HOMEWORK 3 - MATH 325
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Problem 1 *Assume that $\triangle ABC$ and $\triangle EFG$ are similar with ratio k. Let $\overline{AD}$ be an altitude of $\triangle ABC$ and $\overline{EH}$ be an altitude of $\triangle EFG$. Prove that $EH = k \cdot AD$. What can you conclude about the areas of the two triangles?*

Solution: We have $\widehat{ABD} \cong \widehat{EFH}$ and $\widehat{ADB} \cong \widehat{EHF}$, whence $\triangle ABD \cong \triangle EFH$. Thus, since $EF = k \cdot AB$, we have $EH = k \cdot AD$. Hence

$$\text{Area}(EFG) = \frac{1}{2} FG \cdot EH = \frac{1}{2} k \cdot BC \cdot k \cdot AD = k^2 \text{Area}(ABC).$$

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Problem 2 *In triangles $\triangle ABC$ and $\triangle DEF$ assume that $ED = k \cdot AB$, $FE = k \cdot BC$ and that $\widehat{C}$ and $\widehat{F}$ are right angles. Prove that $\triangle ABC$ and $\triangle DEF$ are similar with ratio k.*

Solution: $\frac{ED}{AB} = \frac{FE}{BC} = k$ implies $\frac{ED^2}{AB^2} = \frac{FE^2}{BC^2} = k^2$, whence $\frac{ED^2 - FE^2}{AB^2 - BC^2} = k^2$. Therefore $\frac{DF^2}{AC^2} = k^2$, i.e., $\frac{DF}{AC} = k$. Now SAS for similar triangles applies to yield $\triangle ABC \sim_k \triangle DEF$. ■

Problem 3 *Let $\triangle ABC$ be a right triangle at C and with altitude $\overline{CD}$. prove that $\triangle ABC \sim \triangle ACD \sim \triangle CBD$. Use this to give another proof of the Pythagorean Theorem.*

Solution: We have $\widehat{ACB} \cong \widehat{ADC} \cong \widehat{BDC}$ and also $\widehat{BAC} \cong \widehat{DAC} \cong \widehat{BCD}$, whence $\triangle ACB \sim \triangle ADC \sim \triangle CDB$. These similarities yield $\frac{AC}{AB} = \frac{AD}{AC}$ and $\frac{BC}{AB} = \frac{BD}{BC}$, whence $AC^2 = AB \cdot AD$ and $BC^2 = AB \cdot BD$. Therefore

$$AB^2 = AB(AD + BD) = AB \cdot AD + AB \cdot BD = AC^2 + BC^2.$$

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Problem 4 *Let $\triangle ABC$ and $\triangle DEF$ be such that $\overline{AB} \parallel \overline{DE}, \overline{BC} \parallel \overline{EF}$ and $\overline{AC} \parallel \overline{DF}$. Prove that $\triangle ABC \sim \triangle DEF$.*

Solution: Suppose that P is the point of intersection of $\overline{AC}$ with $\overline{DE}$ and that Q is the point of intersection of $\overline{BC}$ with $\overline{DE}$. Then we have $\widehat{BAC} \cong \widehat{EPC} \cong \widehat{EDF}$ and $\widehat{ABC} \cong \widehat{DQC} \cong \widehat{DEF}$. Therefore $\triangle ABC \sim \triangle DEF$. ■

Problem 5 *Assume that you are given a line segment $\overline{AB}$ of length 1.*
(a) Given line segments of lengths a and b, construct a segment of length $\frac{a}{b}$.
(b) Given line segments of lengths a and b, construct a line segment of length ab.

Solution: (a) Take any angle $\widehat{A}$ and construct on one of its sides $\overline{AB}$, such that $AB = b$ and $\overline{BC}$, such that $BC = a$ and on its other side $\overline{AD}$, such that $AD = 1$. Now draw the line segment $\overline{BD}$ and construct the parallel from C to $\overline{BD}$. Let E be the point of intersection of this parallel with $\overline{AD}$. We claim that $DE = \frac{a}{b}$. Since $\overline{BD} \parallel \overline{CE}$, $\triangle ABD \sim \triangle ACE$, whence $\frac{AB}{AC} = \frac{AD}{AE}$, i.e., $\frac{AB}{BC} = \frac{AD}{DE}$. Therefore $\frac{b}{a} = \frac{1}{DE}$, whence $DE = \frac{a}{b}$.
(b) This is a similar construction. We take on one side of $\widehat{A}$ point B, such that $AB = 1$ and point C, such that $BC = a$ and on its other side point D, such that $AD = b$. Draw $\overline{BD}$ and construct the parallel to $\overline{BD}$ from C. Let E be the point of intersection of this parallel with $\overline{AD}$. Then, it can be shown exactly as in (a) that $DE = ab$. ■

Problem 6 *Given a triangle $\triangle ABC$ and a point P on $\overline{AB}$ with $AP > PB$, show how to construct a point Q on $\overline{AC}$ such that $\triangle APQ$ will have one-half the area of $\triangle ABC$.*

Solution: Let P' be the point on the extension of $\overline{AB}$ on the side of B such that $PP' = AP$. Draw $\overline{P'C}$ and construct the parallel from B to $\overline{P'C}$ intersecting $\overline{AC}$ at Q. Then $\text{Area}(APQ) = \frac{1}{2}\text{Area}(ABC)$. To see this is the case, calculate

$$\begin{aligned} \frac{\text{Area}(APQ)}{\text{Area}(ABC)} &= \frac{AP \cdot AQ}{AB \cdot AC} \\ &= \frac{1}{2}\frac{AP' \cdot AQ}{AB \cdot AC} \\ &= \frac{1}{2}\frac{AC}{AQ}\frac{AQ}{AC} \\ &= \frac{1}{2}. \end{aligned}$$

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Problem 7 *Find the indicated parts in each figure:*

Solution: (a) $\triangle AEC \sim \triangle DEB$, whence $\frac{AE}{EC} = \frac{DE}{EB}$, i.e., $AE \cdot EB = DE \cdot EC$. Therefore $4AE^2 = DE \cdot EC$, and, hence, $AE = \frac{3}{2}\sqrt{2}$. Therefore $EB = 4AE = 6\sqrt{2}$.
(b)

$$\text{Arc}(AD) = 2\widehat{ABD} = 2(90^o - \widehat{BDC}) = 2(90^o - \frac{1}{2}\text{Arc}(BC)) = 2(90^o - \frac{1}{2}40^o) = 140^o.$$

$$\text{Arc}(BD) = 2\widehat{BAD} = 2(90^o - \widehat{CDA}) = 2(90^o - \frac{1}{2}\text{Arc}(CA)) = 2(90^o - \frac{1}{2}30^o) = 150^o.$$

(c) We have $AC \cdot AD = AE \cdot AF = AB^2 = 100$. Also $AC + CD = AD$ which gives $AD = AC + 4$ and $AE + EF = AF$ which gives $AF = AE + 12$. Now we may set up two quadratic equations in AC and AE and solve them to find these lengths. ■

Problem 8 *(a) Prove that the perpendicular bisector of a chord of a circle is a diameter.*
(b) Given a circle, how would you construct the center?

Solution: (a) Let $\overline{AB}$ be a chord of a circle and $\overline{CD}$ its perpendicular bisector intersecting it at M. To show that $\overline{CD}$ is a diameter, it suffices to show that $\widehat{CAD} = 90^o$. We have by SAS, $\triangle ACM \cong \triangle BCM$ and $\triangle ADM \cong \triangle BDM$. Therefore $\widehat{CAM} \cong \widehat{CBM}$ and $\widehat{DAM} \cong \widehat{DBM}$. Hence $\widehat{CAD} \cong \widehat{CAM} + \widehat{DAM} \cong \widehat{CBM} + \widehat{DBM} \cong \widehat{CBD}$. But these two are supplementary angles, whence

$\widehat{CAD} = 90^o$, as was to be shown.
(b) Take any two chords $\overline{AB}$ and $\overline{BC}$ and construct their perpendicular bisectors. The point of intersection of their perpendicular bisectors is the center of the circle. ∎

Problem 9 *Given a quadrilateral $ABCD$ such that each of the four sides is tangent to a circle, prove that $AB + CD = AD + BC$.*

Solution: Let M, N, P and Q be the points of tangency of $\overline{AB}, \overline{BC}, \overline{CD}$ and $\overline{DA}$, respectively, with the given circle. We then have

$$\begin{aligned} AB + CD &= AM + MB + CP + DP \\ &= AQ + BN + CN + DQ \\ &= AQ + DQ + BN + CN \\ &= AD + BC. \end{aligned}$$

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