HOMEWORK 2 - MATH 325
INSTRUCTOR: George Voutsadakis

Problem 1 *Given triangles $\triangle ABC$ and $\triangle DEF$ such that $\overline{AB}$ and $\overline{DE}$ are parallel and congruent and $\overline{BC}$ and $\overline{EF}$ are parallel and congruent, prove that $\overline{AC}$ and $\overline{DF}$ are parallel and congruent.*

Solution: Since $\overline{AB}$ and $\overline{DE}$ are parallel and congruent, the quadrilateral $ADEB$ is a parallelogram, whence $\overline{AD}$ and $\overline{BE}$ are also parallel and congruent. Similarly, since $\overline{BC}$ and $\overline{EF}$ are parallel and congruent, the quadrilateral $BEFC$ is also a parallelogram, whence $\overline{BE}$ and $\overline{CF}$ are parallel and congruent as well. Therefore $\overline{AD}$ is parallel and congruent with $\overline{CF}$. But then the quadrilateral $ACFD$ is a parallelogram and, therefore, $\overline{AC}$ is parallel and congruent to $\overline{DF}$. ■

Problem 2 *Given a quadrilateral $ABCD$ such that $\overline{AB} \parallel \overline{CD}$, prove that $\widehat{C} = \widehat{D}$ or $\widehat{C} = \mathrm{external}(\widehat{D})$ if and only if $\overline{AD} \cong \overline{BC}$.*

Solution: Suppose, first, that $\widehat{C} = \widehat{D}$ or $\widehat{C} = \mathrm{external}(\widehat{D})$. Bring the perpendiculars AP and BQ from A and B, respectively, to $\overline{CD}$. then $ABQP$ is a parallelogram, whence $AP = BQ$. Thus, by AAS for right triangles, $\triangle ADP \cong \triangle BCQ$ and, therefore, $AD = BC$.

Suppose, conversely, that $AD = BC$. Let again P and Q be as above. Now we use ASS for right triangles to conclude that $\triangle ADP \cong \triangle BCQ$. Therefore $\widehat{C} = \widehat{D}$ or $\widehat{C} = \mathrm{external}(\widehat{D})$. ■

Problem 3 *Our proof that parallel lines are at a constant distance apart used the parallel postulate. Prove the following without using the parallel postulate: Assume that the lines l_1 and l_2 are at a constant distance apart. Then, if l_1 and l_2 are cut by a transversal, alternate interior angles must be congruent.*

Solution: Let l be a transversal and suppose that it cuts l_1 at A and l_2 at B. Let C and D be respectively the perpendiculars from A to l_2 and from B to l_1. Then in the right triangles $\triangle ADB$ and $\triangle ACB$ we have $AB = AB, DB = AC$, whence $\triangle ADB \cong \triangle BCA$. Hence $\widehat{DAB} \cong \widehat{CBA}$. ■

Problem 4 *Let $ABCD$ be a parallelogram. Define the base to be $\overline{AB}$ and the height to be the distance between $\overline{AB}$ and $\overline{CD}$. Prove that $ABCD$ has area = base × height.*

Solution: Let CP and DQ be the perpendiculars from C and D, respectively to $\overline{AB}$. Then, by the formula for the area of a rectangle, we have $\mathrm{Area}(DCPQ) = PQ \cdot CP$. In the right triangles $\triangle AQD$ and $\triangle CBP$, we have $DQ = CP$ and $\widehat{DAQ} \cong \widehat{CBP}$, whence $\triangle AQD \cong \triangle BPC$, whence $\mathrm{Area}(AQD) = \mathrm{Area}(BPC)$ and $AQ = BP$. Thus we get

$$
\begin{aligned}
\mathrm{Area}(ABCD) &= \mathrm{Area}(AQD) + \mathrm{Area}(DCBQ) \\
&= \mathrm{Area}(BPC) + \mathrm{Area}(DCBQ) \\
&= \mathrm{Area}(DCPQ) \\
&= PQ \cdot CP \\
&= (QB + BP)CP \\
&= (QB + AQ)CP \\
&= AB \cdot CP.
\end{aligned}
$$

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Problem 5 *Assume that in quadrilateral $ABCD$ $\overline{AB} \parallel \overline{CD}$. Let $AB = b_1, CD = b_2$ and let h be the distance between $\overline{AB}$ and $\overline{CD}$. Prove that $ABCD$ has area $\frac{1}{2}h(b_1 + b_2)$.*

Solution: Suppose that the perpendiculars from A and B to $\overline{CD}$ intersect $\overline{CD}$ at P and Q, respectively. If they intersect the extension of $\overline{CD}$, then the proof has to be modified slightly. We denote $x = DP, y = QC$. Then we have

$$\begin{aligned} \text{Area}(ABCD) &= \text{Area}(ADP) + \text{Area}(ABQP) + \text{Area}(BQC) \\ &= \tfrac{1}{2}xh + b_1h + \tfrac{1}{2}yh \\ &= \tfrac{1}{2}(x+y)h + b_1h \\ &= \tfrac{1}{2}(b_2 - b_1)h + b_1h \\ &= \tfrac{1}{2}(b_1 + b_2)h. \end{aligned}$$

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Problem 6 *Prove that in a right triangle, if the hypotenuse is the base of length c, then the height is $h = \frac{ab}{c}$.*

Solution: Let CP be the altitude corresponding to the right angle. Then $\triangle APC \cong \triangle ACB$, whence $\frac{PC}{CB} = \frac{AC}{AB}$, i.e., $PC = \frac{AC \cdot CB}{AB} = \frac{ab}{c}$. ■

Problem 7 *Given a convex quadrilateral $ABCD$ with $AC \perp BD$, prove that $AB^2 + CD^2 = BC^2 + AD^2$.*

Solution: Let P be the point of intersection of $\overline{AC}$ with $\overline{BD}$. Then we have

$$\begin{aligned} AB^2 + CD^2 &= AP^2 + BP^2 + CP^2 + DP^2 \\ &= AP^2 + DP^2 + BP^2 + CP^2 \\ &= AD^2 + BC^2. \end{aligned}$$

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Problem 8 *(a) Given $\overline{AB}$ and $\overline{CD}$ construct $\overline{EF}$ such that $EF^2 = AB^2 + CD^2$.*
(b) Given $\overline{AB}, \overline{CD}$ and $\overline{EF}$, construct $\overline{GH}$ such that $GH^2 = AB^2 + CD^2 + EF^2$.

Solution: (a) Construct a right angle $\widehat{P}$ and take on one of its sides $\overline{PE}$, such that $PE = AB$ and on its other side $\overline{PF}$, such that $PF = CD$. Then $EF^2 = PE^2 + PF^2 = AB^2 + CD^2$, whence $\overline{EF}$ is the line segment with the property required.
(b) Construct a right angle $\widehat{P}$. Take on one of its sides $\overline{PG}$, such that $PG = AB$. Take on its other side $\overline{PQ}$, such that $PQ = CD$. Now construct the perpendicular to $\overline{QG}$ at Q and take on it segment $\overline{QH}$, such that $QH = EF$. Then we have

$$\begin{aligned} GH^2 &= QH^2 + QG^2 \\ &= EF^2 + QP^2 + PG^2 \\ &= EF^2 + CD^2 + AB^2. \end{aligned}$$

Thus $\overline{GH}$ has the property required. ■

Problem 9 *(a) Assume that in* $\triangle ABC$, $a = 4, b = 9$ *and* $c = 11$. *Calculate the area.*
(b) Calculate the length of each of the three altitudes in the triangle of part (a).
(c) If we let $a = 2, b = 3$ *and* $c = 7$ *in Heron's formula we get a problem. What is the problem and why does it happen?*

Solution: (a) First, the semiperimeter is $s = \frac{a+b+c}{2} = \frac{4+9+11}{2} = 12$. Now the area is

$$\begin{aligned} \text{Area}(ABC) &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{12 \cdot 8 \cdot 3 \cdot 1} \\ &= \sqrt{288} = 12\sqrt{2}. \end{aligned}$$

(b) $h_a = \frac{2\text{Area}(ABC)}{a} = \frac{2 \cdot 12\sqrt{2}}{4} = 6\sqrt{2}$.
$h_b = \frac{2\text{Area}(ABC)}{b} = \frac{2 \cdot 12\sqrt{2}}{9} = \frac{8}{3}\sqrt{2}$.
$h_c = \frac{2\text{Area}(ABC)}{c} = \frac{2 \cdot 12\sqrt{2}}{11} = \frac{24}{11}\sqrt{2}$.
(c) The problem is that $s = \frac{2+3+7}{2} = 6$, whence $s - c = -1$ and Heron's formula requires the extraction of the square root of a negative quantity. This happens because the given lengths cannot be the lengths of the sides of a triangle. They violate the triangle inequality. ■