suppose that A is an upper bound of $\{C, D\}$. Then $C \subseteq A$ and $D \subseteq A$. Then $C \cup D \subseteq A$, whence $C \cup D$ is the least upper bound of $\{C, D\}$.

We next show that $C \cap D$ is the greatest lower bound of $\{C, D\}$. Clearly, $C \cap D \subseteq C$ and $C \cap D \subseteq D$. Thus $C \cap D$ is a lower bound of $\{C, D\}$. Suppose that B is a lower bound of $\{C, D\}$. Then $B \subseteq C$ and $B \subseteq D$. Then $B \subseteq C \cap D$, whence $C \cap D$ is the greatest lower bound of $\{C, D\}$.
(b) We first show that $\bigcup_{B\in\mathcal{B}} B$ is the least upper bound of $\mathcal{B}$. Clearly, $B \subseteq \bigcup_{B\in\mathcal{B}} B$, for all $B \in \mathcal{B}$. Thus $\bigcup_{B\in\mathcal{B}} B$ is an upper bound of $\mathcal{B}$. Suppose that A is an upper bound of $\mathcal{B}$. Then $B \subseteq A$, for all $B \in \mathcal{B}$. Then $\bigcup_{B\in\mathcal{B}} B \subseteq A$, whence $\bigcup_{B\in\mathcal{B}} B$ is the least upper bound of $\mathcal{B}$.

We next show that $\bigcap_{B\in\mathcal{B}} B$ is the greatest lower bound of $\mathcal{B}$. Clearly, $\bigcap_{B\in\mathcal{B}} B \subseteq B$, for all $B \in \mathcal{B}$. Thus $\bigcap_{B\in\mathcal{B}} B$ is a lower bound of $\mathcal{B}$. Suppose that A is a lower bound of $\mathcal{B}$. Then $A \subseteq B$, for all $B \in \mathcal{B}$. Then $A \subseteq \bigcap_{B\in\mathcal{B}} B$, whence $\bigcap_{B\in\mathcal{B}} B$ is the greatest lower bound of $\mathcal{B}$. ■

Problem 9 *For what sets A is $\mathcal{P}(A)$ with set inclusion a linear ordering?*

Solution: It is a linear ordering only for $A = \emptyset$ and for A a singleton, i.e., $\#A = 1$. ■

Problem 10 *Prove that every subset of a well-ordered set is well-ordered.*

Solution: Suppose that A is a well-ordered set and that $B \subseteq A$. Let $X \subseteq B$, such that $X \neq \emptyset$. Then $X \subseteq A$ and $X \neq \emptyset$, whence X has a least element in A. But this is also a least element in B. Thus B is also well-ordered. ■

Problem 11 *P is a preorder for a set A if P is reflexive and transitive relation on A. Define a relation E on A by xEy iff xPy and yPx. Show that E is an equivalence relation on A.*

Solution: We need to show that E is reflexive symmetric and transitive. Since P is reflexive, we have xPx, for all $x \in A$. Therefore xEx, for all $x \in A$, and E is reflexive. Suppose next that xEy. Then xPy and yPx, whence yPx and xPy, which gives yEx and E is symmetric. Finally, let xEy and yEz. Then xPy and yPx and yPz and zPy. But P is transitive, whence the first and third relations yield xPz and the second and fourth yield zPx, i.e., we have xEz and E is also transitive. Thus E is an equivalence relation on A. ■

Problem 12 *If possible, give an example of a graph with order 6 such that*
(a) the vertices have degrees $1, 1, 1, 1, 1, 5$
(b) the vertices have degrees $1, 1, 1, 1, 1, 1$
(c) the vertices have degrees $2, 2, 2, 2, 2, 2$
(d) the vertices have degrees $1, 2, 2, 2, 3, 3$
(e) exactly two vertices have even degree.
(f) exactly two vertices have odd degree.

Solution: We let $V = \{1, 2, 3, 4, 5, 6\}$.
(a) $E = \{\{1, 2\}, \{1, 3)\}, \{1, 4\}, \{1, 5\}, \{1, 6\}$.
(b) $E = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$.
(c) $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}, \{4, 6\}, \{5, 6\}\}$.
(d) Not possible.
(e) $E = \{\{1, 2\}, \{3, 4\}\}$.
(f) $E = \{\{1, 2\}$. ■

Problem 13 *f possible give an example of a graph*
(a) with order 6 and size 6
(b) with order 4 and size 6
(c) with order 3 and size 6
(d) with order 6 and size 3.

Solution: (a) Let $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4, 5\}, \{6, 6\}, \{5, 6\}\}$.
(b) Let $V = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$.
(c) Not possible.
(d) Let $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$. ■

Problem 14 *The* **complement** $\tilde{G}$ *of a graph $G(V, E)$ is the graph with vertex set V in which two vertices are adjacent iff they are not adjacent in G. Give the complements of these graphs.*

Solution:

$\blacksquare$

Problem 15 *Give an example of a graph with 6 vertices having degrees* $1, 1, 2, 2, 2, 2$ *that is (a) connected (b) disconnected*

Solution: Let $V - \{1, 2, 3, 4, 5, 6\}$.
(a) $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}\}$.
(b) $E = \{\{1, 2\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 3\}\}$. $\blacksquare$

Problem 16 *Give an example of a graph with 6 vertices having*
(a) one component (b) two components (c) three components (d) six components.

Solution: Let $V = \{1, 2, 3, 4, 5, 6\}$.
(a) $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}\}$.
(b) $E = \{\{1, 2\}, \{3, 4\}, \{4, 5\}, \{5, 6\}\}$.
(c) $E = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$.
(d) $E = \emptyset$. $\blacksquare$

Problem 17 *Give an example of a graph with order 6 such that*
(a) two vertices u *and* v *have distance 5*
(b) for any two vertices u *and* v, $d(u, v) \leq 2$.

Solution: Let $V = \{1, 2, 3, 4, 5, 6\}$.
(a) $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}\}$.
(b) $E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 1\}, \{1, 4\}, \{2, 5\}, \{3, 6\}\}$. $\blacksquare$